HOPF BIFURCATION IN 3-DIMENSIONAL POLYNOMIAL VECTOR FIELDS

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Abstract. In this work we study the local cyclicity of some planar polynomial vector fields in $\mathbb{R}^3$. In particular, we give a quadratic system with 11 limit cycles, a cubic system with 31 limit cycles, a quartic system with 54 limit cycles, and a quintic system with 92 limit cycles. All limit cycles are small amplitude limit cycles and bifurcate from a Hopf type equilibrium. We introduce how to find Lyapunov constants in $\mathbb{R}^3$ for considering the usual degenerate Hopf bifurcation with a parallelization approach, which enables to prove our results for 4th and 5th degrees.

1. Introduction

In 1900, D. Hilbert published a series of problems which would be very influential for mathematics during the 20th century. Ten of these problems were presented at the International Congress of Mathematicians in Paris. Among them there is the 16th Hilbert Problem, a question related to finding the maximum number of limit cycles $\mathcal{H}(n)$ that a planar polynomial system can have as a function of its degree n. See more details in the review of Y. Ilyashenko in [20]. This problem remains unsolved, but a number of researchers have made a lot of advances in this problem. J. Li did a good review of the state of the problem in [21]. About global lower bounds, the work of C. Christopher and N. Lloyd in [12] is remarkable, improved some years ago by M. Han and J. Li in [19]. Regarding summaries of known lower bounds for $\mathcal{H}(n)$ for low values of the degree, the best ones can be found in [25].

There is a local version of the Hilbert problem that consists on finding the maximum number of limit cycles of small amplitude that bifurcate from an equilibrium point for a planar polynomial vector field of degree n. This number is usually called the cyclicity of the equilibrium. The most standard way to get lower bounds for this number is to analyze the local return map defined in a neighborhood of a monodromic equilibrium point, usually by studying the maximum codimension of a degenerated Hopf bifurcation. The most recent progress in this problem for small degrees can be found in [17], studying this bifurcation near very special centers that have high codimension.

In this paper, we consider the Hopf bifurcation in families of polynomial differential systems of equations in $\mathbb{R}^3$, and we aim to find as many limit cycles as possible for systems of several degrees n. It is widely known that, unlike for planar systems, systems in $\mathbb{R}^3$ can exhibit infinitely many limit cycles, as it is the case, for example, in any vector field with a Shilnikov homoclinic orbit, see [18, 28]. Other interesting bifurcations also exhibiting infinitely many can be found in [6], where a counterexample to a multidimensional version of the weakened Hilbert's 16th problem is presented, or the ones appearing on an infinite family of algebraic invariant surfaces, even for the quadratic case ([10, 30]). There is also the example of the bifurcation of infinitely many limit cycles near a Hopf-zero equilibrium point. Indeed, geometrical arguments to show the existence Shilnikov homoclinic orbits around a Hopf-zero point were already provided by J. Guckenheimer and P. Holmes ([18]).

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Formal statements were proved in [8] and later in [13], but the first rigorous proof was done very recently in [1].

In our case, we will restrict the problem of finding lower bounds for the maximum number of limit cycles of small amplitude to the center manifold. In fact, we will study bifurcations from systems having an equilibrium point such that the corresponding Jacobian matrix has eigenvalues $\{\pm \mathrm{i}, 1\}$, this is having a center in the center manifold and a hyperbolic eigenvalue in the third direction. The fact that the linear part of the third equation $\dot{z}$ is different from 0 and thus hyperbolic makes that the z direction is tightening the solutions towards the center manifold, and in this sense the considered problem is far from the Hopf-zero situation. In this line, the problem is more similar to finding the cyclicity in a two-dimensional case, and therefore makes sense to consider lower bounds for the maximum number of local limit cycles despite the global problem being unbounded. This problem was also considered in [3, 4, 5, 27] and more recently in [15, 16]. We will also take the advantage that, as explained in some of these works, from the computational point of view it is not necessary to do the changes of variables to transform the problem to a planar one. The scenario being considered is then a local cyclicity problem of a certain object inside a particular class of vector fields which is far from the 0 eigenvalue degeneration in the third component. We notice that in the nonpolynomial case this problem makes no sense because also an infinite number of small limit cycles can bifurcate from the origin, see [29].

Let us consider then a three-dimensional system

$$\begin{cases} \dot{x} = \alpha x - y + X(x,y,z), \\ \dot{y} = x + \alpha y + Y(x,y,z), \\ \dot{z} = z + Z(x,y,z), \end{cases} \tag{1}$$

where X, Y, Z are polynomials of degree $n \geq 2$ having no constant nor linear terms in x, y, z. Our main result is as follows.

Theorem 1. *There exist systems of the form* (1) *(with $\alpha \approx 0$) such that at least* 11, 31, 54, 92 *limit cycles of small amplitude bifurcate from the origin for $n = 2, 3, 4, 5$, respectively.*

To the best of our knowledge, the highest number of limit cycles found so far for the degenerate Hopf bifurcation in the quadratic case is 10 (see [30]), and we are not aware of any studies neither on cubic, quartic, nor quintic polynomial vector fields in $\mathbb{R}^3$, probably due to the computational difficulties.

This work is devoted to prove the above main theorem and is structured as follows. First, we present a section which introduces the main tools necessary for the proof of our result: the Lyapunov constants method computation in $\mathbb{R}^3$ and a couple of results that will be useful for the proofs. Sections 3 and 4 use 2-parametric families to study the Hopf bifurcation and prove Theorem 1 for $n = 2$ and $n = 3$, respectively. In the last section, we extend to $\mathbb{R}^3$ the parallelization approach for $\mathbb{R}^2$ presented in [22] for the Lyapunov constants computation and apply it to achieve our above main result for the fourth and fifth degree cases.

2. Preliminary tools

The usual way to study the Hopf bifurcation in $\mathbb{R}^3$ is the restriction to the central manifold where a center-focus type problem can be considered. But, many times, the necessary normal form changes to go further in the computations make the problem impossible to be solved. So, we have opted for the approach of working directly in $\mathbb{R}^3$ as in [27] but with the algorithm described in [9]. Additionally, as we will explain later, we have avoided when the center manifold is the invariant plane $z = 0$. Because the corresponding obtained results are worse than when the center manifold is not a plane. In fact we will prove that the number of limit cycles in the Hopf bifurcation depends on the center manifold when the considered vector field is a family depending on parameters. We will restrict our computations to families with (at most) two parameters because of the computational difficulties.

2.1. **Lyapunov constants in $\mathbb{R}^3$.** The main tool to study the local cyclicity of a Hopf equilibrium point are the Lyapunov constants, and we present here a method to find such quantities in $\mathbb{R}^3$. We remark that the approach that we have used does not need to perform the corresponding transformation to the center manifold in order to consider it as a planar problem. This approach presents computational advantages.

Even though the center notion can be only considered in even dimensional spaces, some authors introduce the notion of center in $\mathbb{R}^3$ to simplify the reading. With this aim we can say that the origin is a center for an analytic system in $\mathbb{R}^3$ when the eigenvalues of the Jacobian matrix are $\{\pm\mathrm{i}, \lambda\}$, with $\lambda \neq 0$, and the system has a center on the 2-dimensional center manifold. The next result is a classical one in the study of the existence of 2-dimensional center varieties having centers in a three-dimensional space. It can be found in [9] and is proved in [2].

Theorem 2 ([9])**.** *The origin is a center for the analytic system* (1) *if and only if* $\alpha = 0$ *and it admits a real analytic local first integral of the form* $H(x, y, z) = x^2 + y^2 + O_3(x, y, z)$ *in a neighborhood of the origin in* $\mathbb{R}^3$, *being* $O_3(x, y, z)$ *a sum of terms of degree at least* 3*. Moreover, when there is a center, the local center manifold is unique and analytic.*

The method we propose to find Lyapunov constants in $\mathbb{R}^3$ consists on using Theorem 2 to construct a first integral $H(x, y, z) = x^2 + y^2 + \cdots$ with unknown coefficients up to a certain degree. It is a well-known fact that condition

$$\dot{H} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} + \frac{\partial H}{\partial z}\dot{z} \equiv 0 \tag{2}$$

is equivalent to the system having a center at the origin. In contrast, if the center does not have a center at the origin then (2) is not identically zero, and it can be proved ([14]) that actually it is an analytic function in $x^2 + y^2$, this is

$$\dot{H} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} + \frac{\partial H}{\partial z}\dot{z} = \sum_{\ell \geq 1} L_\ell (x^2 + y^2)^{\ell+1}, \tag{3}$$

for some coefficients $L_\ell \in \mathbb{R}$. In this context, the coefficients L_ℓ are the so-called Lyapunov constants, and they have the property that they all vanish if and only if the system has a center.

We have developed an algorithm that works as follows. First, we define a first integral up to a certain degree N having the form

$$H = x^2 + y^2 + \sum_{i+j+k=3}^{N} h_{ijk} x^i y^j z^k, \tag{4}$$

being h_{ijk} the unknown coefficients for degrees between 3 and N. We aim to find these coefficients together with Lyapunov constants. First, we calculate $\dot{H} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} + \frac{\partial H}{\partial z}\dot{z}$ for (4). Then, by imposing equality (3), the coefficients of both sides are equated degree by degree starting at 3, and this allows to recursively determine coefficients h_{ijk} together with the Lyapunov constants when the degree is even. Actually, the extra term $L_\ell(x^2 + y^2)^{\ell+1}$ is added so that the resulting systems for even degrees are not underdetermined, but in our case and for simplicity we have equivalently considered $L_\ell x^{2\ell+2}$ as the adding term (see [11] for more details on this change).

This is essentially the usual Lyapunov method used to find Lyapunov constants in the plane but adapted to the three-dimensional case. As we have commented above, this approach decreases the computational time because the restriction of being on the center manifold given by Theorem 2 is not necessary.

Remark 3. *If the linear part of* (1) *was not written in its real Jordan normal form, we could use the same approach but finding which coefficients would make function* $H(x,y,z) = a_{20}x^2 + a_{11}xy + a_{02}y^2 + O_3(x,y,z)$ *be a first integral.*

The algorithm explained here has been computationally implemented with Maple ([23]), and used to calculate the Lyapunov constants throughout the rest of this work. Actually, we will see that for the results we want to prove we will only need those Lyapunov constants up to first order in the perturbative parameters, which highly simplifies the calculations and reduces the computation time.

2.2. **The Poincaré–Miranda's Theorem.** Here we formulate Poincaré–Miranda's Theorem, which will be necessary for the proofs of the results in the following sections. This theorem could be described as a generalization of Bolzano's Theorem to higher dimensions.

Theorem 4 ([24])**.** *(Poincaré–Miranda's Theorem) Let* $\mathcal{B} = \{x \in \mathbb{R}^m : |x_j| \leq h, \text{ for } 1 \leq j \leq m\}$ *and suppose that the mapping* $F = (f_1, f_2, \ldots, f_m) : \mathcal{B} \to \mathbb{R}^m$ *is continuous on* $\mathcal{B}$ *and such that* $F(x) \neq (0,0,\ldots,0)$ *for* x *on the boundary* $\partial\mathcal{B}$ *of* $\mathcal{B}$, *and*

(i) $f_j(x_1, x_2, \ldots, x_{j-1}, -h, x_{j+1}, \ldots, x_m) \geq 0$ *for* $1 \leq j \leq m$, *and*

(ii) $f_j(x_1, x_2, \ldots, x_{j-1}, +h, x_{j+1}, \ldots, x_m) \leq 0$ *for* $1 \leq j \leq m$.

Then, $F(x) = (0,0,\ldots,0)$ *has a solution in* $\mathcal{B}$.

We observe that in all our examples of using the above result, the inequalities on $\partial\mathcal{B}$ are always strict.

2.3. **A result on the number of limit cycles of parametric systems.** The idea behind the proof of Theorem 1 is studying the structure of the Lyapunov constants near centers up to first order Taylor development and analyzing the rank of such linear parts. The last preliminary tool we present is a theorem which, given a family of centers which depends on some parameters and under certain conditions, enables to obtain extra limit cycles to those seen only with the ranks of linear parts. The following result is based on the fact that we study the local cyclicity on a center component, which takes a generical value that can be increased on some special curves. For example, in Proposition 6 we have a 2-dimensional space of center parameters (a,b) such that, generically, 9 limit cycles of small amplitude bifurcate from the origin under quadratic perturbations. But there exists a curve of special centers where 10 limit cycles bifurcate and, on this curve, there exist special points (we prove the existence of at least one) for which, from the corresponding center, 11 limit cycles bifurcate. In fact, this is like describing a bifurcation diagram on the center component because the local cyclicity depends on the parameters of the center family. In all our results we are only providing lower bounds for the local cyclicity value.

Theorem 5 ([17])**.** *We denote by* $L_j^{(1)}(\lambda, b)$ *the first-order development, with respect to* $\lambda \in \mathbb{R}^k$, *of the* j*-Lyapunov constant of system*

$$\begin{cases} \dot{x} = \alpha y + P_c(x,y,\mu) + P(x,y,\lambda), \\ \dot{y} = \alpha x + Q_c(x,y,\mu) + Q(x,y,\lambda), \end{cases} \tag{5}$$

being $(\dot{x}, \dot{y}) = (P_c(x, y, \mu), Q_c(x, y, \mu))$ *a family of polynomial centers of degree* n *depending on a parameter* $\mu \in \mathbb{R}^\ell$ *and having a non-degenerate center equilibrium point at the origin, and being* $P(x, y, \lambda), Q(x, y, \lambda)$ *polynomials of degree* n *having no constant nor linear terms with perturbative parameters* $\lambda \in \mathbb{R}^{n^2+3n-4}$. *We assume that, after a change of variables in the parameter space if necessary, we can write*

$$L_j = \begin{cases} \lambda_j + O_2(\lambda), \text{ for } j = 1, \dots, k-1, \\ \sum\limits_{l=1}^{k-1} g_{j,l}(\mu)\lambda_l + f_{j-k}(\mu)\lambda_k + O_2(\lambda), \text{ for } j = k, \dots, k+\ell, \end{cases}$$

where with $O_2(\lambda)$ *we denote all the monomials of degree higher or equal than* 2 *in* λ *with coefficients analytic functions in* μ. *If there exists a point* μ^* *such that* $f_0(\mu^*) = \cdots = f_{\ell-1}(\mu^*) = 0, f_\ell(\mu^*) \neq 0$, *and the Jacobian matrix of* $(f_0, \dots, f_{\ell-1})$ *with respect to* μ *has rank* ℓ *at* μ^*, *then system* (5) *has* $k+\ell$ *hyperbolic limit cycles of small amplitude bifurcating from the origin.*

This result is proved in [17], where it is used to study the cyclicity of some planar families of vector fields. Even though we aim to study cyclicity in $\mathbb{R}^3$, the same technique from Theorem 5 can be automatically extrapolated to vector fields in the space as the whole problem is analogous.

3. 11 limit cycles for a quadratic system

In this section we present a quadratic system in $\mathbb{R}^3$ and show that it unfolds 11 limit cycles by using the techniques from Section 2. This proves the case $n = 2$ in Theorem 1.

The presented result improves by one the lower bound found in [30]. In such work, the authors use a family of planar centers inspired by a family of Lotka-Volterra systems in $\mathbb{R}^2$, because this family defines a center component (depending on parameters) denoted by Q_3^{LV} according to the classification of quadratic planar centers provided by [31]. But this family does not have the maximum generic local cyclicity unlike family Q_4. Family Q_4 has a first integral of the form F^3G^{-2}, as detailed in [5], but no free parameters. Our goal is to take advantage of Theorem 5 in a family of centers extending Q_4 to $\mathbb{R}^3$. Hence, we add parameters in the third component instead of the first two, and from Theorem 2 we know the existence of a local center manifold. We remark that the next result is saying that the local cyclicity can increase when the parameters change.

From the computational point of view, it is important to remark that the best centers are the ones whose linear part is in the usual Jordan normal form, $(x', y', z') = (-y, x, z)$. But sometimes, as in our case, the changes of variables to achieve this form add square roots in the coefficients, and this increases the computational complexity, so we use Remark 3 to avoid these difficulties.

The following result provides, up to our knowledge, the best lower bound for the local cyclicity near a Hopf point in quadratic vector fields in $\mathbb{R}^3$.

Proposition 6. *The quadratic system*

$$\begin{cases} \dot{x} = -\frac{1}{3}y - 5x^2 - 2xy + \frac{1}{3}y^2, \\ \dot{y} = x - 3x^2 - 10xy + y^2, \\ \dot{z} = z + z^2 + ax^2 + by^2, \end{cases} \tag{6}$$

has a center at the origin and unfolds 11 *limit cycles under a complete quadratic perturbation.*

Proof. The two first equations in (6) define a center in the plane, as it can be trivially checked that the corresponding system has a rational first integral of the form

$$H_2(x,y) = \frac{(36x^2-24xy+4y^2-8y+1)^3}{(108x^3-108x^2y+36xy^2-4y^3-36xy+12y^2-12y+1)^2}.$$

This center in $\mathbb{R}^2$ can be embedded in $\mathbb{R}^3$ by adding a third component that, from Theorem 2, adds a 2-dimensional center manifold where system (6) has a center and the origin is a equilibrium point of Hopf type. Let us add a quadratic perturbation and the trace parameter to (6) in the following way

$$\begin{cases} \dot{x} = \alpha x - \frac{1}{3}y - 5x^2 - 2xy + \frac{1}{3}y^2 + \sum\limits_{i+j+k=2}^{2} a_{ijk}x^iy^jz^k, \\ \dot{y} = x + \alpha y - 3x^2 - 10xy + y^2 + \sum\limits_{i+j+k=2}^{2} b_{ijk}x^iy^jz^k, \\ \dot{z} = z + z^2 + ax^2 + by^2 + \sum\limits_{i+j+k=2}^{2} c_{ijk}x^iy^jz^k, \end{cases} \tag{7}$$

for $\alpha, a_{ijk}, b_{ijk}, c_{ijk} \in \mathbb{R}$ perturbative parameters.

The next step is to find, for $\alpha = 0$, the first 11 Lyapunov constants of (7) up to first order in the perturbative parameters. Notice that, due to Remark 3, in this case we will consider a first integral with the form $H(x,y,z) = \frac{1}{2}x^2 + \frac{1}{6}y^2 + O_3(x,y,z)$. Once we have the linear parts of the 11 first Lyapunov constants of the system, we check that their rank is generically 9. Hence, by the Implicit Function Theorem and adding α, we generically have 9 limit cycles of small amplitude bifurcating from the origin.

After a linear change of coordinates in the parameters space, we obtain that the Lyapunov constants have the following form:

$$\begin{aligned} L_k &= u_k + O_2, \text{ for } k = 1, \ldots, 8, \\ L_9 &= \frac{f_1(a,b)}{d(a,b)}u_9 + O_2, \\ L_{10} &= \frac{f_2(a,b)}{d(a,b)}u_9 + O_2, \\ L_{11} &= \frac{f_3(a,b)}{d(a,b)}u_9 + O_2, \end{aligned}$$

being $f_1(a,b), f_2(a,b), f_3(a,b), d(a,b)$ certain polynomials with integer coefficients in the variables a and b. We do not show the complete polynomials here due to their large size. They have respectively total degree $38, 39, 40$, and 29, their number of monomials are respectively $744, 784, 825$, and 444, and the coefficients are integers having between 72 and 158 figures. Then, by Theorem 5, to prove the bifurcation of 11 limit cycles we just have to check that there exists a point $(\hat{a}, \hat{b})$ in the parameters space such that $f_1(\hat{a},\hat{b}) = f_2(\hat{a},\hat{b}) = 0$, $f_3(\hat{a},\hat{b}) \neq 0, d(\hat{a},\hat{b}) \neq 0$, and the Jacobian determinant $\det \mathrm{Jac}_{(f_1,f_2)}(\hat{a},\hat{b}) \neq 0$.

The situation is represented on the graph in Figure 1, where the zero level curves of the considered polynomials are represented. In it, we can see how the curves $f_1(a,b) = 0$ and $f_2(a,b) = 0$ intersect at a point $(\hat{a},\hat{b})$ which does not belong to the curves $f_3(a,b) = 0$ and $d(a,b) = 0$. We aim to analytically prove the existence of such point by means of Poincaré–Miranda's Theorem (Theorem 4). To do this, we will provide a computer-assisted proof by using rational interval analysis.

Let us start by finding a numerical approximation for an intersection point

$$(\hat{a},\hat{b}) \approx (-15.87687966375324925, 12.23255254136248609) \tag{8}$$

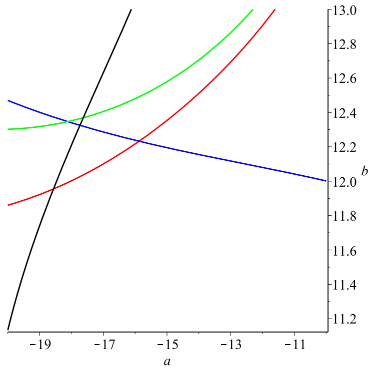

FIGURE 1. Plot of the zero level curves of $f_1(a,b), f_2(a,b), f_3(a,b)$, and $d(a,b)$ in color red, blue, green, and black, respectively.

of $f_1(a,b)=0$ and $f_2(a,b)=0$, which can be seen in Figure 1. To simplify the application of Poincaré–Miranda's Theorem, we will perform a linear change of variables $(a,b)\to(u,v)$ with rational coefficients. To this end, we will consider the approximation $(\hat{a},\hat{b})\approx(-\frac{159}{10},\frac{61}{5})=(-15.9,12.2)$ and define u and v as the numerical tangent lines at this point. Then, one can isolate (a,b) as a function of (u,v), consider the solution with 3 significant digits and convert it to rational values, obtaining

$$(a,b)=\left(-\frac{159}{10}-\frac{577}{100}u+\frac{577}{100}v,\frac{61}{5}+\frac{129}{500}u+\frac{371}{500}\right). \tag{9}$$

If we substitute (9) in $f_1(a,b), f_2(a,b), f_3(a,b)$, and $d(a,b)$ we obtain the polynomials in the new variables, which we will denote by $F_1(u,v), F_2(u,v), F_3(u,v)$, and $D(u,v)$. These new polynomials are represented in Figure 2, where we can see that now the intersection point $(\hat{u},\hat{v})$ of $F_1(u,v)=0$ and $F_2(u,v)=0$ has shifted near $(0,0)$ and the application of Poincaré–Miranda's Theorem will be easier.

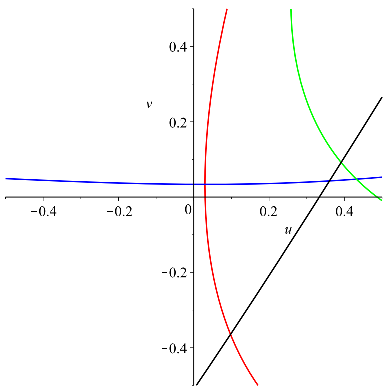

FIGURE 2. Plot of the zero level curves of $F_1(u,v), F_2(u,v), F_3(u,v)$, and $D(u,v)$ in color red, blue, green, and black, respectively.

Taking $h=1/10$ in Theorem 4, we will shown that in the square $[-h,h]^2$ there must be a zero of $F_1(u,v)$ and $F_2(u,v)$. The proof follows by checking also that $F_3(u,v)$, $D(u,v)$, and the Jacobian determinant $J(u,v):=\det \mathrm{Jac}_{(F_1,F_2)}(u,v)$ do not vanish in the whole square. Observe that $F_1(u,v)$ and $F_2(u,v)$ are continuous because they are polynomials. Then, there will be a point $(\hat{u},\hat{v})\in(-h,h)^2$ such that $F_1(\hat{u},\hat{v})=0$ and $F_2(\hat{u},\hat{v})=0$ by applying the Poincaré–Miranda's Theorem because the following conditions hold.

(a) $F_1(h,v)>0$ and $F_1(-h,v)<0$ for $v\in[-h,h]$.

To prove this, we will show that $F_1(h,v)$ is inferiorly bounded by a positive number in

$v \in [-h,h]$ and $F_1(-h,v)$ is superiorly bounded by a negative number in $v \in [-h,h]$. Indeed,

$$4 \cdot 10^{145} < F_1(h,0) - \sum_{i=1}^{38} |A_i| h^i \leq F_1(h,0) + \sum_{i=1}^{38} A_i v^i = F_1(h,v),$$

$$F_1(-h,v) = F_1(-h,0) + \sum_{i=1}^{38} B_i v^i \leq F_1(-h,0) + \sum_{i=1}^{38} |B_i| h^i < -8 \cdot 10^{145},$$

where $A_i, B_i \in \mathbb{Q}$ are the coefficients of the corresponding polynomials. Notice that, due to how Theorem 4 is formulated, it should be applied to $-F_1(u,v)$ rather than $F_1(u,v)$, but the conclusion is exactly the same.

(b) $F_2(u,-h) > 0$ and $F_2(u,h) < 0$ for $u \in [-h,h]$.
Analogously, we will show that $F_2(u,-h)$ is inferiorly bounded by a positive number in $u \in [-h,h]$ and $F_2(u,h)$ is superiorly bounded by a negative number in $u \in [-h,h]$. Indeed,

$$1 \cdot 10^{158} < F_2(0,-h) - \sum_{i=1}^{39} |C_i| h^i \leq F_2(0,-h) + \sum_{i=1}^{39} C_i u^i = F_2(u,-h),$$

$$F_2(u,h) = F_2(0,h) + \sum_{i=1}^{39} D_i u^i \leq F_2(0,h) + \sum_{i=1}^{39} |D_i| h^i < -8 \cdot 10^{157},$$

where $C_i, D_i \in \mathbb{Q}$ are the coefficients of the corresponding polynomials.

The last step of the proof is to ensure that both $F_3(u,v), D(u,v)$, and $J(u,v)$ do not vanish in $[-h,h]^2$. More concretely, we will check that the three functions are negative in the whole square by seeing that they are superiorly bounded by a negative number,

$$F_3(u,v) = F_3(0,0) + \sum_{i+j=1}^{40} G_{ij} u^i v^j \leq F_3(0,0) + \sum_{i+j=1}^{40} |G_{ij}| h^{i+j} < -6 \cdot 10^{171},$$

$$D(u,v) = D(0,0) + \sum_{i+j=1}^{29} H_{ij} u^i v^j \leq D(0,0) + \sum_{i+j=1}^{29} |H_{ij}| h^{i+j} < -7 \cdot 10^{128},$$

$$J(u,v) = J(0,0) + \sum_{i+j=1}^{75} K_{ij} u^i v^j \leq J(0,0) + \sum_{i+j=1}^{75} |K_{ij}| h^{i+j} < -3 \cdot 10^{305},$$

where $G_{ij}, H_{ij}, K_{ij} \in \mathbb{Q}$ are the coefficients of the corresponding polynomials. Hence, all the above bounds are obtained by adding rational numbers with no error in the computations.

All the computations have been done with $\alpha = 0$, and 10 limit cycles are obtained. The proof follows by adding an extra limit cycle which bifurcates from the origin when using the usual Hopf bifurcation moving the trace parameter α adequately. $\square$

From the proof of Proposition 6, we can extract from Figure 1 the bifurcation diagram as we explained at the beginning of Section 2.3, obtaining Figure 3. In fact, there are other intersection points for the zero level curves of f_1 and f_2, but they are more difficult to find and to prove their transversal intersection. See Figure 4 for a better understanding of the difficulty of finding intersection points like (8).

Before finishing this section, we would like to make two final comments about quadratic systems in $\mathbb{R}^3$.

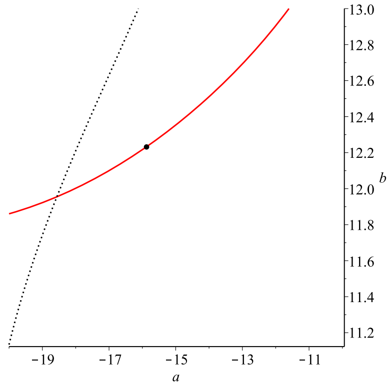

FIGURE 3. Avoiding the dots line, we have (generically) 9 small limit cycles bifurcating from (6), 10 on the red line, and 11 on the black point.

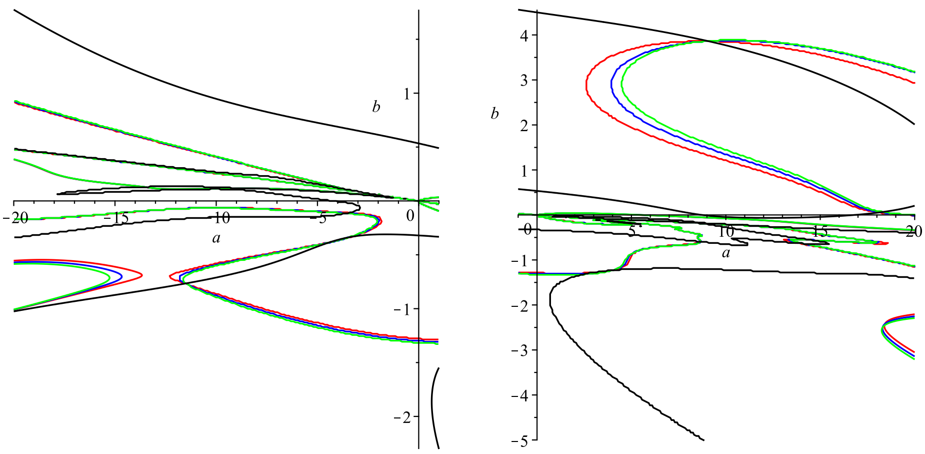

FIGURE 4. Plot of the zero level curves of f_1, f_2, f_3, and d in color red, blue, green, and black, respectively, in different zones of the plane (a, b).

In Proposition 6, we have added a third component in (6) to extend the problem from $\mathbb{R}^2$ to $\mathbb{R}^3$. This procedure can be done in many different ways and for each third component we obtain different center manifolds. We have tested different possibilities, and we have observed that when the center manifold is $z = 0$, the number of limit cycles is the same as in the planar problem. For this reason, we have added some terms including x or y, for increasing such number. Furthermore, we have observed by adding different third components which include x and y that the result does not improve.

We believe that the result obtained in Proposition 6 is the maximum cyclicity that can be obtained by using the presented technique. In (7) we have a total of 18 perturbative parameters, and we have observed in our calculations that only 12 of them actually play a role when finding the ranks of linear parts of the Lyapunov constants. Actually, the ones which play a role are a_{ijk} and b_{ijk}, these are the ones in $\dot{x}$ and $\dot{y}$; parameters c_{ijk} in $\dot{z}$ do not appear in the computation of the linear parts of the Lyapunov constants. Perhaps more limit cycles could be obtained by studying higher-order developments, but this is a very difficult computational problem due to the size of the corresponding polynomials, assuming that such polynomials could be found.

4. 31 LIMIT CYCLES FOR A CUBIC SYSTEM

Here we prove the case $n = 3$ in Theorem 1, by presenting a cubic system in $\mathbb{R}^3$ having a Hopf point at the origin from which 31 limit cycles bifurcate.

Proposition 7. *The cubic system*

$$\begin{cases} \dot{x} = -y(1-68x+1183x^2), \\ \dot{y} = x-58x^2-44xy+30y^2+672x^3+1484x^2y-945xy^2-84y^3, \\ \dot{z} = z+x^2+ax^3+by^3, \end{cases} \tag{10}$$

has a center at the origin and unfolds 31 *limit cycles of small amplitude under a complete cubic perturbation.*

Proof. The two first equations in (10) define a system in the plane with a center at the origin because it has a rational first integral (see [7]). Then, we extend this system to (10) by adding a third equation, and there exists a center manifold tangent to $z=0$ guaranteed by Theorem 2 on which (10) has a center. Let us add a perturbation with cubic and quadratic terms to (10) and the trace parameter as follows,

$$\begin{cases} \dot{x} = \alpha x - y(1-68x+1183x^2) + \sum\limits_{i+j+k=2}^{3} a_{ijk}x^iy^jz^k, \\ \dot{y} = x+\alpha y-58x^2-44xy+30y^2+672x^3+1484x^2y-945xy^2-84y^3+\sum\limits_{i+j+k=2}^{3} b_{ijk}x^iy^jz^k, \\ \dot{z} = z+x^2+ax^3+by^3+\sum\limits_{i+j+k=2}^{3} c_{ijk}x^iy^jz^k, \end{cases} \tag{11}$$

for $\alpha, a_{ijk}, b_{ijk}, c_{ijk} \in \mathbb{R}$ perturbative parameters.

The proof for the unfolding of 31 limit cycles is analogous to that of Proposition 6. We first take $\alpha=0$ and find the linear parts of the first 31 Lyapunov constants of (11) with respect to the perturbative parameters. We see then that generically their rank is 29. After a linear change of coordinates in the parameters space we obtain that the first 28 Lyapunov constants have the following form:

$$L_k = u_k + O_2, \text{ for } k=1,\ldots,28.$$

Next, we consider an analytic change of coordinates, by using the Implicit Function Theorem, such that the Lyapunov constants write as $L_k = v_k$, for $k=1,\ldots,28$. Assuming that $v_k=0$, for $k=1,\ldots,28$, and vanishing the non essential perturbative parameters, we get

$$L_{29} = \frac{f_1(a,b)}{d(a,b)}u_{29} + O_2(u_{29}),$$

$$L_{30} = \frac{f_2(a,b)}{d(a,b)}u_{29} + O_2(u_{29}),$$

$$L_{31} = \frac{f_3(a,b)}{d(a,b)}u_{29} + O_2(u_{29}),$$

being $f_1(a,b), f_2(a,b), f_3(a,b), d(a,b)$ certain polynomials with integer coefficients in the variables a and b. We do not show the complete polynomials here due to their large size, but we see that both $f_1(a,b), f_2(a,b)$, and $f_3(a,b)$ have degree 28 and 434 monomials, and $d(a,b)$ has degree 25 and 350 monomials. To see the bifurcation of 31 limit cycles we can use Theorem 5 and check the conditions on it, this is to find a point $(\hat{a},\hat{b})$ in the parameters space such that $f_1(\hat{a},\hat{b}) = f_2(\hat{a},\hat{b}) = 0$, $f_3(\hat{a},\hat{b}) \neq 0, d(\hat{a},\hat{b}) \neq 0$, and the Jacobian determinant $\det \mathrm{Jac}_{(f_1,f_2)}(\hat{a},\hat{b}) \neq 0$.

We have drawn $f_1(a,b)=0, f_2(a,b)=0, f_3(a,b)=0$, and $d(a,b)=0$ in Figure 5. Although it is not discernible in this graph, what is happening is that curves $f_1(a,b)=0, f_2(a,b)=0$, and $f_3(a,b)=0$ are practically overlapping, so the intersection point that we are searching cannot be appreciated; $d(a,b)=0$ is not shown because it remains out of the plotted region. To sight the intersection point of $f_1(a,b)=0$ and $f_2(a,b)=0$ we will perform a change of variables such that the intersection is transversal, and we will apply Poincaré–Miranda's Theorem (Theorem 4) to analytically show its existence. This will be done by means of a computer-assisted proof and using rational interval analysis.

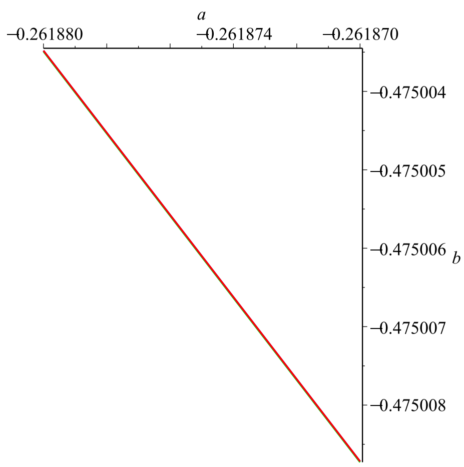

FIGURE 5. Plot of the zero level curves of $f_1(a,b), f_2(a,b)$, and $f_3(a,b)$ in color red, blue and green, respectively.

First we find a numerical approximation

$$(\hat{a},\hat{b}) = (-0.2618746696871324942811545745396788956798,\\ -0.4750062727838396305466509058484908194011)$$

for an intersection point of $f_1(a,b)=0$ and $f_2(a,b)=0$. We need a good rational approximation so that the zero level curves drawn in Figure 5 are separated, and this way the transversality can be appreciated. Let us perform a change of variables $(a,b)\to(u,v)$ such that u and v are the first order Taylor expansion at $(\hat{a},\hat{b})$ of $f_1(a,b)$ and $f_2(a,b)$, respectively. Now we can find the inverse of such change and convert the coefficients to rational numbers, which gives

$$\begin{aligned}(a,b) = \Big(&-\frac{3125780069700516145310827}{11936168066330948602492655}+\frac{1427237216612}{940600247814793427315253720871652057}u\\ &-\frac{6400609904497}{4224724520267912944601259158052139159}v,\\ &-\frac{2950612633153916740411853}{6211734038503208283147559}-\frac{3951139552423}{4971433349089293973487138537373615874}u\\ &+\frac{2249793630741}{2835231811366846850770950557241968074}v\Big).\end{aligned}$$

These expressions can be substituted in $f_1(a,b), f_2(a,b), f_3(a,b)$, and $d(a,b)$ to obtain the polynomials in the new variables, which we will be denoted respectively by $F_1(u,v)$, $F_2(u,v)$, $F_3(u,v)$, and $D(u,v)$. The new polynomials are represented in Figure 6, where we can see that now the intersection point $(\hat{u},\hat{v})$ of $F_1(u,v)=0$ and $F_2(u,v)=0$ has shifted near $(0,0)$ and its transversality can be clearly seen. $F_3(u,v)$ and $D(u,v)$ are not in the graph because they stay out of the plotted region, so they will be nonvanishing at the intersection point.

The proof follows as in the case of Proposition 6, also taking $h=1/10$ and applying the Poincaré–Miranda's Theorem, since the following conditions hold:

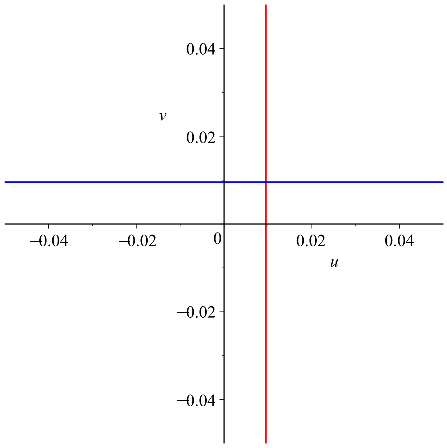

FIGURE 6. Plot of the zero level curves of $F_1(u,v)$ and $F_2(u,v)$ in color red and blue, respectively.

(a) $F_1(h,v)>0$ and $F_1(-h,v)<0$ for $v\in[-h,h]$.
We will see that $F_1(h,v)$ is inferiorly bounded by a positive number in $v\in[-h,h]$ and $F_1(-h,v)$ is superiorly bounded by a negative number in $v\in[-h,h]$. Indeed,

$$\frac{9}{100}<F_1(h,0)-\sum_{i=1}^{28}|A_i|h^i\le F_1(h,0)+\sum_{i=1}^{28}A_iv^i=F_1(h,v),$$
$$F_1(-h,v)=F_1(-h,0)+\sum_{i=1}^{28}B_iv^i\le F_1(-h,0)+\sum_{i=1}^{28}|B_i|h^i<-\frac{1}{10},$$

where $A_i,B_i\in\mathbb{Q}$ are the coefficients of the corresponding polynomials.

(b) $F_2(u,h)>0$ and $F_2(u,-h)<0$ for $u\in[-h,h]$.
Analogously, we will show that $F_2(u,h)$ is inferiorly bounded by a positive number in $u\in[-h,h]$ and $F_2(u,-h)$ is superiorly bounded by a negative number in $u\in[-h,h]$:

$$\frac{9}{100}<F_2(0,h)-\sum_{i=1}^{28}|C_i|h^i\le F_2(0,h)+\sum_{i=1}^{28}C_iu^i=F_2(u,h),$$
$$F_2(u,-h)=F_2(0,-h)+\sum_{i=1}^{28}D_iu^i\le F_2(0,-h)+\sum_{i=1}^{28}|D_i|h^i<-\frac{1}{10},$$

where $C_i,D_i\in\mathbb{Q}$ are the coefficients of the corresponding polynomials.

Notice that, due to how Theorem 4 is formulated, it should be applied to $-F_1(u,v)$ and $-F_2(u,v)$ rather than $F_1(u,v)$ and $F_2(u,v)$, but the conclusion is exactly the same.

Finally, we will check that $F_3(u,v)$ is negative in $[-h,h]^2$, and $D(u,v)$ and $J(u,v)$ are positive, so none of those functions vanish in the whole square.

$$F_3(u,v)=F_3(0,0)+\sum_{i+j=1}^{28}G_{ij}u^iv^j\le F_3(0,0)+\sum_{i+j=1}^{28}|G_{ij}|h^{i+j}<-2\cdot10^{20},$$
$$2\cdot10^{1032}<D(0,0)-\sum_{i+j=1}^{25}|H_{ij}|h^{i+j}\le D(0,0)+\sum_{i+j=1}^{25}H_{ij}u^iv^j=D(u,v),$$
$$\frac{99}{100}<J(0,0)-\sum_{i+j=1}^{54}|K_{ij}|h^{i+j}\le J(0,0)+\sum_{i+j=1}^{54}K_{ij}u^iv^j=J(u,v)$$

where $G_{ij}, H_{ij}, K_{ij} \in \mathbb{Q}$ are the coefficients of the corresponding polynomials.

The proof finishes adding, as in Proposition 6, the trace parameter. □

As we already noticed for quadratic systems, we think that we have almost obtained the maximum cyclicity for the cubic case in $\mathbb{R}^3$ by using this approach. Here we have 48 perturbative parameters and, by considering linear Taylor developments, only 32 do appear.

5. A parallelization approach for quartic and quintic systems

In this section we will show how the cases $n = 4$ and $n = 5$ from Theorem 1 are achieved. The idea, as we have done in the previous sections, is to consider centers in the plane such that generically unfold a high number of limit cycles, and then extend them to $\mathbb{R}^3$ by adding the third equation. For this case, we will not use the technique described in Theorem 5 as we did for quadratic and cubic systems of taking the best known quartic and quintic systems due to the difficulty to deal with the obtained constants because of their huge size. We have considered the cubic system (10) adding one or two straight lines of equilibria. Furthermore, for the Lyapunov constants computation during this section we will consider a parallelization approach in order to reduce the executing times of the processes.

The technique we have used to parallelize the computation of linear parts of Lyapunov constants is inspired by [22], and is described as follows. Let us consider a system with perturbative parameters $\lambda_1, \ldots, \lambda_d$. We select some $k \in \{1, \ldots, d\}$ and consider the same system with $\lambda_l = 0$ for $l \in \{1, \ldots, d\} \backslash \{k\}$, this is a system with only one perturbative parameter λ_k. Then, we find its Lyapunov constants up to first order $L_{j,k}^{(1)}$, and repeat this process for every $k = 1, \ldots, d$. This step can be easily parallelized by assigning to each thread of the execution the computation of the Lyapunov constants of the system with a different nonzero perturbative parameter λ_k, so we would have a parallelization paradigm with d threads, as many as perturbative parameters. Once this has been done, the linear part of the jth Lyapunov constant L_j of the original system with all the perturbative parameters $\lambda_1, \ldots, \lambda_d$ would be

$$L_j^{(1)} = \sum_{k=1}^{d} L_{j,k}^{(1)}.$$

These calculations have been performed using a cluster of servers. The parallelization has been implemented by using PBala ([26]), which is a parallellization interface for single threaded scripts and allows to distribute executions in Parallel Virtual Machine enabled clusters using single program multiple data paradigm. This interface lets the user execute a same script/program over multiple input data in several CPUs located at the cluster. It supports memory management, so nodes do not run out of RAM due to too many processes being started at the same node.

The parallelization approach presented here has proved to be highly efficient. For instance, to find the necessary Lyapunov constants to prove Proposition 8, the computing time has been reduced from 30 hours to 4 hours when parallelizing. However, the computational requirements of the studied problem have caused that we cannot go further than 5th degree, which is the case solved in Proposition 9 and for which about 10 days of computing were needed even with parallelization in a cluster with 5 servers

The first result, related to a quartic system, is as follows.

Proposition 8. *The quartic system*

$$\begin{cases} \dot{x} = -y(1 - 68x + 1183x^2)(1 - x - y), \\ \dot{y} = (x - 58x^2 - 44xy + 30y^2 + 672x^3 + 1484x^2y - 945xy^2 - 84y^3)(1 - x - y), \\ \dot{z} = z + x^2 + x^3 + x^4, \end{cases} \tag{12}$$

has a center at the origin and unfolds 54 *limit cycles of small amplitude under a complete quartic perturbation.*

Proof. The two first equations in (12) define a system in the plane with a center at the origin because they are the same center defined by the two first equations of (10) multiplied by a fixed points straight line. Then, by adding the third equation with a center manifold tangent to $z = 0$ we have a center in $\mathbb{R}^3$. The proof follows as the first part of the proofs of Propositions 6 and 7. First, we consider a perturbation having the trace parameter terms and a quartic perturbation starting with degree 2 terms. Second, we take $\alpha = 0$ and compute the first 54 Lyapunov constants of the perturbed system up to first order with respect to the perturbative parameters by using the parallelization algorithm described above. Finally, we see that generically they have rank 54, which by adding the trace parameter proves the unfolding of 54 limit cycles of small amplitude. We notice that the generical rank does not increase when we compute 6 more Lyapunov constants. □

In all our computations we have observed that, for none of the systems studied so far, the perturbative parameters c_{ijk} from the third equation $\dot{z}$ appear in the expressions of any first-order Taylor series of the Lyapunov constants. Even though this fact has not been proved, we believe that this is a general behavior. Therefore, in this sense, when considering the perturbed system for the following quintic system in Proposition 9 we will ignore the perturbation in the third equation to simplify the computations and reduce the execution times, as we think that this will make no difference. Actually, for this quintic system (13) we have checked that the linear parts of the first 70 Lyapunov constants do not include the perturbative parameters in $\dot{z}$, which confirms what we expected. This fact also reduces by 2/3 the maximum cyclicity that can be obtained by only looking at Lyapunov constants up to first order regarding the number of perturbative parameters, as 1/3 of such parameters will not increase the cyclicity of the system.

For the quintic case we have the following result.

Proposition 9. *The quintic system*

$$\begin{cases} \dot{x} = -y(1 - 68x + 1183x^2)f(x, y), \\ \dot{y} = (x - 58x^2 - 44xy + 30y^2 + 672x^3 + 1484x^2y - 945xy^2 - 84y^3)f(x, y), \\ \dot{z} = z + x^2 + x^3 + x^4 + x^5, \end{cases} \tag{13}$$

with $f(x, y) = (1 - x - y)(1 + 2x - y)$, *has a center at the origin and unfolds* 92 *limit cycles under a complete quintic perturbation.*

Proof. System (13) has a center at the origin for the same reason that system (12), in this case with two straight lines filled with equilibrium points and also having a center manifold tangent to $z = 0$. To simplify the calculations we have considered perturbations only on the first two equations. The proof finishes as the previous one. Here we have computed the first 92 Lyapunov constants, and we have checked that when computing three more the rank does not increase. □

We have also made an attempt to find the linear parts of Lyapunov constants for a degree $n = 6$ system, but as we already commented the problem soon becomes highly demanding computationally speaking. In particular, for the tested sextic case we have reached the memory limit and the process is using 16GB of RAM memory. We have reached the 124th Lyapunov constant, and to find only this constant for only one perturbative parameter the required time has been approximately 3 days. For this reason, we have stopped the problem at 5th degree, as we believe that going higher in the degree is impossible at this stage in computational terms.

It is worth making a final comment about the expected local cyclicity from the used approach. The total number of perturbative parameters –also considering the trace parameter– is $(n^3 + 6n^2 + 11n - 16)/2$ but, as we explained above, the parameters from the third equation do not seem to appear in the linear part of the Lyapunov constants. Hence, the maximum number of essential parameters is $(n^3 + 6n^2 + 11n - 15)/3$ and, consequently, the best lower bound for the number of limit cycles of small amplitude in Hopf point in $\mathbb{R}^3$ will be one less, that is $\mathcal{C}(n) \geq (n^3 + 6n^2 + 11n - 18)/3$. This function takes the values 12 and 32 for degrees $n = 2$ and $n = 3$, respectively, which are very close to the ones obtained in our main result, but the values corresponding to $n = 4$ and $n = 5$ is a bit quite far from the ones we obtained. The reason is that the first two systems are built from optimal planar systems. Our achievement is that we have been able to work with such degrees –4th and 5th– because of the designed parallelized algorithm. We notice that, up to our knowledge, all the obtained values are the highest ones found so far.

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