Symbolic dynamics for degree one circle maps

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Introduction

In the last years, the work of Milnor and Thurston ([**MT**]) where the kneading theory for piecewise monotone maps of the interval is described, has been used to study the dynamics of a map by means of the symbolic dynamics. This method has been applied to the particular case of unimodal and bimodal maps of the interval (maps with two and three pieces of monotonicity, respectively). In this memoir we define a kneading theory to study a certain class of degree one circle maps and, from its study, we derive some results which are valid also for every degree one circle map.

In Chapter I we establish the kneading theory for the class $\mathcal{A}$ of maps which are liftings of a degree one circle maps with a single maximum and a single minimum. The study of these maps arise naturally in different contexts in dynamical systems. For instance, a three parameter family of maps of class $\mathcal{A}$ has been used by Levi (see [**L**]) to study Van der Pol equation. Also, they are relevant in the description of the transition to chaos for area contracting annulus maps. The key point of the kneading theory developed here is a suitable definition of itinerary. With this notion of itinerary we can extend the standard results of the kneading theory to class $\mathcal{A}$. First of all we obtain the kneading determinant as a function of two kneading sequences, the itineraries of the maximum and of the minimum. As usual, the kneading determinant is a power series in $\mathbf{Z}[[t]]$ and gives the growth number of the map. If the growth number is one then the kneading determinant does not vanish in $[0,1)$; otherwise, the inverse of the growth number is the first zero of the kneading determinant in $[0,1)$. As it is is well known, the importance of the kneading determinant comes from the fact that the topological entropy of the map is the logarithm of the growth number. Also, for maps with positive entropy we obtain a piecewise linear model which is semiconjugate to the original map. Lastly we characterize the set of itineraries of a map in class $\mathcal{A}$. This characterization depends only on the two kneading sequences.

One of the main tools to study the dynamics of circle maps of degree one is the rotation interval. It is a generalization of the rotation number introduced by Poincaré to study homeomorphisms of the circle. Roughly speaking the rotation interval is the set of all average angular speeds of points under iteration of the map and it gives a considerable amount of information about the periodic orbits of the map in consideration. Another important notion is the topological entropy. It characterizes the complexity of the maps. Intuitively it measures the exponential growth rate of the number of periodic orbits as we increase their periods. For a piecewise monotone map it is also the logarithm of the growth number which is the exponential growth rate of the number of pieces of monotonicity of the iterates of the map. The fact that both, the rotation interval and the topological entropy have something to do with the periodic orbits of a map suggests that there has to be some relation between both notions. Indeed in Chapter I the best lower and upper bounds of the topological entropy of a map of class $\mathcal{A}$ depending on its rotation interval are given. Hence, the rotation interval can be used, to some extent, as a measure of chaos.

The above considerations lead us to the nontrivial problem of approximating the rotation interval of a given continuous circle map of degree one. This problem has already been considered by several authors (see for instance [**Ba**] and [**Be**]). Our approach is analogous to the one adopted by Bernhardt in [**Be**] for a subclass of $\mathcal{A}$. The main result of this chapter is to characterize the rotation interval of a map depending on the two kneading sequences. From this characterization we derive some consequences. First of all, it gives an algorithm to approximate effectively the rotation interval of a map of class $\mathcal{A}$ and therefore to compute the set of periods of such a map. Also we obtain for each rotation interval models with "maxima" and "minima" dynamics. As a corollary, for the maps of our class, we obtain the mentioned result about the best lower bounds of the topological entropy and also the best lower bound of the number of periodic orbits of each period, depending on the rotation interval.

In Chapter II, we prove that the lower bounds of the entropy and of the number of periodic orbits of each period, obtained in the preceding chapter for a restricted class of circle maps of degree one, are also valid for every circle map of degree one. This result generalizes partial results obtained by other authors (see [**ALMS**]) in which they obtain lower bounds of the topological entropy in the case in which one of the endpoints of the rotation interval is an integer. Also it answers an open question formulated in [**BGMY**].

In Chapter III we study the monotonicity of the entropy for a biparametric family of degree one maps of the circle. The monotonicity of the entropy for particular families of maps of the interval has been considered by several authors for several families (see [**MV**],[**BMT**],[**MT**] and [**DH**]). We consider a similar problem to the one considered in [**MV**]. We deal with a biparametric family of piecewise linear circle maps with two pieces and we prove that the entropy increases when any of the two slopes increases. We also describe the regions of the parameter space where the monotonicity is strict.

In Chapter IV we extend the kneading theory for class $\mathcal{A}$ to a new family of maps of the circle of degree one with one discontinuity. This class includes the family of Lorenz-like maps studied by F. Hofbauer ([**H**]) and L. Alsedà, J. Llibre, M. Misiurewicz and Ch. Tresser ([**ALMT**]). In particular it includes the famous β-transformations ([**R**]). Following a process similar to the one given in Chapter I we obtain an algorithm to compute the rotation interval and models with "minimum" and "maximum" dynamics depending on the rotation interval for maps in this class. Thus we also obtain lower and upper bounds of the topological entropy for maps in this new class. These results extend some results from [**ALMT**] to our family.

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