International Journal of Bifurcation and Chaos

The extended 16-th Hilbert problem for discontinuous piecewise systems formed by linear centers and linear Hamiltonian saddles separated by a non-regular line

Jaume Llibre
Departament de Matemàtiques, Universitat Autònoma de Barcelona,
Edifici C, 08193 Bellaterra, Barcelona, Catalonia, Spain
jaume.llibre@uab.cat

Claudia Valls
Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais
1049–001, Lisboa, Portugal
cvalls@math.ist.utl.pt

We study discontinuous piecewise linear differential systems formed by linear centers and/or linear Hamiltonian saddles and separated by a non-regular straight line. There are two classes of limit cycles: the ones that intersect the separation line in two points and the ones that intersect the separation line in four points, named limit cycles of type II_2 and limit cycles of type II_4, respectively. We prove that the maximum numbers of limit cycles of types II_2 and II_4 are two and one, respectively. We show that all these upper bounds are reached providing explicit examples.

1. Introduction and statement of the main result

The 16*th Hilbert problem* which consists in finding an upper bound for the maximum number of limit cycles (periodic orbits of a differential system isolated in the set of all periodic orbits of that system) that a given class of differential systems can exhibit, is in general a very hard and unsolved problem. Only for very few classes of differential systems this problem has been solved. We note that limit cycles play an important role for explaining physical phenomena, see for instance the limit cycle of van der Pol equation [van der Pol, 1920, 1926], or the one of the Belousov-Zhavotinskii model [Belousov, 1959; Zhabotinsky, 1964], etc. and so there has been an intense active research on the limit cycles for many distinct smooth differential systems.

In recent years the study of the limit cycles has been extended to discontinuous planar piecewise differential systems (that is, differential systems whose vector field is discontinuous on some curve). These limit cycles exhibit more complex dynamical behavior because two kind of limit cycles can appear in discontinuous piecewise linear differential systems due to the existence of the discontinuous curve. Following Filippov's convention [Filippov, 1988] they are the sliding limit cycles (those ones that contain some pieces of the discontinuity curve) and the crossing limit cycles (those ones that only contain isolated points of the discontinuity curve). These causes many differences in their dynamics but since these systems are widely used to model processes appearing in control theory, electric circuits, biology, mechanics, economy, etc., (see for instance the books of di Bernardo et al. [di Bernardo *et al.*, 2008] and Simpson [Simpson, 2010], the survey of Makarenkov and Lamb [Makarenkov & Lamb, 2012], as well as many references quoted in these last works) we will only work with the crossing limit cycle, called here simply limit cycles.

The simplest class of discontinuous piecewise differential systems are the planar ones formed by two pieces separated by a straight line having a linear differential system in each piece. Several authors have tried to determine the maximum number of limit cycles for this class of discontinuous piecewise differential systems. Thus, in one of the first papers dedicated to this problem, Giannakopoulos and Pliete [Giannakopoulos & Pliete, 2001] in 2001, showed the existence of discontinuous piecewise linear differential systems with two limit cycles. Then, in 2010 Han and Zhang [Han & Zhang, 2010] found other discontinuous piecewise linear differential systems with two limit cycles and they conjectured that the maximum number of limit cycles for discontinuous piecewise linear differential systems with two pieces separated by a straight line is two. But in 2012 Huan and Yang [Huan & Yang, 2012] provided numerical evidence of the existence of three limit cycles in this class of discontinuous piecewise linear differential systems. In 2012 Llibre and Ponce [Llibre & Ponce, 2012] inspired by the numerical example of Huan and Yang, proved for the first time that there are discontinuous piecewise linear differential systems with two pieces separated by a straight line having three limit cycles. Later on, other authors obtained also three limit cycles for discontinuous piecewise linear differential systems with two pieces separated by a straight line, see Braga and Mello [Braga & Mello, 2013] in 2013, Buzzi, Pessoa and Torregrosa [Buzzi *et al.*, 2013] in 2013, Liping Li [Li, 2014] in 2014, Freire, Ponce and Torres [Freire *et al.*, 2014] in 2014, and Llibre, Novaes and Teixeira [Llibre *et al.*, 2015] in 2015. But proving that discontinuous piecewise linear differential systems separated by a straight line have at most three limit cycles is an open problem.

However in numerous models of practical problems the discontinuous curve is not always a regular straight line. It is known that the number of limit cycles can change if the shape of the discontinuity curve changes. If the unique straight line of discontinuity becomes a piecewise straight line formed by two semi-straight lines the existence and number of limit cycles have recently deserved the attention of many researchers.

Here we shall study the limit cycles of discontinuous piecewise differential systems defined in the angular regions

$$\mathcal{S}_1 = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq \alpha x\},$$
$$\mathcal{S}_2 = \{(x,y) \in \mathbb{R}^2 : x \leq 0, \text{ or } x \geq 0, y \leq \alpha x\},$$

by

$$(\dot{x}, \dot{y}) = \mathbf{F}(x,y) = \begin{cases} \mathbf{F}_1(x,y) = (f_1(x,y), g_1(x,y)) \text{ if } (x,y) \in \mathcal{S}_1, \\ \mathbf{F}_2(x,y) = (f_2(x,y), g_2(x,y) \text{ if } (x,y) \in \mathcal{S}_2, \end{cases} \tag{1}$$

where f_i, g_i are linear polynomials for $i = 1, 2$.

System (1) is bi-valued on the non-regular separation line

$$\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 = \{(0,\, y) : y \geq 0\} \cup \{(x,\, \alpha x) : x \geq 0\}.$$

In [Huan & Yang, 2019] the authors proved that any piecewise differential system of the form (1) can be transformed into a piecewise differential system with $\alpha = 0$ by means of an invertible linear transformation. Thus it is not restrictive to consider $\alpha = 0$. Now the discontinuity line will be

$$\mathcal{R} = \mathcal{R}_y \cup \mathcal{R}_x = \{(0,\, y) : y \geq 0\} \cup \{(x,\, 0) : x \geq 0\},$$

and the two pieces where we have the differential systems are

$$\mathcal{R}_1 = \{(x,y) \in \mathbb{R}^2 : x \geq 0,\, y \geq 0\},$$
$$\mathcal{R}_2 = \{(x,y) \in \mathbb{R}^2 : x \leq 0, \quad \text{or} \quad x \geq 0,\, y \leq 0\}.$$

We denote by II_1 a crossing limit cycle having two intersection points with either $\mathcal{R}_x$, or $\mathcal{R}_y$; by II_2 a crossing limit cycle having one intersection point with $\mathcal{R}_x$ and another intersection point with $\mathcal{R}_y$; by II_3 a crossing limit cycle having three intersection points with the non-regular line $\mathcal{R}$, and by II_4 a crossing limit cycle having two intersection points with $\mathcal{R}_x$ and two intersection points with $\mathcal{R}_y$.

The study of the existence of limit cycles of type II_1 is the study of limit cycles existing for either two linear Hamiltonian saddles separated by a straight line, or formed by one linear center and one linear

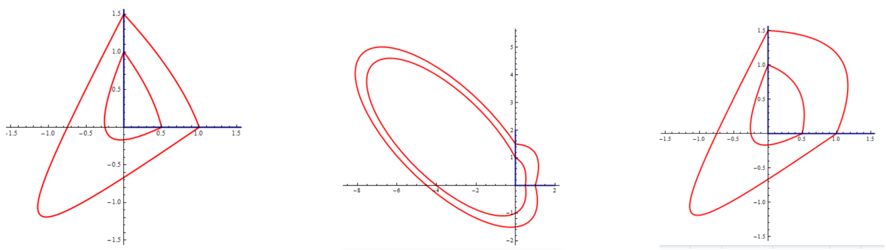
Hamiltonian saddle separated by a straight line. It was proved in [Llibre & Valls, 2021, 2022] that such piecewise differential systems have no limit cycles.

There exist two types of limit cycles of type II_3. The first type in which one of the three points is the origin of coordinates, and the second type in which one of the points is tangent to either $\mathcal{R}_x$, or $\mathcal{R}_y$. The limit cycles in the first case can be considered inside the class of limit cycles of types II_2 or II_4. On the other hand, the limit cycles in the second case can be considered inside the class of type II_1 which we have seen that do not exist.

In short, we restrict our analysis to study the maximum number of limit cycles of type II_2, or of type II_4 of the discontinuous piecewise differential systems (1) with $\alpha = 0$ formed by linear centers or linear Hamiltonian saddles. We note that when in the two pieces of the piecewise differential system we have linear centers these kind of limit cycles already have been studied in [Esteban *et al.*, 2021; Li & Liu, 2022]. Moreover, if the two pieces we have linear Hamiltonian systems without equilibria their limit cycles have been studied in [Zhao *et al.*, 2021]. So here we only consider discontinuous piecewise differential systems (1) with $\alpha = 0$ formed by either a linear center in $\mathcal{R}_1$ and a linear Hamiltonian saddle in $\mathcal{R}_2$, or a linear Hamiltonian saddle in $\mathcal{R}_1$ and a linear center $\mathcal{R}_2$, or two linear Hamiltonian saddles in $\mathcal{R}_1$ and $\mathcal{R}_2$.

The two main results of the present paper are the following.

Theorem 1. *Consider discontinuous piecewise differential systems separated by the non-regular line $\mathcal{R}$ and formed by either arbitrary linear centers or arbitrary linear Hamiltonian saddles. The maximum number of limit cycles of type II_2 of these discontinuous piecewise linear differential systems is two and there exist systems of this form with exactly two limit cycles of type II_2 (see Figures $1(a), (b)$ and (c) respectively).*



(a)The pair of limit cycles of type II_2 for the center (9) in $\mathcal{R}_1$ and the saddle (10) in $\mathcal{R}_2$.

(b)The pair of limit cycles of type II_2 for the saddle (12) in $\mathcal{R}_1$ and the center (13) in $\mathcal{R}_2$.

(c)The pair of limit cycles of type II_2 for the saddles (14) in $\mathcal{R}_1$ and (15) and in $\mathcal{R}_2$.

Fig. 1. All these limit cycles are travelled in counter-clockwise sense.

Theorem 2. *Consider discontinuous piecewise differential systems separated by the non-regular line $\mathcal{R}$ and formed by either arbitrary linear centers or arbitrary linear Hamiltonian saddles. The maximum number of limit cycles of type II_4 of these discontinuous piecewise linear differential systems is one and there exist systems of this form with exactly one limit cycle of type II_4 (see Figure $2(a), (b)$ and (c) respectively).*

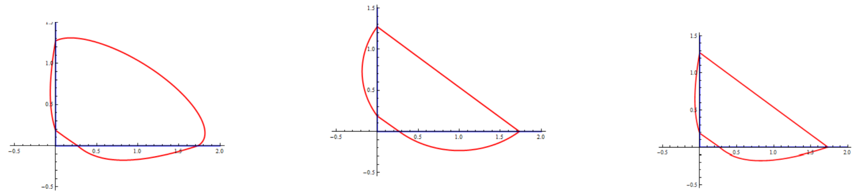
The proof of Theorem 1 is given in section 3 and the proof of Theorem 2 is given in section 4.

2. Preliminaries

Through the proofs of Theorem 1 we will use the following two results which provide a normal form for a linear differential Hamiltonian saddle (for a proof see [Llibre & Valls, 2021, 2022]) and for a linear center (for a proof see [Llibre & Teixeira, 2018]).

Proposition 1. *Any linear differential system having a Hamiltonian saddle can be written as*

$$\dot{x} = -bx - \delta y + d, \quad \dot{y} = \alpha x + by + c, \tag{2}$$



(a)The limit cycle of type II_4 for the center (18) in $\mathcal{R}_1$ and the saddle (19) in $\mathcal{R}_2$.

(b)The limit cycle of type II_4 for the saddle (21) in $\mathcal{R}_1$ and the center (22) in $\mathcal{R}_2$.

(c)The limit cycle of type II_4 for the saddles (23) in $\mathcal{R}_1$ and (24) in $\mathcal{R}_2$.

Fig. 2. All these limit cycle are travelled in counter-clockwise sense.

with $\alpha \in \{0, 1\}$*,* $b, \delta, c, d \in \mathbb{R}$*. Moreover, if* $\alpha = 1$ *then* $\delta = b^2 - \omega$ *with* $\omega > 0$ *and if* $\alpha = 0$ *then* $b = 1$*. A first integral of this system is*

$$H(x, y) = -\frac{\alpha}{2}x^2 - bxy - \frac{\delta}{2}y^2 - cx + dy. \tag{3}$$

Proposition 2. *Any linear differential system having a center can be written as*

$$\dot{x} = -\overline{b}x - \overline{\delta}y + \overline{d}, \quad \dot{y} = x + \overline{b}y + \overline{c}, \tag{4}$$

where $\overline{\delta} = \overline{b}^2 + \overline{\omega}$ *with* $\overline{\omega} > 0$*. A first integral of system* (4) *is*

$$F(x, y) = -\frac{1}{2}x^2 - \overline{b}xy - \frac{\overline{\delta}}{2}y^2 - \overline{c}x + \overline{d}y. \tag{5}$$

Note that any of the Hamiltonians (3) and (5) can be written as

$$G(x, y) = -\frac{A}{2}x^2 - Bxy - \frac{\Delta}{2}y^2 - Cx + Dy,$$

where $A = 1$ and $\Delta = B^2 + \omega$ with $\omega > 0$ if we have a linear center, and in the case that we have a linear Hamiltonian saddle then $A \in \{0, 1\}$, so that $A = 1$ then $\Delta = B^2 - \omega$ with $\omega > 0$ and if $A = 0$ then $B = 1$ and $\Delta \in \mathbb{R}$.

3. Proof of Theorem 1

Without loss of generality we can assume that in $\mathcal{R}_1$ we have either a linear center or a linear Hamiltonian saddle with first integral

$$G_1(x, y) = -\frac{A_1}{2}x^2 - B_1xy - \frac{\Delta_1}{2}y^2 - C_1x + D_1y, \tag{6}$$

and in the region R_2 we have either a linear center or a linear Hamiltonian saddle with first integral

$$G_2(x, y) = -\frac{A_2}{2}x^2 - B_2xy - \frac{\Delta_2}{2}y^2 - C_2x + D_2y. \tag{7}$$

If there exists a crossing limit cycle of type II_2, then it must intersect the non-regular separation curve $\mathcal{R}$ in two points of the form $(x, 0)$ and $(0, y)$, both different from the origin. Since the functions G_1 and G_2 are first integrals, these points must satisfy the equations

$$\begin{aligned} e_1 = G_1(x, 0) - G_1(0, y) = -2C_1x_1 - A_1x_1^2 - 2D_1y_1 + y_1^2\Delta_1 = 0, \\ e_2 = G_2(x, 0) - G_2(0, y) = -2C_2x_1 - A_2x_1^2 - 2D_2y_1 + y_1^2\Delta_2 = 0. \end{aligned} \tag{8}$$

It follows from Bézout theorem that system (8) has at most four real solutions, but since one solution is $(0, 0)$ which cannot produce a limit cycle, we conclude that system (8) has at most three real solutions that without loss of generality we can assume that they are $(x_i, 0), (0, y_i)$ with $i = 1, 2, 3$ satisfying

$$0 < x_1 < x_2 < x_3 \quad \text{and} \quad 0 < y_1 < y_2 < y_3$$

(otherwise the solutions would intersect which is not possible by the uniqueness of solutions of a differential system).

We consider four different cases.

If $A_1 = A_2 = \Delta_1 = \Delta_2 = 0$, then equations $e_1 = 0$ and $e_2 = 0$ are two straight lines that intersect at most in one point, and so there is at most one limit cycle.

If $A_1 = A_2 = 0$ and $\Delta_1^2 + \Delta_2^2 \neq 0$, then $e_1 = 0$ is either a straight line passing through the origin or a parabola symmetric with respect to some horizontal straight line and passing through the origin. Moreover, $E_2 = \Delta_2 e_1 - \Delta_1 e_2 = 0$ become

$$E_2 = 2(\Delta_1 C_2 - \Delta_2 C_1)x_1 + 2(D_2\Delta_1 - D_1\Delta_2)y_1 = 0$$

which is a straight line. Since a parabola and a straight line intersect at most at two points we have that $e_1 = E_2 = 0$ intersect at most in two points satisfying $0 < x_1 < x_2$ and $0 < y_1 < y_2$, and so at most two limit cycles.

If $\Delta_1 = \Delta_2 = 0$ and $A_1^2 + A_2^2 \neq 0$, then $e_1 = 0$ is either a straight line passing through the origin or a parabola symmetric with respect to some vertical straight line and passing through the origin. Moreover, $E_1 = A_2 e_1 - A_1 e_2 = 0$ become

$$E_1 = 2(A_1 C_2 - A_2 C_1)x_1 + 2(D_2 A_1 - D_1 A_2)y_1 = 0,$$

which is a straight line. Since a parabola and a straight line intersect at most at two points we have that $E_1 = e_2 = 0$ intersect at most in two points satisfying $0 < x_1 < x_2$ and $0 < y_1 < y_2$, and so at most two limit cycles.

Finally consider that $A_1^2 + A_2^2 \neq 0$ and $\Delta_1^2 + \Delta_2^2 \neq 0$. It follows from $e_1 = e_2 = 0$ that $E_1 = A_2 e_1 - A_1 e_2 = 0$ and $E_2 = \Delta_2 e_1 - \Delta_1 e_2 = 0$ become

$$\begin{aligned} E_1 &:= 2(A_1 C_2 - A_2 C_1)x_1 + 2(D_2 A_1 - D_1 A_2)y_1 + (A_2\Delta_1 - A_1\Delta_2)y_1^2 = 0, \\ E_2 &:= 2(\Delta_1 C_2 - \Delta_2 C_1)x_1 + 2(D_2\Delta_1 - D_1\Delta_2)y_1 + (A_2\Delta_1 - A_1\Delta_2)x_1^2 = 0, \end{aligned}$$

If $A_2\Delta_1 - A_1\Delta_2 = 0$ then equations $E_1 = E_2 = 0$ have at most two solutions and so there are at most two limit cycles.

Assume now that $A_2\Delta_1 - A_1\Delta_2 \neq 0$.

If $A_1 C_2 - A_2 C_1 = 0$ then $E_1 = 0$ reduces to either one horizontal straight line, or two horizontal parallel straight lines passing one of these two straight lines through the origin. The equation $E_2 = 0$ is either a parabola symmetric with respect to some vertical straight line, or one vertical straight line, or two vertical parallel straight lines passing one of these two straight lines through the origin. Since $E_1 = E_2 = 0$ pass through the origin, there are at most two intersection points satisfying $0 < x_1 < x_2$ and $0 < y_1 < y_2$ and so at most two limit cycles.

If $D_2\Delta_1 - D_1\Delta_2 = 0$ then $E_2 = 0$ reduces to either one vertical straight line, or two vertical parallel straight lines passing one of these two straight lines through the origin. The equation $E_1 = 0$ is either a parabola symmetric with respect to some horizontal straight line, or one horizontal straight line, or two horizontal parallel straight lines passing one of these two straight lines through the origin. Since $E_1 = E_2 = 0$ pass through the origin, there are at most two intersection points satisfying $0 < x_1 < x_2$ and $0 < y_1 < y_2$ and so at most two limit cycles.

Finally, assume that $A_1 C_2 - A_2 C_1 \neq 0$ and $D_2\Delta_1 - D_1\Delta_2 \neq 0$. In this case $E_1 = 0$ is a parabola symmetric with respect to some horizontal straight line and $E_2 = 0$ is a parabola symmetric with respect to some vertical line. Since both parabolas intersect at the origin, there are at most two intersection points satisfying $0 < x_1 < x_2$ and $0 < y_1 < y_2$, and so at most two limit cycles.

In summary, the maximum number of limit cycles of type II_2 for our family of discontinuous piecewise linear differential systems is two. Now we give three examples of discontinuous piecewise linear differential systems in our class having exactly two limit cycles of type II_2, the first having a center in $\mathcal{R}_1$ and a saddle in $\mathcal{R}_2$, the second having a saddle in $\mathcal{R}_1$ and a center in $\mathcal{R}_2$, and the third having two saddles in $\mathcal{R}_1$ and $\mathcal{R}_2$. This will conclude the proof of Theorem 1.

Case 1*: In the region $\mathcal{R}_1$ we have a linear center and in the region $\mathcal{R}_2$ we have a linear Hamiltonian saddle.* In region $\mathcal{R}_1$ we consider the linear differential center

$$\dot{x} = -2x - 4y, \qquad \dot{y} = \frac{7}{2} + 2x + 2y, \tag{9}$$

with the first integral

$$H_1(x, y) = y^2 + \frac{7}{2}x + (x + y)^2;$$

and in region $\mathcal{R}_2$ we consider the linear differential saddle

$$\dot{x} = \frac{31x}{32} - \frac{3y}{4} + \frac{5}{16}, \qquad \dot{y} = x - \frac{31y}{32} - \frac{1}{8}, \tag{10}$$

with the first integral

$$H_2(x, y) = -\frac{x^2}{2} + \frac{31xy}{32} + \frac{x}{8} - \frac{3y^2}{8} + \frac{5y}{16}.$$

The two solutions of equations (8) for the discontinuous piecewise differential system (9)–(10) are

$$(x_1, y_1) = \left(\frac{1}{2}, 1\right), \qquad (x_2, y_2) = \left(1, \frac{3}{2}\right), \tag{11}$$

and the corresponding limit cycles are shown in Figure 1(a).

Case 2*: In the region $\mathcal{R}_1$ we have a linear Hamiltonian saddle and in the region $\mathcal{R}_2$ we have a linear center.* In region $\mathcal{R}_1$ we consider the linear Hamiltonian saddle

$$\dot{x} = 2x - 3y - \frac{1}{4}, \qquad \dot{y} = x - 2y + \frac{13}{4}, \tag{12}$$

with the first integral

$$H_1(x, y) = -\frac{x^2}{2} + 2xy - \frac{13x}{4} - \frac{3y^2}{2} - \frac{y}{4};$$

and in region $\mathcal{R}_2$ we consider the linear differential center

$$\dot{x} = -2x - 4y, \qquad \dot{y} = 2x + 2y + \frac{7}{2}, \tag{13}$$

with the first integral

$$H_2(x, y) = y^2 + \frac{7}{2}x + (x + y)^2.$$

The two solutions of equations (8) for the discontinuous piecewise differential system (12)–(13) are given in (11), and the corresponding limit cycles are shown in Figure 1(b).

Case 3*: In both regions $\mathcal{R}_1$ and $\mathcal{R}_2$ we have a linear Hamiltonian saddle.* In region $\mathcal{R}_1$ we consider the linear Hamiltonian saddle

$$\dot{x} = 2x - 3y - \frac{1}{4}, \qquad \dot{y} = x - 2y + \frac{13}{4}, \tag{14}$$

with the first integral

$$H_1(x, y) = -\frac{x^2}{2} + 2xy - \frac{13x}{4} - \frac{3y^2}{2} - \frac{y}{4};$$

and in region $\mathcal{R}_2$ we consider the linear Hamiltonian saddle

$$\dot{x} = \frac{31x}{32} - \frac{3y}{4} + \frac{5}{16}, \qquad \dot{y} = x - \frac{31y}{32} - \frac{1}{8}, \tag{15}$$

with the first integral

$$H_2(x, y) = -\frac{x^2}{2} + \frac{31xy}{32} + \frac{x}{8} - \frac{3y^2}{8} + \frac{5y}{16}.$$

The two solutions of equations (8) for the discontinuous piecewise differential system (14)–(15) are given in (11), and the corresponding limit cycles are shown in Figure 1(c).

4. Proof of Theorem 2

Without loss of generality we can assume that in $\mathcal{R}_1$ we have either a linear center or a linear Hamiltonian saddle with first integral (6) and in the region R_2 we have either a linear center or a linear Hamiltonian saddle with first integral (7). Note that if there exists a periodic solution candidate to be a limit cycle of type II_4, then this periodic solution has four intersection points on the discontinuity line $\mathcal{R}$ of the form $(x_1, 0)$, $(x_2, 0)$, $(0, y_1)$ and $(0, y_2)$, satisfying $0 < x_1 < x_2$, $0 < y_1 < y_2$ and so the following equations must be satisfied

$$\begin{aligned} G_1(x_1,0) - G_1(0,y_1) &= -2C_1x_1 - 2D_1y_1 - A_1x_1^2 + \Delta_1 y_1^2 = 0,\\ G_1(x_2,0) - G_1(0,y_2) &= -2C_1x_2 - 2D_1y_2 - A_1x_2^2 + \Delta_1 y_2^2 = 0,\\ G_2(x_1,0) - G_2(x_2,0) &= (x_1 - x_2)(2C_2 + A_2x_1 + A_2x_2) = 0,\\ G_2(0,y_1) - G_2(0,y_2) &= (y_1 - y_2)(2D_2 - \Delta_2 y_1 - \Delta_2 y_2) = 0. \end{aligned} \tag{16}$$

We consider three different cases.

Case 1*:* $A_2 = 0$. In this case from the third equation of (16) we get that $C_2 = 0$. If additionally $\Delta_2 = 0$ then it follows from the fourth equation of (16) that $D_2 = 0$ but, then the first and second equations in (16) yield a continuum of solutions. If $\Delta_2 \neq 0$ then from the four equation of (16) we get $y_2 = -y_1 + 2D_2/\Delta_2$, and substituting y_2 into the first and second equations in (16) we get a continuum of solutions. Therefore there are no limit cycles in this case.

Case 2*:* $\Delta_2 = 0$. Then from the third and four equations of (16) we get $D_2 = 0$ and $x_2 = -x_1 - 2C_2/A_2$. Substituting these two conditions into the first and second equations of (16) we get a continuum of solutions. Hence there are no limit cycles in this case.

Case 3*:* $A_2\Delta_2 \neq 0$ In this case from the third and fourth equations of (16) we get that

$$x_2 = -\frac{2C_2}{A_2} - x_1, \qquad y_2 = \frac{2D_2}{\Delta_2} - y_1. \tag{17}$$

Using these last expressions we can write the first and second equations of (16) in terms of x_1 and y_1, and we get

$$\begin{aligned} E_1(x_1,y_1) &= -2C_1x_1 - 2D_1y_1 - A_1x_1^2 + \Delta_1 y_1^2 = 0,\\ E_2(x_1,y_1) &= 4(A_2^2D_2^2\Delta_1 - A_2^2D_1D_2\Delta_2 + A_2C_1C_2\Delta_2^2 - A_1C_2^2\Delta_2^2)\\ &\quad + 2A_2\Delta_2^2(A_2C_1 - 2A_1C_2)x_1 - 2A_2^2\Delta_2(2D_2\Delta_1 - D_1\Delta_2)y_1\\ &\quad - A_1A_2^2\Delta_2^2x_1^2 + A_2^2\Delta_1\Delta_2^2y_1^2 = 0. \end{aligned}$$

Setting $E_3(x,y) = \Delta_2^2A_2^2E_1(x,y) - E_2(x,y) = 0$ we get

$$\begin{aligned} E_3(x,y) = &-4(A_2^2D_2^2\Delta_1 - A_2^2D_1D_2\Delta_2 + A_2C_1C_2\Delta_2^2 - A_1C_2^2\Delta_2^2)\\ &- 4A_2(A_2C_1 - A_1C_2)\Delta_2^2x_1 + 4A_2^2\Delta_2(D_2\Delta_1 - D_1\Delta_2)y_1 = 0. \end{aligned}$$

If $D_2\Delta_1 - D_1\Delta_2 = 0$ it follows from $E_3(x,y) = 0$ that $x_1 = -C_2/A_2$, but then from (17) we get $x_2 = -C_2/A_2$ which is not possible. Hence, $D_2\Delta_1 - D_1\Delta_2 \neq 0$ and solving $E_3(x,y) = 0$ in the variable y_1 we obtain

$$\begin{aligned} y_1 = \frac{1}{A_2^2\Delta_2(D_1\Delta_2 - D_2\Delta_1)}\big(&-A_2^2D_2^2\Delta_1 + A_2^2D_1D_2\Delta_2 - A_2C_1C_2\Delta_2^2\\ &+ A_1C_2^2\Delta_2^2 - A_2\Delta_2^2(A_2C_1 - A_1C_2)x_1\big). \end{aligned}$$

Now we introduce y_1 into $E_1(x,y) = 0$ and we get

$$R(x_1) = C_0 + C_1x_1 + C_2x_1^2 = 0,$$

where

$$\begin{aligned}C_0 =&(A_2^2D_2^2\Delta_1 - A_2^2D_1D_2\Delta_2 + A_2C_1C_2\Delta_2^2 - A_1C_2^2\Delta_2^2)(A_2^2D_2^2\Delta_1^2\\&- 3A_2^2D_1D_2\Delta_1\Delta_2 + 2A_2^2D_1^2\Delta_2^2 + A_2C_1C_2\Delta_1\Delta_2^2 - A_1C_2^2\Delta_1\Delta_2^2),\\C_1 =&2A_2C_2\Delta_2^2(-A_1A_2^2D_2^2\Delta_1^2 + 2A_1A_2^2D_1D_2\Delta_1\Delta_2 - A_1A_2^2D_1^2\Delta_2^2\\&+ A_2^2C_1^2\Delta_1\Delta_2^2 - 2A_1A_2C_1C_2\Delta_1\Delta_2^2 + A_1^2C_2^2\Delta_1\Delta_2^2),\\C_2 =&A_2^2\Delta_2^2(-A_1A_2^2D_2^2\Delta_1^2 + 2A_1A_2^2D_1D_2\Delta_1\Delta_2 - A_1A_2^2D_1^2\Delta_2^2 + A_2^2C_1^2\Delta_1\Delta_2^2\\&- 2A_1A_2C_1C_2\Delta_1\Delta_2^2 + A_1^2C_2^2\Delta_1\Delta_2^2)x_1^2.\end{aligned}$$

The polynomial $R(x_1)$ is quadratic in the variable x_1, whose roots $x_{1,\pm}$, are

$$x_{1,\pm} = -\frac{C_2}{A_2} \pm \frac{A_2^2\sqrt{\Delta}}{C_2},$$

where

$$\begin{aligned}\Delta =&\Delta_2^2(D_2\Delta_1 - D_1\Delta_2)^2(A_1A_2^2D_2^2\Delta_1^2 - 2A_1A_2^2D_1D_2\Delta_1\Delta_2 + (A_1A_2^2D_1^2\\&- (A_2C_1 - A_1C_2)^2\Delta_1)\Delta_2^2)(2A_2C_1C_2\Delta_2^2 - A_1C_2^2\Delta_2^2\\&+ A_2^2D_2(D_2\Delta_1 - 2D_1\Delta_2)).\end{aligned}$$

Note that from (17) we get

$$x_{2,\pm} = -\frac{2C_2}{A_2} - x_{1,\pm} = -\frac{C_2}{A_2} \mp \frac{A_2^2\sqrt{\Delta}}{C_2} = x_{1,\mp}.$$

Since $x_2 > x_1$ there is at most one solution of the two possible solutions $x_{1,\pm}$. Therefore the maximum number of limit cycles of type II_4 that our discontinuous piecewise linear differential system can have is one.

Now we give examples of discontinuous piecewise differential systems in our class having exactly one limit cycle of type II_4. As in the proof of Theorem 1 we consider the following three different cases, the first having a center in $\mathcal{R}_1$ and a saddle in $\mathcal{R}_2$, the second having a saddle in $\mathcal{R}_1$ and a center in $\mathcal{R}_2$, and the third having two saddles in $\mathcal{R}_1$ and $\mathcal{R}_2$. This will conclude the proof of Theorem 2.

Case 1: In the region $\mathcal{R}_1$ we have a linear center and in the region $\mathcal{R}_2$ we have a linear Hamiltonian saddle. In region $\mathcal{R}_1$ we consider the linear differential center

$$\dot{x} = \frac{17}{4} - 2x - 4y, \qquad \dot{y} = -\frac{191}{64} + 2x + 2y, \tag{18}$$

with the first integral

$$H_1(x, y) = y^2 - 2\frac{191}{128}x + \frac{17}{8}y + (x + y)^2;$$

and in region $\mathcal{R}_2$ we consider the linear differential saddle

$$\dot{x} = \frac{561}{1024} + x - \frac{3}{4}y, \qquad \dot{y} = -1 + x - y \tag{19}$$

with the first integral

$$H_2(x, y) = x - \frac{1}{2}x^2 + \frac{561}{1024}y + xy - \frac{3}{8}y^2.$$

In this case the two solutions of equations (16) are

$$\begin{aligned}(x_1, y_1) &= \left(\frac{182528 - 85\sqrt{2474110}}{182528}, \frac{133331 - 63\sqrt{2474110}}{182528}\right),\\(x_2, y_2) &= \left(\frac{182528 + 85\sqrt{2474110}}{182528}, \frac{133331 + 63\sqrt{2474110}}{182528}\right),\end{aligned} \tag{20}$$

and the corresponding limit cycle of the discontinuous piecewise differential system (18)–(19) is shown in Figure 2(a).

Case 2: In the region $\mathcal{R}_1$ we have a linear Hamiltonian saddle and in the region $\mathcal{R}_2$ we have a linear center. In region $\mathcal{R}_1$ we consider the linear Hamiltonian saddle

$$\dot{x} = \frac{52157003}{42917504} - \frac{89}{64}x - \frac{644918}{335293}y, \qquad \dot{y} = -\frac{55}{64} + x + \frac{89}{64}y, \tag{21}$$

with the first integral

$$H_1(x,y) = \frac{55}{64}x + \frac{52157003}{42917504}y - \frac{1}{2}x^2 - \frac{89}{64}xy - \frac{322459}{335293}y^2,$$

and in region $\mathcal{R}_2$ we consider the linear differential center

$$\dot{x} = 8 - \frac{2048}{187}y, \qquad \dot{y} = -8 + 8x, \tag{22}$$

with the first integral

$$H_2(x,y) = 4x^2 - 8(x+y) + \frac{1024}{187}y^2.$$

In this case the two solutions of equations (16) are (x_1, y_1) and (x_2, y_2) given in (20) and the corresponding limit cycle of the discontinuous piecewise differential system (21)–(22) is shown in Figure 2(b).

Case 3: In both regions $\mathcal{R}_1$ and $\mathcal{R}_2$ we have a linear Hamiltonian saddle. In region $\mathcal{R}_1$ we consider the linear Hamiltonian saddle

$$\dot{x} = \frac{52157003}{42917504} - \frac{89}{64}x - \frac{644918}{335293}y, \qquad \dot{y} = -\frac{55}{64} + x + \frac{89}{64}y, \tag{23}$$

with the first integral

$$H_1(x,y) = \frac{55}{64}x + \frac{52157003}{42917504}y - \frac{1}{2}x^2 - \frac{89}{64}xy - \frac{322459}{335293}y^2;$$

and in region $\mathcal{R}_2$ we consider the linear Hamiltonian saddle

$$\dot{x} = \frac{561}{1024} + x - \frac{3}{4}y, \qquad \dot{y} = -1 + x - y, \tag{24}$$

with the first integral

$$H_2(x,y) = x + \frac{561}{1024}y - \frac{1}{2}x^2 + xy - \frac{3}{8}y^2.$$

In this case the two solutions of equations (16) are (x_1, y_1) and (x_2, y_2) given in (20) and and the corresponding limit cycle of the discontinuous piecewise differential system (23)–(24) is shown in Figure 2(c).

Acknowledgements

The first author is partially supported by the Agencia Estatal de Investigación grant PID2019-104658GB-I00, the H2020 European Research Council grant MSCA-RISE-2017-777911, AGAUR (Generalitat de Catalunya) grant 2022-SGR 00113, and by the Acadèmia de Ciències i Arts de Barcelona. The second author is partially supported by FCT/Portugal through CAMGSD, IST-ID, projects UIDB/04459/2020 and UIDP/04459/2020.

References

B.P. Belousov[1959] *A periodic reaction and its mechanism*, Collection of Short Papers on Radiation Medicine for 1958, Moscow: Med. Publ. (in Russian).

D.C. Braga and L.F. Mello[2013] *Limit cycles in a family of discontinuous piecewise linear differential systems with two zones in the plane*, Nonlinear Dyn. **73**, 1283–1288.

C. Buzzi, C. Pessoa and J. Torregrosa[2013] *Piecewise linear perturbations of a linear center*, Discrete Contin. Dyn. Syst., **33**, 3915–3936.

M. di Bernardo, C.J. Budd, A.R. Champneys and P. Kowalczyk[2008] *Piecewise-Smooth Dynamical Systems: Theory and Applications*, Appl. Math. Sci., **163**, Springer-Verlag, London.

M. Esteban, J. Llibre and C. Valls[2021] *The extended* 16*-th problem for discontinuous piecewise linear centers separated by a non-regular line*, Internat. J. Bifurcation and Chaos, **31**(15), 2150225, 15 pp.

A.F. Filippov[1988] *Differential equations with discontinuous right-hand sides*, translated from Russian. Mathematics and its Applications (Soviet Series) vol. **18**, Kluwer Academic Publishers Group, Dordrecht.

E. Freire, E. Ponce and F. Torres[2014] *A general mechanism to generate three limit cycles in planar Filippov systems with two zones*, Nonlinear Dyn. **78**, 251–263.

F. Giannakopoulos and K. Pliete[2014] *Planar systems of piecewise linear differential equations with a line of discontinuity*, Nonlinearity **14**, 1611–1632.

M. Han and W. Zhang[2010] *On Hopf bifurcation in non–smooth planar systems*, J. of Differential Equations **248**, 2399–2416.

S.M. Huan and X.S. Yang[2012] *On the number of limit cycles in general planar piecewise systems*, Discrete Cont. Dyn. Syst., Series A **32**, 2147–2164.

S.M. Huan and X.S. Yang[2019] *Limit cycles in a family of planar piecewise linear differential systems with a nonregular separation line*, Internat. J. Bifurcation and Chaos **29**, 1950109, 22 pp.

L. Li[2014] *Three crossing limit cycles in planar piecewise linear systems with saddle-focus type*, Electron. J. Qual. Theory Differ. Equ. **70**, pp. 14.

Z. Li, and X. Liu[2022] *Limit cycles in discontinuous piecewise linear planar Hamiltonian systems without equilibrium points*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **32**(10), 2250153, 20 pp.

J. Llibre and E. Ponce[2012] *Three nested limit cycles in discontinuous piecewise linear differential systems with two zones*, Dyn. Cont. Disc. Impul. Syst., Series B **19**, 325–335.

J. Llibre, D.D. Novaes and M.A. Teixeira[2015] *Maximum number of limit cycles for certain piecewise linear dynamical systems*, Nonlinear Dyn. **82**, 1159–1175.

J. Llibre and M.A. Teixeira[2018] *Piecewise linear differential systems with only centers can create limit cycles*?, Nonlinear Dyn. **91**, 249–255.

J. Llibre and C. Valls[2021] *Limit cycles of piecewise differential systems with linear Hamiltonian saddles*, Symmetry **13**, 1128, 10 pp.

J. Llibre and C. Valls[2022] *Limit cycles of piecewise differential systems with linear Hamiltonian saddles and linear centers*, *Dynamical Systems* **37**(2), 262–279.

O. Makarenkov and J.S.W. Lamb[2012] *Dynamics and bifurcations of nonsmooth systems: a survey*, Physica D **241**, 1826–1844.

B. van der Pol[1920] *A theory of the amplitude of free and forced triode vibrations*, Radio Review (later Wireless World) **1**, 701–710.

B. van der Pol[1926] *On relaxation-oscillations*, The London, Edinburgh and Dublin Phil. Mag. & J. of Sci. **2**, 978–992.

D.J.W. Simpson[2010] *Bifurcations in Piecewise-Smooth Continuous Systems*, World Sci. Ser. Nonlinear Sci. Ser. A, vol. 69, World Scientific, Singapore.

A.M. Zhabotinsky[1964] *Periodical oxidation of malonic acid in solution (a study of the Belousov reaction kinetics)*, Biofizika **9**, 306–311.

Q. Zhao, C. Wang and J. Yu[2021] *Limit cycles in discontinuous planar piecewise linear systems separated by a nonregular line of center-center type*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **31**(9), 2150136, 17 pp.