TWIN QUADRATIC POLYNOMIAL VECTOR FIELDS IN THE SPACE $\mathbb{C}^3$

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Abstract. In this paper we study quadratic polynomial vector fields on $\mathbb{C}^3$ with 8 isolated singularities. Either two polynomial vector fields share six singularities with the same position and spectra and the remaining two singularities have some relation on their spectra, or two polynomial vector fields share five singularities with the same position and spectra and the remaining three singularities have some other relation on their spectra. Under these conditions we determine the spectra and positions of the remaining singularities. Moreover there exist two three-parametric families of vector fields having the same singular points and for each singular point both vector fields have the same spectrum.

1. Introduction and statement of the results

Consider polynomial vector fields on the affine space $\mathbb{C}^3$. We denote by $\mathcal{P}$ the space of all polynomial vector fields

$$\chi = P(x,y,z)\frac{\partial}{\partial x} + Q(x,y,z)\frac{\partial}{\partial y} + R(x,y,z)\frac{\partial}{\partial z}$$

such that P, Q and R are quadratic. By Bezout's Theorem, a generic element of $\mathcal{P}$ has exactly eight isolated singularities. We denote by $\mathcal{P}_8$ the space of the vector fields in $\mathcal{P}$ that have eight isolated singularities. Since $\chi \in \mathcal{P}_8$ has the maximum number of singularities, the determinant of the linear part of χ at each singular point is nonzero. So the eigenvalues at any singular point are nonzero, i.e. all singular points are non degenerate (see for more details [5]). The space $\mathcal{P}_8$ is endowed with a structure of a complex affine space identifying all the thirty coefficients of the polynomials P, Q and R with a point of $\mathbb{C}^{30}$. This topology in the set $\mathcal{P}_8$ is called the *topology of the coefficients*, and $\mathcal{P}_8$ is an open subset of $\mathcal{P}$.

We say that two vector fields χ and $\widehat{\chi}$ of $\mathcal{P}_8$ are *affine equivalent* if there exists an affine transformation T that maps χ into $\widehat{\chi}$ that is

$$\widehat{\chi}(x,y,z) = DT \cdot \chi(T^{-1}(x,y,z)).$$

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We denote by $\operatorname{sing}(\chi)$ the singular points of χ. If $p \in \operatorname{sing}(\chi)$ we define its spectrum as the (unordered) triple of eigenvalues of the linearization matrix

$$D\chi(p) = \begin{pmatrix} P_x & P_y & P_z \\ Q_x & Q_y & Q_z \\ R_x & R_y & R_z \end{pmatrix} |_{(x,y,z)=p}.$$

Note that the spectrum of the matrix $M = D\chi(p)$ carries the same information as the ordered triple

$$(\operatorname{tr} M, \operatorname{tr}(M^2), \det M). \tag{1}$$

eq:triple

Indeed, the characteristic polynomial is given by

$$-\lambda^3 + \operatorname{tr} M \lambda^2 + \frac{1}{2}\big((\operatorname{tr} M)^2 - (\operatorname{tr}(M^2))\big)\lambda + \det M.$$

We say that two vector fields of $\mathcal{P}_8$ have *the same spectra of singularities* if the set of all triples (1) associated to their singularities coincides. Note that the above definition takes into account the spectra of singularities and does not take into account the position of them. Our ultimate main aim is to understand the pair of vector fields that share both position and spectra of singularities, that is, to provide results in the following direction: consider two polynomial vector fields χ and $\widehat{\chi}$ having each of them 8 singularities that we denote respectively by $p_1, \ldots, p_8$ and $\widehat{p}_1, \ldots, \widehat{p}_8$. Assume that

$$p_i = \widehat{p}_i \text{ for } i = 1, \ldots, M \quad \text{and} \quad D\chi(p_i) \sim D\widehat{\chi}(\widehat{p}_i) \text{ for } i = 1, \ldots, N,$$

where $A \sim B$ denotes that the matrices A and B are *similar*, that is they have the same spectrum. Then for certain values of M and N we want to see when

$$p_i = \widehat{p}_i \quad \text{and} \quad D\chi(p_i) \sim D\widehat{\chi}(\widehat{p}_i) \text{ for } i = 1, \ldots, 8.$$

Once it has been established that two vector fields agree on the positions of their singularities and their corresponding spectra, it is natural to ask whether these two vector fields are identical or not. This question gives rise to the concept of *twin vector fields*. We say that two *different* vector fields are *twin vector fields* if they agree on position and spectra at all their singularities. On our notation this corresponds to $M = N = 8$. The corresponding question for quadratic vector fields in the plane was studied in [6] and for general polynomial vector fields in [4]. There the author proved that if two quadratic vector fields in the plane have the same spectra then after an affine transformation we can achieve that all the points share the same spectra and position and it is proved that a *generic* vector field indeed admits a unique twin vector field. Similar results for the space are not available in the literature and this is the main aim of this paper. The fact that the dimension is greater makes the analysis much more intricate in particular because any two vector fields χ and $\widehat{\chi}$ in general position after a suitable affine map on $\mathbb{C}^3$ can have four singularities in the same position (and for them if in addition both vector fields have the same spectra we can assume without loss of generality that two vector fields that have the same spectra), but there are still four free points. If instead of quadratic vector fields we take polynomial vector fields of higher degree the analysis gets even much more complicated, so a good starting point in the case of the space is to work with quadratic polynomial vector fields. The main tool in the proofs will be the well-known Euler-Jacobi formula (see below for more details) and that is the reason why in the paper we shall focus on polynomial vector fields. Such a formula will be applied in a clever and convenient way that will lead to have some control on the position and spectra of the singularities of a polynomial vector field. The following are our results.

⟨thm.0⟩
Theorem 1. *Assume that two vector fields χ and $\widehat{\chi} \in \mathcal{P}_8$ have the same spectra and there are seven singularities sharing the same position and spectra. Then the eight singularities share position and spectra.*

Related results in the plane were obtained in [6].

The proof of Theorem 1 is given in section 2.

⟨t.1⟩
Theorem 2. *Assume that two vector fields χ and $\widehat{\chi}$ in $\mathcal{P}_8$ have the same spectra, and that they have six singularities sharing positions and spectra denoted by $p_1, \ldots, p_6$. We denote by p_7 and p_8 (respectively $\widehat{p}_7$ and $\widehat{p}_8$) the remaining singularities of χ (respectively $\widehat{\chi}$). Assume that $trD\chi(p_7) \neq trD\chi(p_8)$, then either $\chi = \widehat{\chi}$ or χ and $\widehat{\chi}$ are twin vector fields.*

The proof of Theorem 2 is given in section 3.

⟨t.2⟩
Theorem 3. *Assume that the vector fields χ and $\widehat{\chi}$ in $\mathcal{P}_8$ have the same spectra, and share five singularities $p_1, \ldots, p_5$ with the same positions and spectra. Denote by p_6, p_7, p_8 (respectively $\widehat{p}_6, \widehat{p}_7, \widehat{p}_8$) the remaining singularities of χ (respectively $\widehat{\chi}$) and assume that $trD\chi(p_6) \neq 0$. If $\det D\chi(p_7) \neq -\det D\chi(p_8)$, $trD\chi(p_7) = trD\chi(p_8)$ and $trD\chi(p_6) \neq trD\chi(p_7)$, then there are six singularities sharing positions and spectra and*

(a) *either $\chi = \widehat{\chi}$, or χ and $\widehat{\chi}$ are twin vector fields,*
(b) *or $\widehat{x}_8 = x_8 = \widehat{x}_7 = x_7$, $\widehat{z}_8 = z_8 = \widehat{z}_7 = z_7$, $\widehat{y}_k \neq y_k$ for $k = 7, 8$, and equations* (5) *hold,*
(c) *or $\widehat{y}_8 = y_8 = \widehat{y}_7 = y_7$, $\widehat{z}_8 = z_8 = \widehat{z}_7 = z_7$, $\widehat{x}_k \neq x_k$ for $k = 7, 8$, and equations*(4) *hold,*
(d) *or $\widehat{x}_8 = x_8 = \widehat{x}_7 = x_7$, $\widehat{y}_8 = y_8 = \widehat{y}_7 = y_7$, $\widehat{z}_k \neq z_k$ for $k = 7, 8$, and equations* (6) *hold,*
(e) *or $\widehat{x}_8 = x_8 = \widehat{x}_7 = x_7$, $\widehat{y}_k \neq y_k$, $\widehat{z}_k \neq z_k$ for $k = 7, 8$, and equations* (5) *and* (6) *hold,*
(f) *or $\widehat{y}_8 = y_8 = \widehat{y}_7 = y_7$, $\widehat{x}_k \neq x_k$, $\widehat{z}_k \neq z_k$ for $k = 7, 8$, and equations* (4) *and* (6) *hold,*
(g) *or $\widehat{z}_8 = z_8 = \widehat{z}_7 = z_7$, $\widehat{x}_k \neq x_k$, $\widehat{y}_k \neq y_k$ for $k = 7, 8$, and equations* (4) *and* (5) *hold,*

(h) *or* $\widehat{x}_k \neq x_k$, $\widehat{y}_k \neq y_k$, $\widehat{z}_k \neq z_k$ *for* $k = 7, 8$, *and equations* (4), (5) *and* (6) *hold.*

The proof of Theorem 3 is given in section 4.

⟨thms.1⟩

Theorem 4. *Assume that two vector fields* χ *and* $\widehat{\chi}$ *of* $\mathcal{P}_8$ *have* $p_1, \ldots, p_6$ *singular points with the same positions and spectra. We denote by* p_7 *and* p_8 *(respectively* $\widehat{p}_7$, $\widehat{p}_8$*) the remaining singularities of* χ *(respectively* $\widehat{\chi}$*). If* p_7 *and* $\widehat{p}_7$ *have the same spectra, then the following statements hold.*

(a) *The spectra of* p_8 *and* $\widehat{p}_8$ *are the same.*
(b) *If* $trD\chi(p_7) \neq trD\chi(p_8)$, *then either* $\chi = \widehat{\chi}$, *or* χ *and* $\widehat{\chi}$ *are twin vector fields.*
(c) *If* $trD\chi(p_7) = trD\chi(p_8)$, *then*
  (c.1) *either* $\chi = \widehat{\chi}$, *or* χ *and* $\widehat{\chi}$ *are twin vector fields,*
  (c.2) *or* $\widehat{x}_k = x_k$, $\widehat{z}_k = z_k$, $\widehat{y}_k \neq y_k$ *for* $k = 7, 8$, *and equations* (5) *hold,*
  (c.3) *or* $\widehat{y}_k = y_k$, $\widehat{z}_k = z_k$, $\widehat{x}_k \neq x_k$ *for* $k = 7, 8$, *and equations* (4) *hold,*
  (c.4) *or* $\widehat{x}_k = x_k$, $\widehat{y}_k = y_k$, $\widehat{z}_k \neq z_k$ *for* $k = 7, 8$, *and equations* (6) *hold,*
  (c.5) *or* $\widehat{x}_k = x_k$, $\widehat{y}_k \neq y_k$, $\widehat{z}_k \neq z_k$ *for* $k = 7, 8$, *and equations* (5) *and* (6) *hold,*
  (c.6) *or* $\widehat{y}_k = y_k$, $\widehat{x}_k \neq x_k$, $\widehat{z}_k \neq z_k$ *for* $k = 7, 8$, *and equations* (4) *and* (6) *hold,*
  (c.7) *or* $\widehat{z}_k = z_k$, $\widehat{x}_k \neq x_k$, $\widehat{y}_k \neq y_k$ *for* $k = 7, 8$, *and equations* (4) *and* (5) *hold,*
  (c.8) *or* $\widehat{x}_k \neq x_k$, $\widehat{y}_k \neq y_k$, $\widehat{z}_k \neq z_k$ *for* $k = 7, 8$, *and equations* (4), (5) *and* (6) *hold.*

The proof of Theorem 4 is given in section 5.

We remark that Theorems 2, 3 and 4 are similar to Theorem 1 of [6], and all these results are related in spirit to the ones for foliations on $\mathbb{P}^2$ of degree two in [3], and to the ones for foliations on $\mathbb{P}^2$ coming from a generic quadratic vector field on $\mathbb{C}^2$ in [2]. In all of them we provide conditions and study the situation of having two vector fields agree on position and spectra at all their singularities. Now we want to study whether the two vector fields are identical or not, or in other words, if twin vector fields really exist. In the following theorem we provide a family of twin vector fields.

⟨thm.3⟩

Theorem 5. *There exist two three-parametric families of twin vector fields in* $\mathcal{P}_8$.

Note that in Theorem 5 we have constructed a positive-dimensional family of twin vector fields, but since this family is very special it is not known that for generic vector fields, twin vector fields exist and are unique as it happens in the plane where the author in [6] proves that generic quadratic vector fields admit always a twin vector field.

2. Proof of Theorem 7

⟨sec.2⟩

The main result that we use for proving Theorems 7, is the Euler-Jacobi formula restricted to polynomials of degree 2 in $\mathbb{C}^3$, which can be stated as follows. For a proof see [1].

?⟨EJF⟩?

Theorem 6 (Euler-Jacobi formula)**.** *If P, Q and R are polynomials in $\mathbb{C}[x, y, z]$ of degree 2 such that the systems $P(x, y, z) = Q(x, y, z) = R(x, y, z) = 0$ has exactly 8 solutions, denoted by $p_1, \ldots, p_8 \in \mathbb{C}^3$ and let $g(x, y, z)$ be an arbitrary polynomial of degree at most 2, then*

$$\sum_{k=1}^{8} \frac{g(p_k)}{J(p_k)} = 0,$$

where $J(x, y, z)$ is the Jacobian determinant of P, Q and R, that is,

$$J(x, y, z) = \det \frac{\partial(P, Q, R)}{\partial(x, y, z)} = \det \begin{pmatrix} P_x & P_y & P_z \\ Q_x & Q_y & Q_z \\ R_x & R_y & R_z \end{pmatrix}.$$

Moreover $J(p_k) \neq 0$ for $k = 1, \ldots, 8$.

Before proving Theorem 1 we state and proof a proposition that will immediately imply the proof of the theorem.

⟨t.0⟩

Proposition 7. *Let χ be a vector field in $\mathcal{P}_8$. Denote by $p_1, \ldots, p_8$ their singularities. Then the position and the spectra of the singular point p_8 is completely determined by the position and the spectra of $p_1, \ldots, p_7$.*

Proof. Let $\chi = (P, Q, R) \in \mathcal{P}_8$ be a vector field having $p_1, \ldots, p_8$ singularities with their spectra. Applying the Euler-Jacobi formula with $g(x, y, z) = 1$ we get that

$$\sum_{k=1}^{8} \frac{1}{J(p_k)} = \sum_{k=1}^{8} \frac{1}{\det D\chi(p_k)} = 0,$$

and then we can obtain $J(p_8)$ in function of $J(p_k)$ for $k = 1, \ldots, 7$.

Now applying the Euler-Jacobi formula with $g(x, y, z) = \operatorname{tr}(D\chi(x, y, z))$ we get

$$\sum_{k=1}^{8} \frac{\operatorname{tr}(D\chi(p_k))}{J(p_k)} = 0,$$

and so we obtain $\operatorname{tr}(D\chi(p_8))$ in function of $\operatorname{tr}(D\chi(p_k))$ and $J(p_k)$ for $k = 1, \ldots, 7$.

Applying the Euler-Jacobi formula with $g(x,y,z) = \mathrm{tr}(D\chi(x,y,z)^2)$ we get

$$\sum_{k=1}^{8} \frac{\mathrm{tr}(D\chi(p_k)^2)}{J(p_k)} = 0,$$

and so we obtain $\mathrm{tr}(D\chi(p_8)^2)$ in function of $\mathrm{tr}(D\chi(p_k)^2)$ and $J(p_k)$ for $k = 1, \ldots, 7$.

Finally applying the Euler-Jacobi formula with $g_1(x,y,z) = x$, $g_2(x,y,z) = y$ and $g_3(x,y,z) = z$ we get

$$\sum_{k=1}^{8} \frac{g_1(p_k)}{J(p_k)} = 0 \quad \sum_{k=1}^{8} \frac{g_2(p_k)}{J(p_k)} = 0 \quad \text{and} \quad \sum_{k=1}^{8} \frac{g_3(p_k)}{J(p_k)} = 0$$

which determines completely the position of p_8 in function of the positions of p_k and $J(p_k)$ for $k = 1, \ldots, 7$. This completes the proof of the proposition. □

Note that the proof of Theorem 1 follows directly from the proof of Proposition 7.

3. Proof of Theorem 2

⟨sec.3⟩

We denote by a_k and $\widehat{a}_k$ the determinant of $D\chi(p_k)$ and $D\widehat{\chi}(p_k)$, respectively, and by b_k and $\widehat{b}_k$ the traces of $D\chi(p_k)$ and $D\widehat{\chi}(p_k)$, respectively. Moreover we denote by $p_k = (x_k, y_k, z_k)$ the positions of the points for the vector field χ, and by $\widehat{p}_k = (\widehat{x}_k, \widehat{y}_k, \widehat{z}_k)$ the positions of the points for the vector field $\widehat{\chi}$.

Note that by assumptions $p_k = \widehat{p}_k$, $a_k = \widehat{a}_k$ and $b_k = \widehat{b}_k$ for $k = 1, \ldots, 6$, either $b_7 \neq b_8$ with $b_7 \neq 0$ and $b_8 \neq 0$, or $b_8 = 0$ and $b_7 \neq 0$, or $b_7 = 0$ and $b_8 \neq 0$. Note that without loss of generality we can assume that $a_7 = \widehat{a}_7$, $b_7 = \widehat{b}_7$, $a_8 = \widehat{a}_8$ and $b_8 = \widehat{b}_8$.

The Euler-Jacobi formula with $g_1(x,y,z) = x$ and $g_2(x,y,z) = \mathrm{tr}(D\chi(x,y,z))x$ yield

$$\frac{x_7}{a_7} + \frac{x_8}{a_8} = \frac{\widehat{x}_7}{a_7} + \frac{\widehat{x}_8}{a_8},$$

and

$$\frac{b_7 x_7}{a_7} + \frac{b_8 x_8}{a_8} = \frac{b_7 \widehat{x}_7}{a_7} + \frac{b_8 \widehat{x}_8}{a_8}.$$

If $b_8 = 0$ and $b_7 \neq 0$ then $x_7 = \widehat{x}_7$ and so $x_8 = \widehat{x}_8$. If $b_7 = 0$ and $b_8 \neq 0$ then $x_8 = \widehat{x}_8$ and so $x_7 = \widehat{x}_7$. If $b_7 \neq b_8$ with $b_7 \neq 0$ and $b_8 \neq 0$, then

$$x_7 - \widehat{x}_7 = -\frac{a_7}{a_8}(x_8 - \widehat{x}_8) \quad \text{and} \quad x_7 - \widehat{x}_7 = -\frac{b_8 a_7}{b_7 a_8}(x_8 - \widehat{x}_8)$$

and so $x_7 = \widehat{x}_7$ and $x_8 = \widehat{x}_8$.

Proceeding analogously taking $g_3(x,y,z)=y$, $g_4(x,y,z)=\mathrm{tr}(D\chi(x,y,z))y$, $g_5(x,y,z)=z$ and $g_6(x,y,z)=\mathrm{tr}(D\chi(x,y,z))z$ we get that $y_7=\widehat{y}_7$, $y_8=\widehat{y}_8$, $z_7=\widehat{z}_7$ and $z_8=\widehat{z}_8$. In short, χ and $\widehat{\chi}$ are twin vector fields.

4. Proof of Theorem 3

⟨sec.4⟩

We denote by a_k and $\widehat{a}_k$ the determinant of $D\chi(p_k)$ and $D\widehat{\chi}(p_k)$, respectively, and by b_k and $\widehat{b}_k$ the traces of $D\chi(p_k)$ and $D\widehat{\chi}(p_k)$, respectively. Moreover we denote by $p_k=(x_k,y_k,z_k)$ the positions of the points for the vector field χ, and by $\widehat{p}_k=(\widehat{x}_k,\widehat{y}_k,\widehat{z}_k)$ the positions of the points for the vector field $\widehat{\chi}$.

Without loss of generality we can assume that $a_k=\widehat{a}_k$ and $b_k=\widehat{b}_k$ for $k=1,\ldots,8$. By assumptions $b_6\neq 0$, $a_7+a_8\neq 0$, $b_7=b_8$ and $b_6\neq b_7$.

The Euler-Jacobi formula with $g_1(x,y,z)=x$ and $g_2(x,y,z)=\mathrm{tr}(D\chi(x,y,z))x$ yield

$$\frac{x_6}{a_6}+\frac{x_7}{a_7}+\frac{x_8}{a_8}=\frac{\widehat{x}_6}{a_6}+\frac{\widehat{x}_7}{a_7}+\frac{\widehat{x}_8}{a_8},$$

and

$$\frac{b_6x_6}{a_6}+\frac{b_7x_7}{a_7}+\frac{b_8x_8}{a_8}=\frac{b_6\widehat{x}_6}{a_6}+\frac{b_7\widehat{x}_7}{a_7}+\frac{b_8\widehat{x}_8}{a_8}.$$

Hence

$$x_6-\widehat{x}_6=-\frac{a_6}{a_7}(x_7-\widehat{x}_7)-\frac{a_6}{a_8}(x_8-\widehat{x}_8),$$

and

$$x_6-\widehat{x}_6=-\frac{a_6b_7}{a_7b_6}(x_7-\widehat{x}_7)-\frac{a_6b_8}{a_8b_6}(x_8-\widehat{x}_8).$$

Since $b_8=b_7\neq b_6$ we get that

$$x_6-\widehat{x}_6=\frac{b_7}{b_6}\left(-\frac{a_6}{a_7}(x_7-\widehat{x}_7)-\frac{a_6}{a_8}(x_8-\widehat{x}_8)\right),$$

and so $x_6=\widehat{x}_6$. Moreover it follows that

(2) eq:burra
$$(x_7-\widehat{x}_7)=-\frac{a_7}{a_8}(x_8-\widehat{x}_8).$$

Applying the Euler-Jacobi formula with $g_1(x,y,z)=x^2$ taking into account that $x_6=\widehat{x}_6$ we get

$$\frac{x_7^2}{a_7}+\frac{x_8^2}{a_8}=\frac{\widehat{x}_7^2}{a_7}+\frac{\widehat{x}_8^2}{a_8},$$

and so

$$\frac{1}{a_7}(x_7-\widehat{x}_7)(x_7+\widehat{x}_7)=-\frac{1}{a_8}(x_8-\widehat{x}_8)(x_8+\widehat{x}_8).$$

Using also (2) we get

$$-\frac{1}{a_8}(x_8-\widehat{x}_8)\big(x_7+\widehat{x}_7-x_8-\widehat{x}_8\big)=0.$$

Then either $x_8 = \widehat{x}_8$ and then also $x_7 = \widehat{x}_7$, or $x_8 \neq \widehat{x}_8$ and $x_7 \neq \widehat{x}_7$ and

(3) eq:relations.re
$$\widehat{x}_7 + x_7 = \widehat{x}_8 + x_8$$

This equation together with (2) and the fact that $a_7 + a_8 \neq 0$ imply

(4) eq:relations.0
$$\widehat{x}_7 = \frac{(a_7 - a_8)x_7 - 2a_7x_8}{a_7 + a_8}, \quad \widehat{x}_8 = -\frac{(a_8 - a_7)x_8 - 2a_8x_7}{a_7 + a_8}.$$

Proceeding in the same manner using $g_2(x, y, z) = y$, $g_3(x, y, z) = \mathrm{tr}(D\chi(x, y, z))y$, $g_4(x, y, z) = y^2$, we get that $y_6 = \widehat{y}_6$, and either $y_7 = \widehat{y}_7$ and $y_8 = \widehat{y}_8$, or $y_7 \neq \widehat{y}_7$ and $y_8 \neq \widehat{y}_8$ and

(5) eq:relations.01
$$\widehat{y}_7 = \frac{(a_7 - a_8)y_7 - 2a_7y_8}{a_7 + a_8}, \quad \widehat{y}_8 = -\frac{(a_8 - a_7)y_8 - 2a_8y_7}{a_7 + a_8},$$

Finally proceeding in the same way using $g_5(x, y, z) = z$, $g_6(x, y, z) = \mathrm{tr}(D\chi(x, y, z))z$, $g_7(x, y, z) = z^2$ we get that $z_6 = \widehat{z}_6$ and either $z_7 = \widehat{z}_7$ and $z_8 = \widehat{z}_8$ or $z_7 \neq \widehat{z}_7$ and $z_8 \neq \widehat{z}_8$ and

(6) eq:relations.02
$$\widehat{z}_7 = \frac{(a_7 - a_8)z_7 - 2a_7z_8}{a_7 + a_8}, \quad \widehat{z}_8 = -\frac{(a_8 - a_7)z_8 - 2a_8z_7}{a_7 + a_8}.$$

Note that $p_6 = \widehat{p}_6$.

We consider the following cases.

Case (a) : $x_i = \widehat{x}_i$, $y_i = \widehat{y}_i$ and $z_i = \widehat{z}_i$ for $i = 7, 8$. So statement (a) holds.

Case (b) : $x_i = \widehat{x}_i$, $z_i = \widehat{z}_i$ for $i = 7, 8$, $y_7 \neq \widehat{y}_7$ and $y_8 \neq \widehat{y}_8$. Applying the Euler-Jacobi formula with $g_8(x, y, z) = xy$ we get

$$\frac{x_7y_7}{a_7} + \frac{x_8y_8}{a_8} = \frac{x_7\widehat{y}_7}{a_7} + \frac{x_8\widehat{y}_8}{a_8}$$

but then

$$\frac{x_7}{a_7}(y_7 - \widehat{y}_7) = -\frac{x_8}{a_8}(y_8 - \widehat{y}_8)$$

which yields (in view of the corresponding equation (2) with x replaced by y) that $x_7 = x_8$. Doing the same with $g_9(x, y, z) = yz$ we get that $z_7 = z_8$ and from (5) we get statement (b).

Case (c) : $y_i = \widehat{y}_i$, $z_i = \widehat{z}_i$ for $i = 7, 8$, $x_7 \neq \widehat{x}_7$ and $x_8 \neq \widehat{x}_8$. Following similar arguments to the case (b), statement (c) follows.

Case (d) : $x_i = \widehat{x}_i$, $y_i = \widehat{y}_i$ for $i = 7, 8$, $z_7 \neq \widehat{z}_7$ and $z_8 \neq \widehat{z}_8$. Following similar arguments to the case (b), statement (d) follows.

Case (e) : $x_i = \widehat{x}_i$, for $i = 7, 8$, $y_7 \neq \widehat{y}_7$, $y_8 \neq \widehat{y}_8$, $z_7 \neq \widehat{z}_7$ and $z_8 \neq \widehat{z}_8$. Following similar arguments to the case (b), statement (e) follows.

Case (f) : $y_i = \widehat{y}_i$, for $i = 7, 8$, $x_7 \neq \widehat{x}_7$, $x_8 \neq \widehat{x}_8$, $z_7 \neq \widehat{z}_7$ and $z_8 \neq \widehat{z}_8$. Following similar arguments to the case (b), statement (f) follows.

Case (g) : $z_i = \widehat{z}_i$, for $i = 7, 8$, $x_7 \neq \widehat{x}_7$, $x_8 \neq \widehat{x}_8$, $y_7 \neq \widehat{y}_7$ and $y_8 \neq \widehat{y}_8$. Following similar arguments to the case (b), statement (g) follows.

Case (h) : $x_7 \neq \widehat{x}_7$, $x_8 \neq \widehat{x}_8$, $y_7 \neq \widehat{y}_7$, $y_8 \neq \widehat{y}_8$, $z_7 \neq \widehat{z}_7$ and $z_8 \neq \widehat{z}_8$. From (4), (5) an (6), it follows statement (h).

5. Proof of Theorem 4

⟨secs.2⟩

We denote by a_8 and $\widehat{a}_8$ the determinant of $D\chi(p_8)$ and $D\widehat{\chi}(p_8)$, respectively, by b_8 and $\widehat{b}_8$ the traces of $D\chi(p_8)$ and $D\widehat{\chi}(p_8)$, respectively, and by c_8 and $\widehat{c}_8$ the traces of $D\chi(p_8)^2$ and $D\widehat{\chi}(p_8)^2$, respectively. Moreover, we denote by $p_8 = (x_8, y_8, z_8)$ and $p_7 = (x_7, y_7, z_7)$ the positions of the points for the vector field χ, and by $\widehat{p}_8 = (\widehat{x}_8, \widehat{y}_8, \widehat{z}_8)$ and $\widehat{p}_7 = (\widehat{x}_7, \widehat{y}_7, \widehat{z}_7)$ the positions of the points for the vector field $\widehat{\chi}$. Note that from the Euler-Jacobi formula with $g_1(x,y,z) = 1$, $g_2(x,y,z) = \operatorname{tr}(D\chi(x,y,z))$ and $g_3(x,z,y) = \operatorname{tr}(D\chi(x,y,z)^2)$ we get

$$\frac{1}{a_8} = \frac{1}{\widehat{a}_8}, \qquad \frac{b_8}{a_8} = \frac{\widehat{b}_8}{\widehat{a}_8}, \qquad \frac{c_8}{a_8} = \frac{\widehat{c}_8}{\widehat{a}_8},$$

so $a_8 = \widehat{a}_8$, $b_8 = \widehat{b}_8$ and $c_8 = \widehat{c}_8$. This completes the proof of statement (a).

Applying the Euler Jacobi formula with $g_1(x,y,z) = x$, $g_2(x,y,z) = y$, $g_3(x,y,z) = z$, $g_4(x,y,z) = \operatorname{tr}(D\chi(p))x$, $g_5(x,y,z) = \operatorname{tr}(D\chi(p))y$, $g_6(x,y,z) = \operatorname{tr}(D\chi(p))z$ and taking into account that $a_i = \widehat{a}_i$ and $b_i = \widehat{b}_i$ for $i = 1, \ldots, 8$, we get

$$\begin{aligned}
\frac{x_7}{a_7} + \frac{x_8}{a_8} &= \frac{\widehat{x}_7}{a_7} + \frac{\widehat{x}_8}{a_8},\\
\frac{y_7}{a_7} + \frac{y_8}{a_8} &= \frac{\widehat{y}_7}{a_7} + \frac{\widehat{y}_8}{a_8},\\
\frac{z_7}{a_7} + \frac{z_8}{a_8} &= \frac{\widehat{z}_7}{a_7} + \frac{\widehat{z}_8}{a_8},\\
\frac{b_7 x_7}{a_7} + \frac{b_8 x_8}{a_8} &= \frac{b_7 \widehat{x}_7}{a_7} + \frac{b_8 \widehat{x}_8}{a_8},\\
\frac{b_7 y_7}{a_7} + \frac{b_8 y_8}{a_8} &= \frac{b_7 \widehat{y}_7}{a_7} + \frac{b_8 \widehat{y}_8}{a_8},\\
\frac{b_7 z_7}{a_7} + \frac{b_8 z_8}{a_8} &= \frac{b_7 \widehat{z}_7}{a_7} + \frac{b_8 \widehat{z}_8}{a_8}.
\end{aligned} \tag{7}$$

eq:relaciones

Multiplying by b_7 the first equality in (7) and subtracting the fourth from it, and doing the same with the second and the fifth equalities and the same with the third and sixth we get

$$\frac{(b_7 - b_8)x_8}{a_8} = \frac{(b_7 - b_8)\widehat{x}_8}{a_8}, \ \frac{(b_7 - b_8)y_8}{a_8} = \frac{(b_7 - b_8)\widehat{y}_8}{a_8}, \ \frac{(b_7 - b_8)z_8}{a_8} = \frac{(b_7 - b_8)\widehat{z}_8}{a_8}.$$

We have two cases: either $b_7 - b_8 \neq 0$ or $b_7 - b_8 = 0$.

If $b_7 - b_8 \neq 0$, then $x_8 = \widehat{x}_8$, $y_8 = \widehat{y}_8$ and $z_8 = \widehat{z}_8$. Then the second, fourth and sixth equalities in (7) imply that $x_7 = \widehat{x}_7$, $y_7 = \widehat{y}_7$ as well as $z_7 = \widehat{z}_7$. In short $p_7 = \widehat{p}_7$, $p_8 = \widehat{p}_8$ and the spectrum of p_8 and $\widehat{p}_8$ is the same. This concludes the proof of statement (b).

If $b_7 = b_8$ then we apply the Euler-Jacobi formula with $g_1(x,y,z) = x^2$, $g_2(x,y,z) = y^2$, and $g_3(x,y,z) = z^2$ and we get

eq:relations.bis

$$\begin{aligned}
&\frac{x_7^2}{a_7} + \frac{x_8^2}{a_8} = \frac{\widehat{x}_7^2}{a_7} + \frac{\widehat{x}_8^2}{a_8}, \quad \text{i.e} \quad \frac{x_7^2 - \widehat{x}_7^2}{a_7} = \frac{\widehat{x}_8^2 - x_8^2}{a_8},\\
&\frac{y_7^2}{a_7} + \frac{y_8^2}{a_8} = \frac{\widehat{y}_7^2}{a_7} + \frac{\widehat{y}_8^2}{a_8}, \quad \text{i.e} \quad \frac{y_7^2 - \widehat{y}_7^2}{a_7} = \frac{\widehat{y}_8^2 - y_8^2}{a_8},\\
&\frac{z_7^2}{a_7} + \frac{z_8^2}{a_8} = \frac{\widehat{z}_7^2}{a_7} + \frac{\widehat{z}_8^2}{a_8}, \quad \text{i.e} \quad \frac{z_7^2 - \widehat{z}_7^2}{a_7} = \frac{\widehat{z}_8^2 - z_8^2}{a_8}.
\end{aligned} \tag{8}$$

Note that the first, second and third relations in (7) can be written as

eq:relations.biss

$$\frac{x_7 - \widehat{x}_7}{a_7} = \frac{\widehat{x}_8 - x_8}{a_8}, \quad \frac{y_7 - \widehat{y}_7}{a_7} = \frac{\widehat{y}_8 - y_8}{a_8}, \quad \frac{z_7 - \widehat{z}_7}{a_7} = \frac{\widehat{z}_8 - z_8}{a_8}. \tag{9}$$

It follows from the first identity in (9) (which is also (2)) that either $\widehat{x}_7 = x_7$ and so $\widehat{x}_8 = x_8$, or $\widehat{x}_7 \neq x_7$ and $\widehat{x}_8 \neq x_8$. In this last case from the first and second relations in (8) we get (3). Proceeding exactly as in the proof of Theorem 3 that equation (4) holds. Proceeding exactly in the same way for the coordinates y and z we get that either $\widehat{y}_7 = y_7$ and so $\widehat{y}_8 = y_8$, or $\widehat{y}_7 \neq y_7$ and $\widehat{y}_8 \neq y_8$ and (5) holds, or that $\widehat{z}_7 = z_7$ and so $\widehat{z}_8 = z_8$, or $\widehat{z}_7 \neq z_7$ and $\widehat{z}_8 \neq z_8$ and (6) holds. We consider the following cases.

Case (a) : $x_i = \widehat{x}_i$, $y_i = \widehat{y}_i$ and $z_i = \widehat{z}_i$ for $i = 7, 8$ and statement (c.1) holds.
Case (b) : $x_i = \widehat{x}_i$, $z_i = \widehat{z}_i$ for $i = 7, 8$, $y_7 \neq \widehat{y}_7$ and $y_8 \neq \widehat{y}_8$. From (5) we get statement (c.2).
Case (c) : $y_i = \widehat{y}_i$, $z_i = \widehat{z}_i$ for $i = 7, 8$, $x_7 \neq \widehat{x}_7$ and $x_8 \neq \widehat{x}_8$. From (4) we get statement (c.3).
Case (d) : $x_i = \widehat{x}_i$, $y_i = \widehat{y}_i$ for $i = 7, 8$, $z_7 \neq \widehat{z}_7$ and $z_8 \neq \widehat{z}_8$. From (6) we get statement (c.4).
Case (e) : $x_i = \widehat{x}_i$, for $i = 7, 8$, $y_7 \neq \widehat{y}_7$, $y_8 \neq \widehat{y}_8$, $z_7 \neq \widehat{z}_7$ and $z_8 \neq \widehat{z}_8$. From (5) and (6) we get statement (c.5).
Case (f) : $y_i = \widehat{y}_i$, for $i = 7, 8$, $x_7 \neq \widehat{x}_7$, $x_8 \neq \widehat{x}_8$, $z_7 \neq \widehat{z}_7$ and $z_8 \neq \widehat{z}_8$. From (4) and (6) we get statement (c.6).
Case (g) : $z_i = \widehat{z}_i$, for $i = 7, 8$, $x_7 \neq \widehat{x}_7$, $x_8 \neq \widehat{x}_8$, $y_7 \neq \widehat{y}_7$ and $y_8 \neq \widehat{y}_8$. From (4) and (5) we get statement (c.7).
Case (h) : $x_7 \neq \widehat{x}_7$, $x_8 \neq \widehat{x}_8$, $y_7 \neq \widehat{y}_7$, $y_8 \neq \widehat{y}_8$, $z_7 \neq \widehat{z}_7$ and $z_8 \neq \widehat{z}_8$. From (4), (5) an (6), it follows statement (c.8).

This concludes the proof of the theorem.

6. Proof of Theorem 5

Let

$$\chi = (P(x), Q(y), R(z)) := (P, Q, R)$$

be a quadratic polynomial vector field with 8 singularities, that is, with P, Q and R having two simple complex roots. Consider now another quadratic vector field of the form

$$\widehat{\chi} = (aP + bQ + cR)\frac{\partial}{\partial x} + (dP + eQ + fR)\frac{\partial}{\partial y} + (gP + hQ + iR)\frac{\partial}{\partial z}$$

for some complex numbers $a, \ldots, i$. Let

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \quad \text{and note that} \quad D\widehat{\chi}(x,y,z) = A \cdot D\chi(x,y,z).$$

Moreover note that

$$\chi = A\widehat{\chi}.$$

Assume that $\det A = 1$. Then $\det D\widehat{\chi}(x,y,z) = \det D\chi(x,y,z)$. Moreover note that since A is invertible the singularities of χ are the same as the singularities of $\widetilde{\chi}$. Furthermore the relation $\mathrm{tr} D\widehat{\chi}(x,y,z) = \mathrm{tr}(A \cdot D\chi(x,y,z))$ becomes

$$P'(x) + Q'(y) + R'(z) = aP'(x) + eQ'(y) + iR'(z)$$

that is

$$(1-a)P'(x) + (1-e)Q'(y) + (1-i)R'(z) = 0,$$

whose solution is $a = e = i = 1$. Finally, the relation $\mathrm{tr} D\widehat{\chi}(x,y,z)^2 = \mathrm{tr}(A \cdot D\chi(x,y,z^2)$ which yields

$$P'(x)^2 + Q'(y)^2 + R'(z)^2 = P'(x)^2 + 2bdQ'P' + 2cgR'P' + Q'(y)^2 + 2fhQ'R' + R'(z)^2$$

and so $bd = cg = fh = 0$. Since $\det A = 1$ we get

$$dch + bfg = 0.$$

So, we have a family of three parametric vector fields all of them being twin vector fields (the elements in the family are basically in bijection with 3×3 unipotent triangular matrices). In short, we have proven the existence and non-uniqueness of twin vector fields.

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