Periodic orbits and entropy for maps of the n-star.

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Introduction

Thirty years ago Sharkovskiĭ [31] proved his beautiful theorem (see Theorem 1.3.1). Unfortunately, this theorem remained unknown to western (english-speaking) mathematicians until 1977, when Štefan published his paper [32] about it. In spite of this, it quickly became a mainstay of the theory of one dimensional dynamical systems, strongly stimulating its development.

In the meantime (1975), the famous paper of Li and Yorke *Period three implies chaos*[25] came out. It also contributed very much to popularize the study of the discrete dynamics and chaos.

As it is well known, Sharkovskiĭ's Theorem characterizes the possible sets of periods of all periodic orbits of continuous maps of a closed interval (or the real line) into itself. This is done by means of a curious total ordering of the natural numbers. From this, in particular, it follows trivially part of the result of Li and Yorke. Moreover, Sharkovskiĭ's Theorem opened new, interesting and more general problems. For instance, the theorem says nothing about the behaviour of the relevant periodic orbits, which, as it has been shown later by several authors, gives a lot of information in the study of the dynamics of the system under consideration.

One of the important questions in the theory of dynamical systems is to be able to somehow measure the complexity of a system or, in other words, the degree of "chaos" present in it. Various notions of chaotic behaviour can be considered. However, maybe the better way to estimate chaos is to compute the topological entropy of a system.

The study of certain classes of periodic orbits may shed light on these problems. Formerly they were considered simple and minimal orbits (see, for instance, [32], [11], [17], [7], [19]). Afterwards the notion of primary orbit arose. This kind of orbits differs from the previous ones in the fact that they are independent on the a priori knowledge of the Sharkovskiĭ's Theorem. They depend only on the class of maps under consideration, and this is why they are useful. For instance, the characterization of the primary orbits of interval maps, together with the usual graph techniques, allows to give a new proof of Sharkovskiĭ's Theorem, independent on the a priori knowledge of the Sharkovskiĭ's ordering. The characterization of the primary orbits also yields good estimates of the topological entropy, by using the method introduced in [13].

The notion of a primary orbit was stated formally in [2] and, simultaneously, in [8] where it was called $\rightarrow$-minimal. The aim of the first paper was, precisely, to develop and use the above technique to generalize Sharkovskiĭ's Theorem to the case of continuous self maps of the space consisting of three intervals joined by a common endpoint —the **Y** or the 3-star—, with this point fixed. This class of maps will be called $\mathcal{Y}$ in the memory.

In this memory, we try to continue and generalize the work done for interval maps and maps of $\mathcal{Y}$ in these directions. In particular we try to understand the primary orbits in the n-star (n intervals joined by a common endpoint), and derive the dynamical properties of this understanding. Since our task relies heavily on the study done for the **Y** in [2], we first do a quick review of this paper in Chapter 1. Mainly, we summarize in it all the definitions and results from [2] we are using in the rest of the memory. Also we take our chance to state them in the more general setting of the n-stars.

Chapter 2 continues the study of the **Y** done in [2], in the sense that we apply the characterization of the primary orbits given there to the study of the topological entropy. Of course, we use also the graph techniques of [13]. In this way, the best lower bounds of the topological entropy for maps of $\mathcal{Y}$, depending on the set of periods, are obtained.

The description of the sets of periods for maps of the interval is done in terms of the Sharkovskiĭ's ordering. Baldwin, in [9], has shown that the situation in the n-star (he calls it the n-od) is similar. That is, the sets of periods of continuous self maps of the n-star can be described by means of a finite set of orderings. In [2] the sets of periods of maps of $\mathcal{Y}$ are also described by means of three orderings. In both cases Sharkovskiĭ's ordering appears. However the coincidences finish there. The essential difference between Baldwin's orderings (except Sharkovskiĭ's one) and the orderings given in [2] is that these are linear (total), whereas those are partial. In the first part of Chapter 3 we show that the sets of periods of continuous self maps of the n-star can also be described by means of a finite set of linear orderings, which can be associated to some rational numbers. These orderings are constructed in a "number theoretical" way and they do the same job as Baldwin's orderings.

Intimately related with the description of the sets of periods, and with the notion of chaos, is a feature of maps we call full periodicity. We say that a map has *full periodicity* if its set of periods is the whole set of natural numbers. Sharkovskiĭ's Theorem (and Li and Yorke's result) state, in particular, that if a continuous self map of the interval has a periodic point of period 3 then it has full periodicity. Mumbrú, in [28], solves the analogous problem in the $\mathbf{Y}$ (several years before the publication of [2], where the solution of this problem was obtained as a corollary). Namely he finds that, for a map of $\mathcal{Y}$ to have full periodicity, it suffices that its set of periods contains the numbers 2, 3, 4, 5 and 7.

These results give rise to the question of finding minimal subsets of natural numbers with the property that, for any map of certain class to have full periodicity, it suffices that its set of periods contains some of those subsets. Sets of numbers with those properties are called *full periodicity kernels.* In Chapter 3 we also characterize the full periodicity kernels for continuous self maps of the n-stars, by using Baldwin's characterization of their sets of periods.

In the last chapter we deal again with primary orbits. We try to generalize the characterization of primary orbits done in [2] to the n-star. There, the primary orbits were classified into two categories: directed and undirected. Roughly speaking, we can say that directed orbits are the "genuine" primary orbits in the $\mathbf{Y}$, whereas the undirected ones are "inherited" in some way from the interval. On trying to generalize the characterization of the primary orbits, we see at once with suitable examples that for $n \geq 4$ directed primary orbits no longer have the good properties they had in the $\mathbf{Y}$. The most striking fact about this class of orbits is that there is no bound for the number of coloured arrows they can have. This point will be discussed in detail in Section 4.5. Therefore we restrict ourselves to consider the class of periodic orbits we call *strongly directed.* In Chapter 4 we completely characterize the primary strongly directed orbits for self maps of the 4-star with the branching point fixed. These orbits are in turn classified into several families, some of them not easy to describe. For the two simplest ones we give a characterization which is independent of n.

By looking at examples of primary orbits of the n-star with $n \geq 4$, we still have the impression that the strongly directed orbits are the "genuine" primary orbits of star maps, while the other primary orbits (undirected and directed not strongly directed) are "inherited" somehow from stars of smaller n. It would be interesting to understand this point in depth and, of course, to be able to extend the characterization of the primary orbits to these families for the 4-star. We believe that the main difficulties of the characterization of the primary orbits of continuous self maps of the n-star ($n > 4$) having the branching point fixed, are already present in the case $n = 4$. Thus, the knowledge of the full characterization for the case $n = 4$, will give us good clues on how to prosecute this task to the general case.

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Axioma 0.0.0 *Lluís me fecit.*

Moltíssimes gràcies, Lluís.