On the limit cycles of a class of discontinous piecewise differential systems formed by two rigid centers governed by odd degree polynomials

Tiago Carvalho

Departamento de Computação e Matemática; Faculdade de Filosofia, Ciências e Letras de Ribeirão Preto; Universidade de São Paulo, Av. Bandeirantes, 3900, CEP 14040-901, Ribeirão Preto, SP, Brazil.

e-mail adress: tiagocarvalho@usp.br

Luiz Fernando Gonçalves*

Instituto Federal de Educação, Ciência e Tecnologia de Minas Gerais Rua São Luiz Gonzaga, zip code 35577-020, Formiga, MG, Brazil

e-mail adress: luiz.goncalves@ifmg.edu.br

Jaume Llibre

Departament de Matemàtiques, Facultat de Ciències; Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Spain.

e-mail adress: jllibre@mat.uab.cat

Abstract

We provide an upper bound for the maximum number of limit cycles for the class of discontinuous piecewise differential systems formed by two differential systems separated by the straight line $x = 0$, one of which is a linear rigid center while the other is a rigid center formed of a linear part plus a homogeneous polynomial of odd degree. We solve the extended 16th Hilbert problem for this class of discontinuous piecewise differential systems.

Keywords: Piecewise smooth vector fields, rigid centers, limit cycles

2010 MSC: 34A36, 37G15, 34C05, 34C07

1. Introduction and main results

The study of the limit cycles is one of the most important objectives in the qualitative theory of the planar ordinary differential equations. Already in 1900 Hilbert [1] proposed a list of 23 relevant problems to be solved during the XX century. In his 16th problem Hilbert asked for an upper bound for the number of limit cycles for the class of planar polynomial vector fields of degree n. This problem remains unsolved for $n \geq 2$. We remark that to provide an upper bound for the maximum number of limit cycles in general is a very difficult problem for any class of given differential systems, non necessarily polynomial differential systems.

The study of the discontinuous piecewise differential systems, more recently also called Filippov systems, has attracted the attention of the mathematicians during these past decades due to their applications. These piecewise differential systems in the plane are formed by different differential systems defined in distinct regions separated by a curve. A pioneering work on this subject was due to Andronov, Vitt and Khaikin [2] in 1920's, and later on to Filippov [3] in 1988 who provide the theoretical bases for this kind of differential systems. Nowadays a vast literature on these differential systems is available. See for instance [4] for the main theory and some applications, [5]

*Corresponding author

for applications in electrical circuits, [6] for applications in mechanical models, [7] for applications in relay systems, [**?** 8] for applications in biological models, among others. As for the smooth differential systems the study of the existence and location of limit cycles in the piecewise differential systems is also of great importance.

Let $p \in \mathbb{R}^2$ be a singular point of an analytic differential system in the plane. The singularity p is a *center* if there exists an open neighborhood U of p such that all the solutions in $U \setminus \{p\}$ are periodic. Denote by $\mathcal{T}_q$ the period of the periodic orbit through the point $q \in U \setminus \{p\}$. We say that p is an *isochronous center* if $\mathcal{T}_q$ is constant for all $q \in U \setminus \{p\}$. An isochronous center is *uniform* or *rigid* if the angular velocity of the vector field is the same for all periodic orbits in $U \setminus \{p\}$. In other words, an isochronous center is rigid if in polar coordinates (r, θ) defined by $x = r\cos\theta$, $y = r\sin\theta$ it can be written as $\dot{r} = G(r, \theta)$, $\dot{\theta} = k$, where $k \neq 0$ is a constant. See [9, 10] for details. After scaling the time (if necessary) it is always possible to consider $\dot{\theta} = 1$ in the previous expression.

Isochronicity in the regular case has been widely studied in the last decades, see for instance [11, 12, 13, 14, 15, 16, 17] due to its importance in applications involving physical phenomena. In recent years isochronicity has also been explored for the discontinuous piecewise differential systems, see for instance [18, 19, 20], by considering the coupling of two or more centers and investigating their dynamics.

The main goal of this paper is to provide the maximum number of limit cycles that can exhibit the discontinuous piecewise differential systems formed by the coupling of two special rigid centers whose discontinuous curve is the straight line $x = 0$. More precisely, in one half-plane we consider the linear rigid center

$$\dot{x} = -y, \qquad \dot{y} = x, \tag{1}$$

while in the other half-plane, given an odd n, we consider a rigid center of the form

$$\dot{x} = -y + x\sum_{i=0}^{n-1} a_i x^{n-i-1} y^i, \qquad \dot{y} = x + y\sum_{i=0}^{n-1} a_i x^{n-i-1} y^i, \tag{2}$$

but both centers after an arbitrary affine change of variables.

Our main result is the following one.

Theorem 1.1. *Consider the discontinuous piecewise differential systems formed by system* (1) *in* $x \leq 0$ *after the affine change of variables*

$$(X, Y) = (b_1 x + b_2 y + d_1, b_3 x + b_4 y + d_2), \qquad b_i \in \mathbb{R},\ d_j \in \mathbb{R},\ \textit{with } i = 1,2,3,4 \textit{ and } j = 1,2,$$

and system (2) *in* $x \geq 0$ *after the affine change of variables*

$$(X, Y) = (c_1 x + c_2 y + M_1, c_3 x + c_4 y + M_2), \qquad c_i \in \mathbb{R},\ M_j \in \mathbb{R},\ \textit{with } i = 1,2,3,4 \textit{ and } j = 1,2,$$

such that their centers continue being rigid centers. Then such discontinuous piecewise differential systems have at most $n - 2$ *limit cycles for* n *odd.*

The paper is organized as follows. In section 2 we recall the basic theory of the piecewise smooth vector fields. Section 3 brings some considerations about the rigid centers considered in this work. Theorem 1.1 is proved in section 4. Finally section 5 closes the paper with concluding remarks.

2. Preliminary definitions and results

In this section we present the basic results of the theory on piecewise smooth vector fields that we need. A *discontinuous piecewise smooth vector field* on an open set $U \subset \mathbb{R}^2$ is a pair of C^r-vector fields X and Y with $r \geq 1$, defined on U separated by a smooth codimension one manifold Σ. The *switching manifold* Σ or *line of discontinuity* of the discontinuous piecewise differential system is obtained by considering $\Sigma = h^{-1}(0)$, where $h : U \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a differentiable function having 0 as a regular value. Note that Σ is the separating boundary of the regions $\Sigma^+ = \{(x,y) \in U \,|\, h(x,y) > 0\}$ and $\Sigma^- = \{(x,y) \in U \,|\, h(x,y) < 0\}$. So, a piecewise smooth vector field is provided by

$$Z(x,y) = \begin{cases} X(x,y), & \text{if } h(x,y) \geq 0, \\ Y(x,y), & \text{if } h(x,y) \leq 0. \end{cases} \tag{3}$$

As usual, system (3) is denoted by $Z = (X, Y, \Sigma)$ or simply by $Z = (X, Y)$, when the separation line Σ is well understood. In order to establish a definition for the trajectories of Z and investigate its behavior, we need a criterion for the transition of the orbits between Σ^+ and Σ^- across Σ. The contact between the vector field X (or Y) and the switching manifold Σ is characterized by the derivative of h in the direction of the vector field X (also known as the Lie derivative of h with respect to X), that is by the expression

$$Xh(p) = \langle \nabla h(p), X(p) \rangle ,$$

and for $i \geq 2$ we define $X^i h(p) = \left\langle \nabla X^{i-1} h(p), X(p) \right\rangle$, where $\langle .,. \rangle$ is the usual inner product in $\mathbb{R}^2$. The basic results of the discontinuous piecewise differential systems in this context were stated by Filippov [3]. We can divide the switching manifold Σ in the following sets:

(*a*) *Crossing set*: $\Sigma^c : \{p \in \Sigma : Xh(\mathbf{x}) \cdot Yh(\mathbf{x}) > 0\}$.

(*b*) *Escaping set*: $\Sigma^e : \{p \in \Sigma : Xh(\mathbf{x}) > 0 \text{ and } Yh(\mathbf{x}) < 0\}$.

(*c*) *Sliding set*: $\Sigma^s : \{p \in \Sigma : Xh(\mathbf{x}) < 0 \text{ and } Yh(\mathbf{x}) > 0\}$.

The *escaping* Σ^e or *sliding* Σ^s regions are respectively defined on points of Σ where both vector fields X and Y simultaneously point outwards or inwards from Σ while the interior of its complement in Σ defines the *crossing region* Σ^c (see Figure 1). The complementary of the union of these regions is the set formed by the *tangency* points between X or Y with Σ.

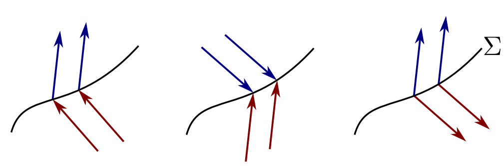

Figure 1: Crossing, sliding and escaping regions, respectively.

A point $p \in \Sigma$ is called a *tangency* point of X (resp. Y) if it satisfies $Xh(p) = 0$ (resp. $Yh(p) = 0$). A tangency point is called a *fold* point of X if $X^2h(p) \neq 0$. Moreover, $p \in \Sigma$ is a *visible* (resp. *invisible*) fold point of X if $X^2h(p) > 0$ (resp. $X^2h(p) < 0$).

In order to define a trajectory of a discontinuous piecewise differential system passing through a crossing point, it is enough to concatenate the trajectories of X and Y through that point. However in the sliding and escaping sets we need to define an auxiliary vector field. So we consider the Filippov's convention (see [3]) and a new vector field is defined on $\Sigma^s \cup \Sigma^e$. This new vector field, called *sliding vector field*, is a convex linear combination of $X(p)$ and $Y(p)$ in a way that Z^s is tangent to Σ in the cone generated by $X(p)$ and $Y(p)$. In this scenario the trajectories of Z are considered as a concatenation of trajectories of X, Y and Z^s.

Furthermore given a discontinuous piecewise vector field $Z = (X, Y)$ we say that an equilibrium point p of X is *real* if $p \in \Sigma^+$ and it is *virtual* if $p \in \Sigma^-$. Similar definitions for the equilibria of Y.

Given a vector field $F(x, y) = (F_1(x, y), F_2(x, y))$, defined on an open set $U \subset \mathbb{R}^2$, we consider the corresponding ordinary differential equations

$$\dot{x} = \frac{dx}{dt} = F_1(x, y), \qquad \dot{y} = \frac{dy}{dt} = F_2(x, y), \tag{4}$$

where the independent variable t is called the *time*. Denote the flow of (4) by φ_F or simply by φ when there is no danger of confusion. Also denote by $\varphi_F(t, p)$ or by $\varphi(t, p)$ the solution of system (4) by the point p such that $\varphi(0, p) = p$.

When the trajectory of the vector field X through $p \in \Sigma$ returns to Σ (by the first time) after a positive time $t_1(p)$, called X–flight time, we define the *half return map associated with* X by $\pi_X(p) = \phi_X(t_1(p), p) = p_1 \in \Sigma$. When the trajectory of Y through $p_1 \in \Sigma$ returns to Σ (by the first time) after a positive time $t_2(p_1)$, called Y–flight time, we define the *half return map associated with* Y by $\pi_Y(p_1) = \phi_Y(t_2(p_1), p_1) \in \Sigma$. The *first return map* associated with $Z = (X, Y)$ is defined by the composition of these two transition maps, that is,

$$\pi_Z(p) = \pi_Y \circ \pi_X(p) = \phi_Y(t_2(p_1), \phi_X(t_1(p), p))$$

or the reverse, applying first the flow of Y and after the flow of X. See Figure 2.

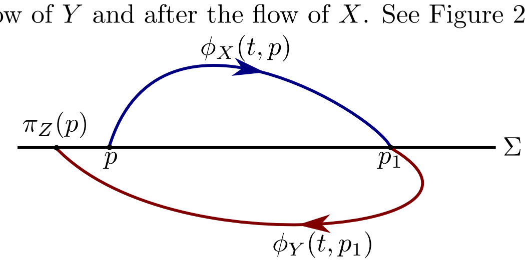

Figure 2: First return map of a discontinuous piecewise differential system.

When the vector fields X and Y associated to $Z = (X, Y)$ have a first integral (see [21]), the solution curves of the respective differential equations are contained in the level sets of the first integrals. In this scenario the first return map can be handily computed by seeking for points in Σ that are on the same level curves of these first integral functions. In this case we avoid working with flight times.

3. Considerations about the rigid centers

We consider a polynomial differential system in $\mathbb{R}^2$, $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ of degree n. Assume that it has a center at the origin of coordinates. Then it is well known, see for instance [10, 22], that this center is a uniform isochronous if and only if by doing a linear change of variables and a rescaling of time it can be written as

$$\dot{x} = -y + xf(x,y), \qquad \dot{y} = x + yf(x,y) \tag{5}$$

where $f(x,y)$ is a polynomial of degree $n-1$ in the variables x and y, and $f(0,0) = 0$. Consider system (5). Conti [10] proved the following result.

Proposition 3.1. *Let $f(x,y) = \sum_{i=0}^{n-1} a_i x^{n-i-1} y^i$ be a homogeneous polynomial of degree $n-1$. Then the origin is a uniform isochronous center of system* (5) *if either n is even, or n is odd and*

$$\sum_{i=0}^{n-1} a_i \int_0^{2\pi} \cos^{n-i-1}\theta \sin^i \theta \, d\theta = \sum_{i=0}^{n-1} a_i I_{n-i-1,i} = 0. \tag{6}$$

When n is even, $\sum_{i=0}^{n-1} a_i \, I_{n-i-1,i} = 0$ always holds, because $f(\cos(\theta+\pi), \sin(\theta+\pi)) = -f(\cos\theta, \sin\theta)$, where

$$f(\cos\theta, \sin\theta) = \sum_{i=0}^{n-1} a_i \cos^{n-i-1}\theta \sin^i \theta.$$

If n odd, from the formulas of sections 2.511 e 2.512 of [23] we get that $I_{n-i-1,i} = 0$ when i odd, and consequently,

$$\begin{aligned}\sum_{i=0}^{n-1} a_i \int_0^{2\pi} \cos^{n-i-1}\theta \sin^i \theta \, d\theta = a_0 &\left[\frac{2\pi(n-2)!!}{2^{\frac{n-1}{2}} \left(\frac{n-1}{2}\right)!}\right] + \\ &+ \sum_{k=1}^{\frac{n-3}{2}} a_{2k} \left[\frac{2\pi(2k-1)!!(n-2k-2)!!}{(n-1)(n-3)\cdots(n-2k+1)\, 2^{\frac{n-2k-1}{2}} \left(\frac{n-2k-1}{2}\right)!}\right] + \\ &+ a_{n-1} \left[\frac{2\pi(n-2)!!}{2^{\frac{n-1}{2}} \left(\frac{n-1}{2}\right)!}\right],\end{aligned}$$

where $(2p)!! = p(p-2)(p-4)\cdots 2$ and $(2p+1)!! = (2p+1)(2p-1)\cdots 3\cdot 1$. Thus, when n is odd, hypothesis (6) is equivalent to take

$$a_{n-1} = -a_0 - \frac{2^{\frac{n-1}{2}} \left(\frac{n-1}{2}\right)!}{(n-2)!!} \sum_{j=1}^{\frac{n-3}{2}} a_{2j} \left[\frac{(2j-1)!!(n-2j-2)!!}{2^{\frac{n-2j-1}{2}} \left(\frac{n-2j-1}{2}\right)! \prod_{i=1}^{j} (n-2i+1)}\right]. \tag{7}$$

So in this paper under the conditions of Proposition 3.1 we shall work with rigid centers of the form (2).

We denote by

$$Y_n(x,y) = \left(-y + x\sum_{i=0}^{n-1} a_i x^{n-i-1} y^i, x + y\sum_{i=0}^{n-1} a_i x^{n-i-1} y^i\right)$$

the vector field associated to system (2). Equivalently in polar coordinates system (2) writes

$$\begin{aligned}\dot{r} &= r^n \sum_{i=0}^{n-1} a_i \cos^{n-i-1}\theta \sin^i \theta \\ \dot{\theta} &= 1.\end{aligned} \tag{8}$$

System (8) has the first integral

$$H_n(r,\theta) = \frac{1}{(1-n)r^{n-1}} - \sum_{i=0}^{n-1} a_i \int \cos^{n-i-1}\theta \sin^i \theta d\theta. \tag{9}$$

So in Cartesian coordinates the first integral of system (2) becomes

$$H_n(x,y) = \frac{1}{(1-n)(x^2+y^2)^{\frac{n-1}{2}}} - \sum_{i=0}^{n-1} a_i \int \frac{x^{n-i}y^i}{(x^2+y^2)^{\frac{n+1}{2}}} dy.$$

By coupling the two rigid centers (1) and (2) we can consider the piecewise smooth vector field

$$Z(x,y) = \begin{cases} Y_n(x,y), & x \geq 0, \\ X(x,y), & x \leq 0, \end{cases} \tag{10}$$

where $X(x,y) = (-y,x)$. Observe that the switching manifold is the straight line $\Sigma = h^{-1}(0)$, where $h(x,y) = x$. Since the vector fields X and Y_n associated with the piecewise smooth vector field $Z = (Y_n, X)$ are integrable, then the solution curves of the respective differential equation are contained in the level sets of their respective first integrals. Thus the first return map associated with the discontinuous piecewise vector field Z can be computed by seeking for points in the switching manifold Σ that are on the same level curves of these first integrals. In this way for a limit cycle of the discontinuous piecewise differential system associated with Z given by (10) which has two intersecting points $(0,y_1)$ and $(0,y_2)$ with the line of discontinuity $\Sigma = \{x=0\}$, its coordinates y_1 and y_2 must be an isolated zero of the set of equations

$$\begin{cases} F(0,y_1) = F(0,y_2), \\ H_n(0,y_1) = H_n(0,y_2), \end{cases} \quad \text{or, in polar coordinates,} \quad \begin{cases} F(r_0,\pi/2) = F(R_0,3\pi/2), \\ H_n(r_0,\pi/2) = H_n(R_0,3\pi/2), \end{cases}$$

where $(r_0,\pi/2)$ and $(R_0,3\pi/2)$ are the respective intersecting points $(0,y_1)$ and $(0,y_2)$ in polar coordinates. Now, we move towards finding the integral $\int \cos^{n-i-1}\theta \sin^i \theta d\theta$ in order to have a complete characterization of the first integral H_n which is of paramount importance in this work.

3.1. n odd

We consider that n is odd. Then we have that

$$\sum_{i=0}^{n-1} a_i \int \cos^{n-i-1}\theta \sin^i\theta d\theta = a_0 \int \cos^{n-1}\theta d\theta + \sum_{j=1}^{\frac{n-1}{2}} a_{2j-1} \int \cos^{n-2j}\theta \sin^{2j-1}\theta d\theta +$$
$$+ \sum_{j=1}^{\frac{n-3}{2}} a_{2j} \int \cos^{n-1-2j}\theta \sin^{2j}\theta d\theta + a_{n-1} \int \sin^{n-1}\theta d\theta.$$

By formulas 2.512(2), 2.512(4), 2.512(1), and 2.511(2) of [23] we get

$$\Omega_0 = \int \cos^{n-1}\theta d\theta = \frac{\sin\theta}{n-1}\left[\cos^{n-2}\theta + \sum_{k=1}^{\frac{n-3}{2}} \frac{(n-2)(n-4)\cdots(n-2k)\cos^{n-2k-2}\theta}{2^k\left(\frac{n-3}{2}\right)\left(\frac{n-5}{2}\right)\cdots\left(\frac{n-1}{2}-k\right)}\right] + \frac{(n-2)!!\theta}{2^{\frac{n-1}{2}}\left(\frac{n-1}{2}\right)!},$$

$$\Omega_{2j-1} = \int \cos^{n-2j}\theta \sin^{2j-1}\theta d\theta = -\frac{\cos^{n-2j+1}\theta}{n-1}\left[\sin^{2j-2}\theta + \sum_{k=1}^{j-1} \frac{2^k(j-1)(j-2)\cdots(j-k)\sin^{2j-2k-2}\theta}{(n-3)(n-5)\cdots(n-2k-1)}\right],$$

$$
\begin{aligned}
\Omega_{2j} &= \int \cos^{n-1-2j}\theta \sin^{2j}\theta d\theta \\
&= -\frac{\cos^{n-2j}\theta}{n-1}\left[\sin^{2j-1}\theta + \sum_{k=1}^{j-1}\frac{(2j-1)(2j-3)\cdots(2j-2k+1)\sin^{2j-2k-1}\theta}{(n-3)(n-5)\cdots(n-2k-1)}\right] + \\
&+ \frac{(2j-1)!!}{(n-1)(n-3)\cdots(n-2j+1)}\left[\frac{(n-2j-2)!!\,\theta}{2^{\frac{n-2j-1}{2}}(\frac{n-2j-1}{2})!} + \frac{\sin\theta}{n-1-2j}\cdot\right.\\
&\left.\cdot\left(\cos^{n-2j-2} + \sum_{k=1}^{\frac{n-2j-3}{2}}\frac{(n-2j-2)(n-2j-4)\cdots(n-2j-2k)\cos^{n-2j-2k-2}\theta}{2^k\left(\frac{n-2j-3}{2}\right)\left(\frac{n-2j-5}{2}\right)\cdots\left(\frac{n-2j-1}{2}-k\right)}\right)\right],
\end{aligned}
$$

and

$$
\Omega_{n-1} = \int \sin^{n-1}\theta d\theta = -\frac{\cos\theta}{n-1}\left[\sin^{n-2}\theta + \sum_{k=1}^{\frac{n-3}{2}}\frac{(n-2)(n-4)\cdots(n-2k)\sin^{n-2k-2}\theta}{2^k\left(\frac{n-3}{2}\right)\left(\frac{n-5}{2}\right)\cdots\left(\frac{n-1}{2}-k\right)}\right] + \frac{(n-2)!!\theta}{2^{\frac{n-1}{2}}(\frac{n-1}{2})!}.
$$

Therefore the first integral H_n given by (9) for n odd can be written as

$$
H_n(r,\theta) = \frac{1}{(1-n)r^{n-1}} - a_0\Omega_0 - \sum_{j=1}^{\frac{n-1}{2}} a_{2j-1}\Omega_{2j-1} - \sum_{j=1}^{\frac{n-3}{2}} a_{2j}\Omega_{2j} - a_{n-1}\Omega_{n-1}.
$$

In Cartesian coordinates we obtain that H_n can be expressed in the form

$$
\begin{aligned}
H_n(x,y) &= \frac{-1}{(n-1)(x^2+y^2)^{\frac{n-1}{2}}} - \\
& a_0\left\{\frac{y}{(x^2+y^2)^{\frac{1}{2}}(n-1)}\left(\frac{x^{n-2}}{(x^2+y^2)^{\frac{n-2}{2}}} + \sum_{k=1}^{\frac{n-3}{2}}\frac{x^{n-2k-2}\prod_{i=1}^k(n-2i)}{2^k(x^2+y^2)^{\frac{n-2k-2}{2}}\prod_{i=1}^k\left(\frac{n-1}{2}-i\right)}\right) + \frac{(n-2)!!\arctan(y/x)}{2^{\frac{n-1}{2}}(\frac{n-1}{2})!}\right\} \\
& - \sum_{j=1}^{\frac{n-1}{2}} a_{2j-1}\left\{\frac{-x^{n-2j+1}}{(x^2+y^2)^{\frac{n-2j+1}{2}}(n-1)}\left(\frac{y^{2j-2}}{(x^2+y^2)^{j-1}} + \sum_{k=1}^{j-1}\frac{2^k y^{2j-2k-2}\prod_{i=1}^k(j-i)}{(x^2+y^2)^{\frac{2j-2k-2}{2}}\prod_{i=1}^k(n-2i-1)}\right)\right\} - \\
& \sum_{j=1}^{\frac{n-3}{2}} a_{2j}\left\{\frac{-x^{n-2j}}{(x^2+y^2)^{\frac{n-2j}{2}}(n-1)}\left(\frac{y^{2j-1}}{(x^2+y^2)^{\frac{2j-1}{2}}} + \sum_{k=1}^{j-1}\frac{y^{2j-2k-1}\prod_{i=1}^k(2j-2i+1)}{(x^2+y^2)^{\frac{2j-2k-1}{2}}\prod_{i=1}^k(n-2i-1)}\right) + \right.\\
& \frac{(2j-1)!!}{\prod_{i=1}^j(n-2i+1)}\left(\frac{y}{(x^2+y^2)^{\frac{1}{2}}(n-2j-1)}\left[\frac{x^{n-2j-2}}{(x^2+y^2)^{\frac{n-2j-2}{2}}} + \sum_{k=1}^{\frac{n-2j-3}{2}}\frac{x^{n-2j-2k-2}\prod_{i=1}^k(n-2j-2i)}{2^k(x^2+y^2)^{\frac{n-2j-2k-2}{2}}\prod_{i=1}^k\left(\frac{n-2j-1}{2}-i\right)}\right]\right.\\
& \left.\left. + \frac{(n-2j-2)!!\arctan(y/x)}{2^{\frac{n-2j-1}{2}}(\frac{n-2j-1}{2})!}\right)\right\} - a_{n-1}\left\{\frac{-x}{(x^2+y^2)^{\frac{1}{2}}(n-1)}\left(\frac{y^{n-2}}{(x^2+y^2)^{\frac{n-2}{2}}} + \right.\right.\\
& \left.\left. + \sum_{k=1}^{\frac{n-3}{2}}\frac{y^{n-2k-2}\prod_{i=1}^k(n-2i)}{2^k(x^2+y^2)^{\frac{n-2k-2}{2}}\prod_{i=1}^k\left(\frac{n-1}{2}-i\right)} + \frac{(n-2)!!\arctan(y/x)}{2^{\frac{n-1}{2}}(\frac{n-1}{2})!}\right)\right\}.
\end{aligned}
$$

By considering hypothesis (7) of Conti we have that the arctangent functions will cancel out and, after straightforward calculations, we get

$$
\begin{aligned}
H_n(x,y) = {} & \frac{1}{(x^2+y^2)^{\frac{n-1}{2}}} \left\{ -\frac{1}{n-1} - \frac{a_0 y x^{n-2}}{n-1} - \frac{a_0}{n-1} \sum_{j=1}^{\frac{n-3}{2}} \frac{y x^{n-2j-2} \left(x^2+y^2\right)^j \prod_{i=1}^{j}(n-2i)}{2^j \prod_{i=1}^{j} \left(\frac{n-1}{2}-i\right)} + \right. \\
& + \sum_{j=1}^{\frac{n-1}{2}} a_{2j-1} \frac{y^{2(j-1)} x^{n-2j+1}}{n-1} + \sum_{j=1}^{\frac{n-1}{2}} \frac{a_{2j-1}}{n-1} \left(\sum_{k=1}^{j-1} \frac{2^k y^{2j-2k-2} x^{n-2j+1} \left(x^2+y^2\right)^k \prod_{i=1}^{k}(j-i)}{\prod_{i=1}^{k}(n-1-2i)} \right) + \\
& + \sum_{j=1}^{\frac{n-3}{2}} a_{2j} \frac{y^{2j-1} x^{n-2j}}{n-1} + \sum_{j=1}^{\frac{n-3}{2}} \frac{a_{2j}}{n-1} \left(\sum_{k=1}^{j-1} \frac{y^{2j-2k-1} x^{n-2j} \left(x^2+y^2\right)^k \prod_{i=1}^{k}(2j-2i+1)}{\prod_{i=1}^{k}(n-1-2i)} \right) \\
& - \sum_{j=1}^{\frac{n-3}{2}} a_{2j} \frac{(2j-1)!! y x^{n-2j-2} \left(x^2+y^2\right)^j}{(n-1-2j) \prod_{i=1}^{j}(n-2i+1)} - \sum_{j=1}^{\frac{n-3}{2}} \frac{a_{2j}(2j-1)!!}{(n-1-2j) \prod_{i=1}^{j}(n-2i+1)} \cdot \\
& \cdot \left(\sum_{k=1}^{\frac{n-2j-3}{2}} \frac{y x^{n-2j-2k-2} \left(x^2+y^2\right)^{j+k} \prod_{i=1}^{k}(n-2j-2i)}{2^k \prod_{i=1}^{k} \left(\frac{n-1-2j}{2}-i\right)} \right) - \frac{a_0 x y^{n-2}}{n-1} \\
& - \frac{a_0}{n-1} \sum_{j=1}^{\frac{n-3}{2}} \frac{x y^{n-2j-2} \left(x^2+y^2\right)^j \prod_{i=1}^{j}(n-2i)}{2^j \prod_{i=1}^{j} \left(\frac{n-1}{2}-i\right)} + \left(\frac{2^{\frac{n-1}{2}} \left(\frac{n-1}{2}\right)!}{(n-1)(n-2)!!} \right) \cdot \\
& \left. \left(\sum_{j=1}^{\frac{n-3}{2}} \frac{a_{2j}(2j-1)!!(n-2j-2)!!}{2^{\frac{n-2j-1}{2}} \left(\frac{n-2j-1}{2}\right)! \prod_{i=1}^{j}(n-2i+1)} \right) \cdot \left(-x y^{n-2} - \sum_{j=1}^{\frac{n-3}{2}} \frac{x y^{n-2j-2} \left(x^2+y^2\right)^j \prod_{i=1}^{j}(n-2i)}{2^j \prod_{i=1}^{j} \left(\frac{n-1}{2}-i\right)} \right) \right\}.
\end{aligned}
\tag{11}
$$

4. Systems (1) and (2) after an affine change of variables

The discontinuous piecewise vector field (10) has the centers of both smooth vector fields placed at the origin of coordinates. Now, we give the expression of the differential systems (1) and (2) and their first integrals after the respective general affine change of variables

$$(X,Y) = (b_1 x + b_2 y + d_1, b_3 x + b_4 y + d_2), \qquad b_i \in \mathbb{R},\ d_j \in \mathbb{R},\ \text{with } i = 1,2,3,4 \text{ and } j = 1,2, \tag{12}$$

and

$$(X,Y) = (c_1 x + c_2 y + M_1, c_3 x + c_4 y + M_2), \qquad c_i \in \mathbb{R},\ M_j \in \mathbb{R},\ \text{with } i = 1,2,3,4 \text{ and } j = 1,2. \tag{13}$$

We want to investigate the number of limit cycles of the discontinuous piecewise vector field (10) after these changes of variables. However after these changes of variables we still want that the centers of each smooth vector field are rigid.

First we consider system (1) in $x \leq 0$ after the change of variables (12). The new linear system has a center at $P = (d_1, d_2)$. Doing the change of variables $(X,Y) = (z + d_1, w + d_2)$ we get

$$
\begin{aligned}
\dot z &= -\frac{(b_1 b_3 + b_2 b_4) z}{b_2 b_3 - b_1 b_4} + \frac{\left(b_1^2 + b_2^2\right) w}{b_2 b_3 - b_1 b_4}, \\
\dot w &= -\frac{\left(b_3^2 - + b_4^2\right) z}{b_2 b_3 - b_1 b_4} + \frac{(b_1 b_3 + b_2 b_4) w}{b_2 b_3 - b_1 b_4}.
\end{aligned}
\tag{14}
$$

Equivalently in polar coordinates we obtain

$$\dot{r} = \frac{r\left((b_1^2+b_2^2-b_3^2-b_4^2)\sin(2\theta)-2(b_3b_1+b_2b_4)\cos(2\theta)\right)}{2(b_2b_3-b_1b_4)},$$
$$\dot{\theta} = \frac{-(b_1^2+b_2^2)\sin^2\theta+(b_1b_3+b_2b_4)\sin(2\theta)-\left(b_3^2+b_4^2\right)\cos^2\theta}{b_2b_3-b_1b_4}.$$

Using polar coordinates we conclude that (14) has a rigid center at the origin if and only if $b_1 = b_4$ and $b_2 = -b_3$ and (14) writes

$$\dot{z} = -w \qquad \dot{w} = z.$$

So

$$\begin{cases} \dot{X} = -Y + d_2 \\ \dot{Y} = X - d_1 \end{cases} \quad \text{or, in polar coordinates} \quad \begin{cases} \dot{r} = d_2\cos\theta - d_1\sin\theta, \\ \dot{\theta} = \dfrac{r - d_2\sin\theta - d_1\cos\theta}{r}. \end{cases} \tag{15}$$

Observe that a first integral of (15) is

$$F_1(X,Y) = (X-d_1)^2 + (Y-d_2)^2 \text{ or, in polar coordinates, } F_1(r,\theta) = (r\cos\theta - d_1)^2 + (r\sin\theta - d_2)^2. \tag{16}$$

On the other hand we consider in $x \geq 0$ system (2) with the affine change of variables (13). The new system has a center at $Q = (M_1, M_2)$. Doing the change of variables $(X,Y) = (z + M_1, w + M_2)$ we get

$$\begin{aligned} \dot{z} &= \frac{\left(c_1^2+c_2^2\right)w - (c_1c_3+c_2c_4)\,z}{c_2c_3-c_1c_4} + z\,(c_2c_3-c_1c_4)^{\,1-n}\left(\sum_{i=0}^{n-1} a_i\,(c_3z-c_1w)^{\,i}\,(c_2w-c_4z)^{\,n-i-1}\right), \\ \dot{w} &= \frac{(c_1c_3+c_2c_4)\,w - \left(c_3^2+c_4^2\right)z}{c_2c_3-c_1c_4} + w\,(c_2c_3-c_1c_4)^{\,1-n}\left(\sum_{i=0}^{n-1} a_i\,(c_3z-c_1w)^{\,i}\,(c_2w-c_4z)^{\,n-i-1}\right), \end{aligned} \tag{17}$$

or equivalently in polar coordinates

$$\dot{r} = \frac{r\left(\left(c_1^2+c_2^2-c_3^2-c_4^2\right)\sin(2\theta) - 2\,(c_1c_3+c_2c_4)\cos(2\theta) + 2\,(c_2c_3-c_1c_4)^{\,2-n}\cdot S\right)}{2c_2c_3-2c_1c_4},$$
$$\dot{\theta} = -\frac{(c_1\sin\theta - c_3\cos\theta)^2 + (c_2\sin\theta - c_4\cos\theta)^{\,2}}{c_2c_3-c_1c_4},$$

where

$$S = \sum_{i=0}^{n-1} a_i\,(c_3r\cos\theta - c_1r\sin\theta)^{\,i}\,(c_2r\sin\theta - c_4r\cos\theta)^{\,n-i-1}.$$

Again using polar coordinates we conclude that this system has a rigid center at the origin if and only if $c_4 = c_1$ and $c_3 = -c_2$. Under these conditions system (17) writes

$$\dot{z} = -w + z\left(c_1^2+c_2^2\right)^{\,1-n}\left(\sum_{i=0}^{n-1} a_i\,(c_2z+c_1w)^{\,i}\,(c_1z-c_2w)^{\,n-i-1}\right),$$
$$\dot{w} = z + w\left(c_1^2+c_2^2\right)^{\,1-n}\left(\sum_{i=0}^{n-1} a_i\,(c_2z+c_1w)^{\,i}\,(c_1z-c_2w)^{\,n-i-1}\right),$$

or equivalently

$$\dot{r} = r^n\left(c_1^2+c_2^2\right)^{\,1-n}\sum_{i=0}^{n-1} a_i\,(c_1\sin\theta + c_2\cos\theta)^{\,i}\,(c_1\cos\theta - c_2\sin(\theta))^{\,n-i-1},$$

$$\dot{\theta} = 1.$$

Now if we go back through the change of variables $(X,Y)=(z+M_1,w+M_2)$ we get

$$\begin{aligned}\dot{X} &= M_2 - Y + \left(c_1^2+c_2^2\right)^{1-n}(X-M_1)\left(\sum_{i=0}^{n-1} a_i\left(c_2(X-M_1)+c_1(Y-M_2)\right)^{i}\left(c_1(X-M_1)-c_2(Y-M_2)\right)^{n-i-1}\right),\\ \dot{Y} &= X - M_1 + \left(c_1^2+c_2^2\right)^{1-n}(Y-M_2)\left(\sum_{i=0}^{n-1} a_i\left(c_2(X-M_1)+c_1(Y-M_2)\right)^{i}\left(c_1(X-M_1)-c_2(Y-M_2)\right)^{n-i-1}\right).\end{aligned} \tag{18}$$

Denote by $Y_n^1(X,Y)$ the vector field associated with system (18). Observe that the first integral of system (18) is the first integral of the original vector field evaluated at the change of variables, that is,

$$F_2(X,Y) = H_n\left(\frac{c_1(X-M_1)+c_2(M_2-Y)}{c_1^2+c_2^2}, \frac{c_1(Y-M_2)-c_2(M_1-X)}{c_1^2+c_2^2}\right)$$

where H_n is given by (11). Notice that $F_2(X,Y)$ can be written as

$$F_2(X,Y) = \frac{f_1(X,Y)}{g_1(X,Y)},$$

where $g_1(X,Y) = \left[(X-M_1)^2+(Y-M_2)^2\right]^{\frac{n-1}{2}}$ and $f_1(X,Y) = F_2(X,Y)g_1(X,Y)$ is a polynomial of degree $n-1$. From equation (16) we have that $F_1(X,Y) = (X-d_1)^2+(Y-d_2)^2$ is a first integral of $X_1(X,Y) = (-Y+d_2, X-d_1)$. So the limit cycles for the discontinuous piecewise vector field

$$Z_1(X,Y) = \begin{cases} Y_n^1(X,Y), & x \geq 0,\\ X_1(X,Y), & x \leq 0,\end{cases} \tag{19}$$

are given by the solutions of the following system of equations

$$F_1(0,y_0) = F_1(0,y_1), \qquad F_2(0,y_0) = F_2(0,y_1). \tag{20}$$

From the first equation of (20) we obtain that $y_1 = -y_0+2d_2$, for every $y_0, y_1 \in \mathbb{R}$ with $y_0 \neq y_1$. Substituting this expression into the second equation of (20) we get $F_2(0,y_0) = F_2(0,-y_0+2d_2)$, that is,

$$f_1(0,y_0)\,g_1(0,-y_0+2d_2) - g_1(0,y_0)\,f_1(0,-y_0+2d_2) = 0. \tag{21}$$

Since n is odd, we get that $(n-1)/2$ is an integer and g_1 is also a polynomial of degree $n-1$. Therefore equation (21) is a polynomial of degree $2(n-1)$ and therefore has at most $2(n-1)$ zeros. So we conclude that the discontinuous piecewise differential system Z_1 can have at most $n-1$ limit cycles because if y_0 is a solution of (21) then $y_1 = -y_0+2d_2$ is also a solution of (21), and every pair $(y_0, y_1 = -y_0+2d_2)$ can determine a limit cycle. However, $y_0 = d_2$ is always a solution of (21) so that this solution corresponds to the periodic orbit of X_1 that is tangent to the separation line given by the y-axis. Therefore this solution does not correspond to a limit cycle. So, we can reduce the upper bound to $n-2$.

5. Conclusion and Further Directions

In this paper we have studied the upper bound for the maximum number of limit cycles of discontinuous piecewise differential systems formed by two differential systems separated by the straight line $x=0$, one of which is a linear rigid center while the other is a rigid center formed of a linear part with a homogeneous polynomial nonlinear part of degree n odd. We have proved that they can have at most $n-2$ limit cycle. So the extended 16th Hilbert problem to these class of piecewise differential systems has been solved.

Funding

Tiago Carvalho is partially supported by São Paulo Research Foundation (FAPESP grant 2019/10450-0) and by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq Grant 304809/2017-9). J. Llibre is partially supported by the Ministerio de Ciencia, Innovación y Universidades, Agencia Estatal de Investigación grants PID2019-104658GB-I00 (FEDER), the Agència de Gestió d'Ajuts Universitaris i de Recerca grant 2017SGR1617, and the H2020 European Research Council grant MSCA-RISE-2017-777911. L. F. Gonçalves thanks Faculdade de Filosofia, Ciências e Letras de Ribeirão Preto of Universidade de São Paulo.

Conflict of interest

The authors declare that they have no conflict of interest.

References

[1] D. Hilbert, Mathematical problems, Bull. Amer. Math. Soc. 8 (10) (1902) 437–479.

[2] A. Andronov, A. Vitt, S. Khaikin, Theory of Oscillators, Adiwes International Series in Physics, Pergamon Press, 1966. doi:https://doi.org/10.1016/B978-1-4831-6724-4.50005-8.

[3] A. F. Filippov, Differential Equations with Discontinuous Righthand Sides, 1st Edition, Vol. 18 of Mathematics and its Applications, Springer Netherlands, 1988. doi:https://doi.org/10.1007/978-94-015-7793-9.

[4] M. di Bernardo, C. J. Budd, A. R. Champneys, P. Kowalczyk, Piecewise-smooth Dynamical Systems: Theory and Applications, 1st Edition, no. 163 in Applied Mathematical Sciences, Springer-Verlag London, 2008.

[5] L. Chua, M. Komuro, T. Matsumoto, The double scroll family, IEEE Transactions on Circuits and Systems 33 (11) (1986) 1072–1118. doi:https://doi.org/10.1109/TCS.1986.1085869.

[6] R. Leine, H. Nijmeijer, Dynamics and Bifurcations of Non-Smooth Mechanical Systems, 1st Edition, Vol. 18 of Lecture Notes in Applied and Computational Mechanics, Springer-Verlag, 2004. doi:https://doi.org/10.1007/978-3-540-44398-8.

[7] M. di Bernardo, K. H. Johansson, F. Vasca, Self-oscillations and sliding in relay feedback systems: Symmetry and bifurcations, International Journal of Bifurcation and Chaos 11 (04) (2001) 1121–1140. doi:https://doi.org/10.1142/S0218127401002584.

[8] T. Carvalho, D. D. Novaes, L. F. Gonçalves, Sliding shilnikov connection in prey switching model, preprint. Available in https://arxiv.org/abs/1809.02060 (2020).

[9] J. Chavarriga, M. Sabatini, A survey of isochronous centers, Qualitative Theory of Dynamical Systems 1. doi:https://doi.org/10.1007/BF02969404.

[10] R. Conti, Uniformly isochronous centers of polynomial systems in $\mathbb{R}^2$, in: Differential Equations, Lecture Notes in Pure and Applied Mathematics, Routledge New York, 1994, p. 984. doi:https://doi.org/10.1201/9781315141237.

[11] A. Algaba, M. Reyes, Computing center conditions for vector fields with constant angular speed, Journal of Computational and Applied Mathematics 154 (1) (2003) 143–159. doi:https://doi.org/10.1016/S0377-0427(02)00818-X.

[12] A. Algaba, M. Reyes, Characterizing isochronous points and computing isochronous sections, Journal of Mathematical Analysis and Applications 355 (2) (2009) 564–576. doi:https://doi.org/10.1016/j.jmaa.2009.02.007.

[13] F. S. Dias, L. F. Mello, The center–focus problem and small amplitude limit cycles in rigid systems, Discrete & Continuous Dynamical Systems - A 32 (5) (2012) 1627–1637. doi:10.3934/dcds.2012.32.1627.

[14] A. Gasull, R. Prohens, J. Torregrosa, Limit cycles for rigid cubic systems, Journal of Mathematical Analysis and Applications 303 (2) (2005) 391–404. doi:https://doi.org/10.1016/j.jmaa.2004.07.030.

[15] A. Gasull, J. Torregrosa, Exact number of limit cycles for a family of rigid systems, Proc. Amer. Math. Soc. 133 (2005) 751–758. doi:https://doi.org/10.1090/S0002-9939-04-07542-2.

[16] M. Han, V. G. Romanovski, Isochronicity and normal forms of polynomial systems of odes, Journal of Symbolic Computation 47 (10) (2012) 1163–1174. doi:https://doi.org/10.1016/j.jsc.2011.12.039.

[17] J. Llibre, R. Rabanal, Center conditions for a class of planar rigid polynomial differential systems, Discrete & Continuous Dynamical Systems - A 35 (3) (2015) 1075–1090. doi:10.3934/dcds.2015.35.1075.

[18] R. Benterki, J. Llibre, The solution of the second part of the 16th Hilbert problem for nine families of discontinuous piecewise differential systems, preprint (2020).

[19] J. Itikawa, J. Llibre, A. Mereu, R. Oliveira, Limit cycles in uniform isochronous centers of discontinuous differential systems with four zones, Discrete & Continuous Dynamical Systems - B 22 (9) (2017) 3259–3272. doi:10.3934/dcdsb.2017136.

[20] J. Llibre, M. A. Teixeira, Piecewise linear differential systems with only centers can create limit cycles?, Nonlinear Dynamics 91 (2018) 249–255. doi:https://doi.org/10.1007/s11071-017-3866-6.

[21] V. I. Arnold, Ordinary Differential Equations, 1st Edition, Universitext, Springer-Verlag Berlin Heidelberg, 1992.

[22] J. Itikawa, J. Llibre, Phase portraits of uniform isochronous quartic centers, Journal of Computational and Applied Mathematics 287 (2015) 98 – 114. doi:https://doi.org/10.1016/j.cam.2015.02.046.

[23] I. Gradshteyn, I. Ryzhik, Table of Integrals, Series, and Products, seventh Edition, Academic Press, 2007. doi:https://doi.org/10.1016/C2009-0-22516-5.