QUALITATIVE STUDY OF THE SELKOV MODEL

JAUME LLIBRE[1] AND CHARA PANTAZI[2]

Abstract. The Selkov oscillator was formulated in 1968 and now it is a classical model for studying the glycolysis. It is a differential system of two equations depending on two parameters in dimensionless variables. When the two equations are polynomials we prove that the Selkov system is not Liouvillian integrable. Additionally, we prove that the polynomial Selkov system for any integer $n \geq 1$ has nine distinct phase portraits in the Poincaré disk.

1. Introduction and statement of the main results

The living organisms obtain energy from sugar using a process called glycolysis. Experimental observations detected that when the input rate of sugar is constant then the subproducts of the glycolysis oscillate on time. Based on these observations Higgins [13] in 1964 provided a mathematical model in order to understand better this phenomenon. Higgins' model was improved in 1968 by Selkov [20]. Thus the Selkov model is given by the differential system of two equations

$$\begin{aligned} \dot{x} &= 1 - xy^{\gamma}, \\ \dot{y} &= ay(-1 + xy^{\gamma-1}), \end{aligned}$$

depending on two parameters $a > 0$ and γ in dimensionless variables. In order to avoid technical complications with differentiability we take $\gamma = n$ a positive integer. The variables x and y are the dimensionless concentrations of ATP (adenosine triphosphate) and ADP (adenosine diphosphate), respectively, while the dot represents the derivative with respect to a dimensionless time variable.

From now on we will work with the system

$$\begin{aligned} \dot{x} &= 1 - xy^{n}, \\ \dot{y} &= ay(-1 + xy^{n-1}), \end{aligned} \tag{1}$$

where n is a positive integer and a is a positive real number.

In his seminal paper Selkov proved that his model admits a Hopf bifurcation, showing the existence of a periodic motion which allowed to explain the oscillations observed experimentally. In 2010 d'Onofrio [10] studied the stability and uniqueness of these periodic orbits.

Since some of the solutions of the Selkov model are unbounded in order to understand them it is necessary to study the neighborhood of the infinity. In 2018 Brechmann and Rendall [3] used the technique of Poincaré compactification for studying those unbounded solutions. Additionally these authors also showed that if the unique equilibrium point of the Selkov model is stable, then any bounded solution converges to it in forward time. If this equilibrium is unstable and there exist a periodic orbit then this periodic orbit is unique and all bounded solutions different from the equilibrium converge to the periodic solution in forward time. If the equilibrium is unstable and does not exist a periodic orbit then all solutions distinct from the equilibrium are unbounded.

Selkov in [20] claimed that his system admits solutions which oscillate with an amplitude which grows without limit in forward time. These solutions are called solutions with unbounded oscillations. In 2020 Brechmann and Rendall [4] proved the existence of solutions with unbounded oscillations. In [5] the authors prove the existence and uniqueness of a limit cycle trasforming system (1) into Liénard system.

We associate to the differential system (1) the vector field

$$X = (1 - xy^n)\frac{\partial}{\partial x} + ay(-1 + xy^{n-1})\frac{\partial}{\partial y}. \tag{2}$$

The two main objectives of this paper on the Selkov model (1) are essentially mathematical, but of course they have biological implications. The first one is to decide if system (1) is Liouvillian integrable for some values of its parameters, the answer is negative, see Theorem 1. The second objective is to classify, for all $n \geq 1$, the topological phase portraits of system (1) in the Poincaré disc, see Theorem 2. This second objective was solved in the particular cases $n = 2$ by Artés et al [2] and Chen and Tang [6], and for $n = 3, 4, 5, 6$ see [15].

Theorem 1. *System* (1) *for* $a > 0$ *is not Liouvillian integrable.*

Theorem 1 is proved in Section 2.

Theorem 1 says that the differential system (1) has neither a first integral nor an integrating factor given by a Darboux function, see for more details the subsection 4.1 of the Appendix.

In order to understand the behaviour of system (1) we need to draw the phase portraits in the Poincaré disc, see subsection 4.2 of the Appendix. Roughly speaking, the *Poincaré disc* is the closed unit disc centered at the origin of $\mathbb{R}^2$, its interior is identified with the whole plane $\mathbb{R}^2$ and the circle of its boundary is identified with the infinity of $\mathbb{R}^2$. In the plane we can go to infinity in as many directions as points has the circle. There is a unique analytic way to extend a polynomial differential system defined in $\mathbb{R}^2$ to the Poincaré disc. Working with this extended system defined in the Poincaré disc, we can study how the orbits of the polynomial differential system goes or come from infinity. For more details on the so called *Poincaré compactification* see for instance chapter 5 of [11].

We recall that our second objective is to present the topological classification of all phase portraits of system (1) for all $n \geq 1$. Thus, we follow the works of Markus, Neumann and Peixoto [16, 17, 18] and the notion of separatrix configuration (see also subsection 4.3 of the Appendix). In what follows we denote by S the number of separatrices and by R the number of the canonical regions. If there does not exist a homeomorphism to bring the separatrix configuration of one phase portrait to the separatrix configuration of the other, we say that the two phase portraits are *not topological equivalent.* Next theorem classify the topological phase portraits of system (1).

Theorem 2. *For the Selkov system* (1) *with* $a > 0$ *and* $n \geq 1$*, there are exactly nine non–topological equivalent phase portraits in the Poincaré disc:*

- (a) *For* $n = 1$ *see Figure* 1(*e*) *with* $S = 15$ *and* $R = 2$.
- (b) *For* $n \geq 2$ *and* n *odd (resp. even)*
  - (i) *Figure* 1(*a*) *(resp.* 2(*a*)*) with* $a \in (0,\ 1/(n-1))$ *and* $S = 17$ *and* $R = 4$.
  - (ii) *Figure* 1(*b*)*(resp.* 2(*b*)*) with* $a \in (1/(n-1),\ a^*)$ *and* a^* *is a unique constant in the interval* $\left(\frac{1}{n-1},\ \frac{2^n-1}{2^n-2}\right)$. *In this case,* $S = 18$ *and* $R = 5$.
  - (iii) *Figure* 1(*c*) *(resp.* 2(*c*)*) with* $a = a^*$ *and* $S = 16$ *and* $R = 4$.
  - (iv) *Figure* 1(*d*) *(resp.* 2(*d*)*) with* $a > a^*$ *and* $S = 17$ *and* $R = 4$.

Theorem 2 is proved in Section 3.

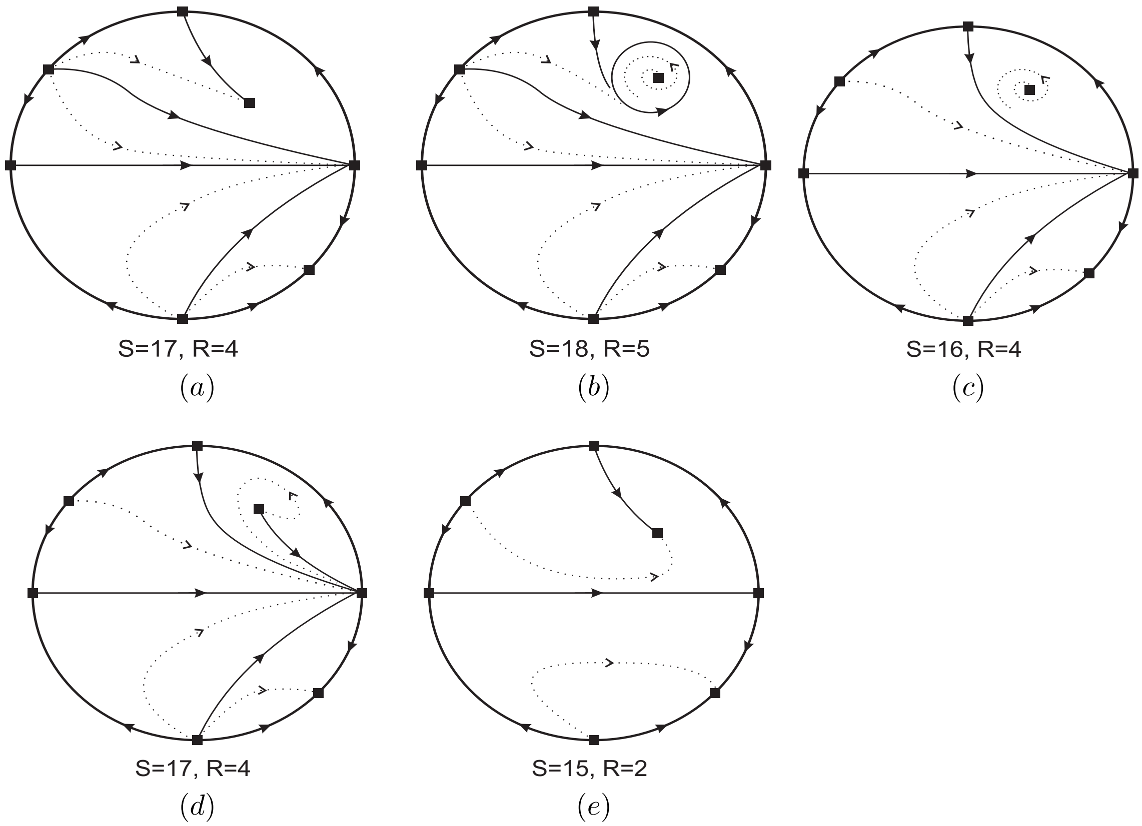

FIGURE 1. The phase portraits of systems (1) for n odd, $n \geq 1$.

The applications to biology from the phase portraits given in Figures 1 and 2 are restricted to the positive quadrant inside the Poincaré disk because x and y are concentrations. The more interesting results corresponds to (a),(b) and (e) of Figure 1 and of (a) and (b) of Figure 2 where are attractors formed by an equilibrium point or a limit cycle. Hence, in the case of an attractor equilibrium point the concentrations tends to a fixed value, while in the case of a limit cycle the concentrations change periodically.

2. Proof of Theorem 1

In the proof of Theorem 1 we shall use Lemmas 3 and 4.

Lemma 3. *The unique irreducible invariant algebraic curve of system* (1) *is* $y = 0$.

Proof. From (1) we have that $\dot{y}|_{y=0} = 0$, therefore the straight line $y = 0$ is invariant under the flow of system (1). Now we must prove that there is no other irreducible invariant algebraic curve.

Consider an irreducible polynomial $F(x, y)$ distinct of the polynomial y. We write $F(x, y) = F_0(x) + F_1(x)y + F_2(x)y^2 + \cdots + F_k(x)y^k$ where F_i is a polynomial in the variable x for $i = 0, \cdots, k$ with $F_k(x) \neq 0$. Note that $F_0(x) \neq 0$, otherwise F is not irreducible because y would be a factor of F. We assume that $F = 0$ is an invariant algebraic curve of system (1) with cofactor $K = K_0(x) + K_1(x)y + \cdots + K_{n-1}(x)y^{n-1} + k_n y^n$ with K_i polynomials in the variable x for $i = 0, \cdots, n-1$. Since system (1) is of degree $n+1$ we have that the degree of the cofactor K is at most n and consequently k_n must be a constant.

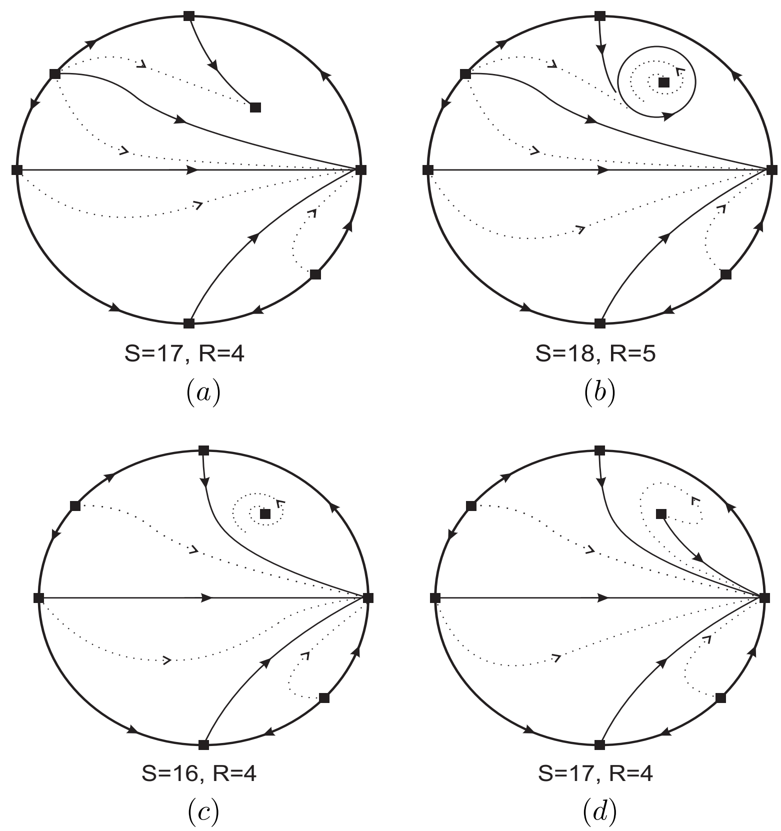

FIGURE 2. The phase portraits of systems (1) for n even, $n \geq 2$.

Since $F = 0$ is an invariant algebraic curve of system (1) we have that

$$(1 - xy^n)F_0'(x) = (K_0(x) + K_1(x)y + \cdots + K_{n-1}(x)y^{n-1} + k_n y^n)F_0(x), \tag{3}$$

if $k = 0$; and

$$\begin{aligned}(1 - xy^n)(F_0'(x) + F_1'(x)y + \cdots + F_k'(x)y^k) - a(y - xy^n)(F_1(x) + 2F_2(x)y + \cdots + kF_k(x)y^{k-1}) \\ = (K_0(x) + K_1(x)y + \cdots + K_{n-1}(x)y^{n-1} + k_n y^n)(F_0(x) + F_1(x)y + \cdots + F_k(x)y^k).\end{aligned} \tag{4}$$

if $k \geq 1$.

We consider the coefficients of y^0 in (3) and (4) and we obtain $F_0'(x) = K_0(x)F_0(x)$, and so $K_0(x) = 0$ and $F_0(x) = f_0 \in \mathbb{R}$ with $f_0 \neq 0$. Therefore, $F(x, y) = f_0 + yH(x, y)$ and since $K_0(x) = 0$ the cofactor K is divisible by y.

In summary if $k = 0$ then $F(x, y) = F_0(x) = f_0 \neq 0$, in contradiction with the fact that $F(x, y) = 0$ is an invariant algebraic curve. Hence $k \geq 1$.

We separate the rest of the proof in two cases.

Case 1: $n = 1$. Then from (4) we have

$$\begin{aligned}(1 - xy)(F_1'(x)y + \cdots + F_k'(x)y^k) - a(y - xy)(F_1(x) + 2F_2(x)y + \cdots + kF_k(x)y^{k-1}) \\ = k_1 y(f_0 + F_1(x)y + \cdots + F_k(x)y^k).\end{aligned} \tag{5}$$

The coefficient of y in (5) is

$$F_1'(x) - a(1 - x)F_1(x) = k_1 f_0.$$

The solution of this linear differential equation is

$$F_1(x) = e^{ax-\frac{ax^2}{2}}\left(C_1 + \sqrt{\frac{\pi}{2a}}k_1 f_0 e^{-\frac{a}{2}}\operatorname{erfi}\left(\frac{\sqrt{a}(x-1)}{\sqrt{2}}\right)\right),$$

where C_1 is a constant and erfi is the imaginary error function, see [22]. Since $F_1(x)$ must be a polynomial, we have that the $C_1 = k_1 = 0$ and $F_1(x) = 0$. Then (5) becomes

$$(1-xy)(F_2'(x)y^2 + \cdots + F_k'(x)y^k) - a(y-xy)(2F_2(x)y + \cdots + kF_k(x)y^{k-1}) = 0. \tag{6}$$

Now the coefficient of y^2 in (6) is

$$F_2'(x) - 2a(1-x)F_2(x) = 0.$$

Therefore $F_2(x) = C_2 e^{ax(2-x)}$ where C_2 is a constant. Since $F_2(x)$ must be a polynomial, we have that $C_2 = 0$ and $F_2(x) = 0$. Repeating this process we obtain that $F_j(x) = 0$ for $j = 3, \ldots, k$, a contradiction with the assumption that $F_k(x) \neq 0$. So the lemma is proved if $n = 1$.

Case 2: $n > 1$. Now we write the irreducicle invariant algebraic curve $F(x,y) = 0$ of system (1) as $F(x,y) = G_0(y) + G_1(y)x + G_2(y)x^2 + \cdots + G_\ell(y)x^\ell$, where G_i is a polynomial in the variable y for $i = 0, \ldots, \ell$ with $G_\ell(y) \neq 0$. Since F is irreducible $G_0(y) \neq 0$. And we write its cofactor as $K(x,y) = k_0(y) + k_1(y)x + \cdots + k_{n-1}(y)x^{n-1} + k_n x^n$ with k_i a polynomial in the variable y for $i = 0, \cdots, n$. Since the degree of the cofactor K is at most n we have that k_n is a constant. But since the cofactor K is divisible by y it follows that $k_n = 0$ and y divides $k_i(y)$ for $i = 0, 1, \ldots, k-1$. Moreover, since $F(x,y) = f_0 + yH(x,y)$ we have that y divides $G_j(y)$ for $j = 1, \ldots, \ell$ and $G_0(y) = f_0 + g(y)$ with $f_0 \neq 0$.

Since $F = 0$ is an invariant algebraic curve of system (1) we have that

$$-a(y - xy^n)G_0'(y) = (k_0(y) + k_1(y)x + \cdots + k_{n-1}(y)x^{n-1})G_0(y), \tag{7}$$

if $\ell = 0$; and

(8)

$$\begin{aligned}(1-xy^n)(G_1(y) + 2G_2(y)x + \cdots + \ell G_\ell(y)x^{\ell-1}) - a(y-xy^n)(G_0'(y) + G_1'(y)x + \cdots + G_\ell'(y)x^\ell) \\ = (k_0(y) + k_1(y)x + \cdots + k_{n-1}(y)x^{n-1})(G_0(y) + G_1(y)x + \cdots + G_\ell(y)x^\ell),\end{aligned}$$

if $\ell \geq 1$.

Assume $\ell = 0$, then from (7) we get

$$-a(y - xy^n)G_0'(y) = (k_0(y) + k_1(y)x)G_0(y), \tag{9}$$

or equivalently

$$-ayG_0'(y) = k_0(y)G_0(y), \qquad ay^nG_0'(y) = k_1(y)G_0(y). \tag{10}$$

Dividing the second of these equations by the first one we have that $k_1(y) = -k_0(y)y^{n-1}$. Solving (10) we obtain

$$G_0(y) = C_0 e^{-\int \frac{k_0(y)}{ay}\,dy},$$

where C_0 is a constant. Since $G_0(y) \neq 0$ and it must be a polynomial we have that $k_0(y) = k_0$ must be a constant. Then $G_0(y) = C_0 y^{-k_0/a}$, and since $G_0(y) = f_0 + g(y)$ with $f_0 \neq 0$ we have that $k_0 = 0$ and so $G_0 = C_0 = f_0$ and $g(y) = 0$. But then $F(x,y) = G_0 = f_0$, a contradiction, the invariant curve cannot be a constant number.

From now on we assume that $\ell \geq 1$.

The coefficient of $x^{n+\ell-1}$ in the polynomial (8) must satisfy

$$k_{n-1}(y)G_\ell(y) = 0 \quad \text{if } n+\ell-1 > 1+\ell, \tag{11}$$

or

$$k_{n-1}(y)G_\ell(y) = ay^n G'_\ell(y) \quad \text{if } n+\ell-1 = 1+\ell. \tag{12}$$

If (12) holds, then $n=2$ and

$$G_\ell(y) = C_\ell e^{\int \frac{k_1(y)}{ay^2}}\, dy,$$

where C_ℓ is a constant. Since $G_\ell(y) \neq 0$ must be a polynomial, we get that $k_1(y) = ky$ and so $G_\ell(y) = C_\ell y^{k/a}$ with $k/a \geq 1$ a non-negative integer and $C_\ell \neq 0$, because y divides $G_\ell(y)$. Therefore from (8) we obtain

$$\begin{aligned}
&(1-xy^2)(G_1(y) + 2G_2(y)x + \cdots + \ell C_\ell y^{k/a} x^{\ell-1}) \\
&\quad -a(y-xy^2)(G'_0(y) + G'_1(y)x + \cdots + G'_{\ell-1}(y)x^{\ell-1} + G'_\ell(y)x^\ell) \\
&= (k_0(y) + kyx)(G_0(y) + G_1(y)x + \cdots + C_\ell y^{k/a} x^\ell).
\end{aligned} \tag{13}$$

Here $k_0(y) = k_{01}y + k_{02}y^2$, because $n=2$ and the cofactor must be divisible by y.

The coefficient of x^0 of the polynomial (13) is

$$G_1(y) - ayG'_o(y) = (k_{01}y + k_{02}y^2)G_0(y).$$

So

$$G_0(y) = e^{-\frac{1}{a}\left(k_{01}y + \frac{1}{2}k_{02}y^2\right)} \left(C_0 + \frac{1}{a} \int y^{-1} G_1(y) e^{\frac{1}{a}\left(k_{01}y + \frac{1}{2}k_{02}y^2\right)} dy \right).$$

where C_0 is a constant. Since $G_0(y)$ and $G_1(y)$ are polynomials we have that $k_{01} = k_{02} = 0$. Therefore

$$G_0(y) = C_0 + \frac{1}{a} \int y^{-1} G_1(y) dy.$$

Consequently $C_0 = f_0 \neq 0$ and we know that $G_1(y)$ is divisible by y.

The coefficient of x^ℓ of the polynomial (13) is $-\ell C_\ell y^{k/a+2} - kC_\ell y^{k/a} + ay^2 G'_{\ell-1}(y) = kyG_{\ell-1}(y)$. Therefore we obtain

$$G_{\ell-1}(y) = \frac{C_\ell}{a}\left(\ell y - \frac{k}{y}\right) y^{\frac{k}{a}} + C_{\ell-1} y^{\frac{k}{a}}.$$

Hence if $\ell > 1$ the integer $k/a > 1$ because y divides $G_{\ell-1}(y)$, and if $\ell = 1$ then $k/a = 1$ because $G_0(y) = f_0 + yg(y)$ and consequently $f_0 = -kC_\ell/a = -C_\ell = -C_1$.

Assume $\ell > 1$. Denoting the integer $k/a = m > 1$. Then (13) becomes

(14)

$$\begin{aligned}
&(1-xy^2)\left(G_1(y) + 2G_2(y)x + \cdots + (\ell-1)\left(C_\ell\left(\frac{\ell}{a}y^{m+1} - my^{m-1}\right) + C_{\ell-1}y^m\right)x^{\ell-2}\right. \\
&\left. +\ell C_\ell y^m x^{\ell-1}\right) - a(y-xy^2)\left(G'_0(y) + G'_1(y)x + \cdots + \left(C_\ell\left(\frac{\ell(m+1)}{a}y^m - m(m-1)y^{m-2}\right)\right.\right. \\
&\left.\left. +mC_{\ell-1}y^{m-1}\right)x^{\ell-1} + mC_\ell y^{m-1}x^\ell\right) \\
&= mayx\left(G_0(y) + G_1(y)x + \cdots + \left(C_\ell\left(\frac{\ell}{a}y^{m+1} - my^{m-1}\right) + C_{\ell-1}y^m\right)x^{\ell-1} + C_\ell y^m x^\ell\right).
\end{aligned}$$

The coefficient of $x^{\ell-1}$ in (14) is

$$\begin{aligned}
&\ell C_\ell y^m - (\ell-1)\left(C_\ell\left(\frac{\ell}{a}y^{m+3} - my^{m+1}\right) + C_{\ell-1}y^{m+2}\right) \\
&\quad -a\left(C_\ell\left(\frac{\ell(m+1)}{a}y^{m+1} - m(m-1)y^{m-1}\right) + mC_{\ell-1}y^m\right) + ay^2G'_{\ell-2}(y) = mayG_{\ell-2}(y).
\end{aligned}$$

So

$$G_{\ell-2}(y) = \quad C_{\ell-2}y^m + \frac{y^{m-2}}{2a^2}\Big((\ell-1)\ell C_\ell y^4 + a^2 m(C_\ell(m-1) - 2C_{\ell-1}y)$$
$$+2ay(\ell C_\ell + (\ell-1)C_{\ell-1}y^2) + 2aC_\ell(\ell+m)y^2\log y\Big).$$

Since the coefficient of $\log y$ cannot be zero, and $G_{\ell-2}(y)$ must be a polynomial we have a contradiction.

Assume $\ell = 1$. Then recall that $k/a = 1$ and so $F(x,y) = G_0(y) + C_1yx$ and equality (13) becomes

$$C_1y(1-xy^2) - a(y-xy^2)(G_0'(y) + C_1x) = ayx(G_0(y) + C_1yx).$$

Considering the terms without x and we have that $C_1y - ayG_0'(y) = 0$ and so $G_0(y) = (C_1/a)y + C$ with C a constant. Now the terms of x gives $-C_1y^3 - aC_1y + ay^2G_0'(y) = ayG_0(y)$ and substituting the expression of G_0 we obtain $C_1 = 0$, a contradiction. Hence the lemma is proved for $n = 2$.

Now we can assume that (11) holds. Therefore $k_{n-1}(y) = 0$ because $G_\ell(y) \neq 0$. Then the coefficient of $x^{n+\ell-2}$ in the polynomial (8) must satisfy

$$k_{n-2}(y)G_\ell(y) = 0 \quad \text{if } n+\ell-2 > 1+\ell, \tag{15}$$

or

$$k_{n-2}(y)G_\ell(y) = ay^nG_\ell'(y) \quad \text{if } n+\ell-2 = 1+\ell. \tag{16}$$

If (16) holds, then $n = 3$ and

$$G_\ell(y) = C_\ell e^{\int \frac{k_1(y)}{ay^3}}\, dy,$$

where C_ℓ is a constant. Since $G_\ell(y) \neq 0$ must be a polynomial, we get that $G_\ell(y) = C_\ell \neq 0$ and $k_1(y) = ky^2$. Therefore $G_\ell(y) = C_\ell y^{k/a}$ with $k/a \geq 1$ a non-negative integer and $C_\ell \neq 0$, because y divides $G_\ell(y)$. Therefore from (8) we obtain

$$\begin{aligned}
&(1-xy^3)(G_1(y) + 2G_2(y)x + \cdots + \ell C_\ell y^{k/a}x^{\ell-1})\\
&\quad -a(y-xy^3)\left(G_0'(y) + G_1'(y)x + \cdots + G_{\ell-1}'(y)x^{\ell-1} + \frac{k}{a}C_\ell y^{k/a-1}x^\ell\right)\\
&= (k_0(y) + ky^2x)(G_0(y) + G_1(y)x + \cdots + C_\ell y^{k/a}x^\ell).
\end{aligned} \tag{17}$$

Here $k_0(y) = k_{01}y + k_{02}y^2 + k_{03}y^3$, because $n = 3$ and the cofactor must be divisible by y.

The coefficient of x^0 of the polynomial (17) is

$$G_1(y) - ayG_o'(y) = (k_{01}y + k_{02}y^2 + k_{03}y^3)G_0(y).$$

So

$$G_0(y) = e^{-\frac{1}{a}\left(k_{01}y + \frac{1}{2}k_{02}y^2 + \frac{1}{3}k_{03}y^3\right)}\left(C_0 + \frac{1}{a}\int y^{-1}G_1(y)e^{\left(k_{01}y + \frac{1}{2}k_{02}y^2 + \frac{1}{3}k_{03}y^3\right)}dy\right).$$

where C_0 is a constant. Since $G_0(y)$ and $G_1(y)$ are polynomials we have that $k_{01} = k_{02} = k_{03} = 0$. Therefore

$$G_0(y) = C_0 + \frac{1}{a}\int y^{-1}G_1(y)dy.$$

Consequently $C_0 = f_0 \neq 0$ and we know that $G_1(y)$ is divisible by y.

The coefficient of x^ℓ in the polynomial (17) is $-\ell C_\ell y^{k/a+3} - kC_\ell y^{k/a} + ay^3G_{\ell-1}'(y) = ky^2G_{\ell-1}(y)$. Therefore we obtain

$$G_{\ell-1}(y) = \frac{C_\ell}{a}\left(\ell y - \frac{k}{2y^2}\right)y^{\frac{k}{a}} + C_{\ell-1}y^{\frac{k}{a}}$$

Hence if $\ell > 1$ the integer $k/a > 2$ because y divides $G_{\ell-1}(y)$, and if $\ell = 1$ then $k/a = 2$ because $G_0(y) = f_0 + yg(y)$ and consequently $f_0 = -kC_1/(2a) = -C_1$.

Assume $\ell > 1$. The coefficient of $x^{\ell-1}$ in the polynomial (17) is

$$\ell C_\ell y^{k/a} - (\ell-1)y^3 G_{\ell-1}(y) - ayG'_{\ell-1}(y) + ay^3 G'_{\ell-2}(y) = ky^2 G_{\ell-2}(y).$$

Solving this equation with respect to $G_{\ell-2}(y)$ we obtain

$$\begin{aligned} G_{\ell-2}(y) = \quad & \frac{y^{k/a-4}}{8a^2}\Big((-C_\ell(2a-k)k + 4a(\ell C_\ell - kC_{\ell-1})y^2 - 4C_\ell(2a\ell + k + \ell k)y^3 \\ & \qquad +8a^2 C_{\ell-2}y^4 + 8aC_{\ell-1}(-1+\ell)y^5 + 4C_\ell(-1+\ell)\ell y^6)\Big). \end{aligned}$$

Since $k > 0$ then $C_\ell(k + k^2) \neq 0$, and consequently the integer $k/a \geq 4$ because $G_{\ell-2}(y)$ must be a polynomial. We note that $\ell \neq 2$, otherwise we have a contradiction with $G_0(y) = f_0 + yg(y)$. Hence $\ell > 2$.

The coefficient of $x^{\ell-2}$ in the polynomial (17) is

$$(\ell-1)G_{\ell-1}(y) - (\ell-2)y^3 G_{\ell-2}(y) - ayG'_{\ell-2}(y) + ay^3 G'_{\ell-3}(y) = ky^2 G_{\ell-3}(y).$$

Solving this differential equation with respect to $G_{\ell-3}(y)$ we obtain

$$G_{\ell-3}(y) = \quad \frac{y^{k/a-6}}{48a^3}\Big(\text{a polynomial in the variable } y\Big) + \frac{1}{a^3}C_\ell(a\ell + k)y^{k/a}\log y.$$

Since $a\ell + k > 0$ we get a contradiction with the fact that $G_{\ell-3}(y)$ is a polynomial.

Assume $\ell = 1$. Then recall that $k/a = 2$ and so $F(x, y) = G_0(y) + C_1 y^2 x$ and equality (17) becomes

$$C_1 y^2(1 - xy^3) - a(y - xy^3)(G'_0(y) + 2C_1 yx) = 2ay^2 x(G_0(y) + C_1 y^2 x).$$

Taking the coefficient without x we have $C_1 y^2 - ayG'_0(y) = 0$ and so $G_0(y) = (C_1/(2a))y^2 + C_0$ with C_0 a constant. From the coefficients of x we have

$$-C_1 y^5 + ay^3 G'_0(y) - 2aC_1 y^2 = 2ay^2 G_0(y),$$

and substituting the expression of G_0 we have that $C_1 = 0$, a contradiction. Hence the lemma is proved for $n = 3$.

We have $k_{n-1}(y) = 0$ and we can assume that $n > 3$ and that (15) holds. Therefore $k_{n-2}(y) = 0$. Now the coefficient of of $x^{n+\ell-3}$ in the polynomial (8) must satisfy

$$k_{n-3}(y)G_\ell(y) = 0 \quad \text{if } n + \ell - 3 > 1 + \ell, \tag{18}$$

or

$$k_{n-3}(y)G_\ell(y) = ay^n G'_\ell(y) \quad \text{if } n + \ell - 3 = 1 + \ell. \tag{19}$$

In a similar way as in the previous cases we get a contradiction. Repeating this process until the iteration $r = n - 1$ we shall arrive to the case $n + \ell - r = 1 + \ell$ with

$$k_{n-r}(y)G_\ell(y) = ay^n G'_\ell(y).$$

and we get again a contradiction. So the lemma is proved. □

Lemma 4. *The invariant line at infinity has multiplicity $n + 1$. The only exponential factors of system* (1) *are $G_i = \exp\big((ax + y)^i\big)$ for $i = 1, \cdots, n$.*

Proof. Applying the definition of the exponential factor (see Appendix) we can see directly that $G_i = \exp\left((ax+y)^i\right)$ are exponential factors for the vector field (2) with cofactors $L_i = -ia(y-1)(ax+y)^{i-1}$ for $i = 1, \cdots, n$.

We claim that the algebraic multiplicity of the line at infinity for the vector field (2) is $n+1$. Consider the expression of the vector field (2) in the chart (U_1, F_1) (see for details chapter 5 of [11])

$$Y = \left(-{z_2}^{n+1} z_1 + {z_1}^{n+1} - {z_2}^{n} a z_1 + a {z_1}^{n}\right) \frac{\partial}{\partial z_1} + z_2 \left({z_1}^{n} - {z_2}^{n+1}\right) \frac{\partial}{\partial z_2}.$$

We set $v_1 = 1$, $v_2 = z_1$ and $v_3 = z_2$. Then the extactic curve $\mathcal{E}_1$ of Y is

$$\begin{aligned} \mathcal{E}_1 &= \begin{vmatrix} v_1 & v_2 & v_3 \\ Y(v_1) & Y(v_2) & Y(v_3) \\ Y(Y(v_1))) & Y(Y(v_2)) & Y(Y(v_3)) \end{vmatrix} \\ &= a z_2^{n+1} \left[z_2\, {z_1}^{2\,n-1} n a + {z_1}^{2\,n+1} + \left(a\,(n-1) - z_2\right) {z_2}^{n} {z_1}^{n+1} \right. \\ &\quad \left. + \left((-n+1)\,a - z_2\right) {z_1}^{2\,n} {z_2}^{2\,n} z_2\, z_1\, a + \left((-n-1)\, z_2\, a {z_2}^{n} + {z_2}^{n+2}\right) {z_1}^{n} \right] = 0. \end{aligned} \tag{20}$$

Note that z_2^{n+1} divides $\mathcal{E}_1$ and z_2^{n+2} does not divide $\mathcal{E}_1$. Then by Proposition 7 of the Appendix, the lemma follows. ☐

Proof of Theorem 1. Clearly the cofactor of $y = 0$ is $K = a(-1+xy^{n-1})$. Additionally, by Lemma 3 there is no other irreducible invariant algebraic curves of system (1). Since the algebraic multiplicity of $y = 0$ is one it turns out that there is no exponential factors associated to the invariant straight line $y = 0$, see Proposition 7 of the Appendix. According to Lemma 4 the straight line at infinity has multiplicity $n+1$. So the infinity provides n exponential factors $G_i = \exp\left((ax+y)^i\right)$ with cofactors $L_i = -ia(y-1)(ax+y)^{i-1}$ for $i = 1, \cdots, n$. System (1) has divergence $\operatorname{div}(X) = anxy^{n-1} - y^n - a$. According to statement (a) of Theorem 8 of the Appendix there is a linear combination between the cofactors K and L_i, if and only if there exists a Darboux first integral, or from its statement (b) there is a linear combination between the cofactors K and L_i and the divergence of the system, if and only if there exists a Darboux integrating factor (24). But we can check easily that there is no such linear combinations. Indeed, considering the linear combination of the cofactors we obtain

$$0 = \lambda K + \sum_{i=1}^{n} \mu_i L_i = \lambda a(-1+xy^{n-1}) - \sum_{i=1}^{n} \mu_i i a(y-1)(ax+y)^{i-1}.$$

For $n \geq 2$, the coefficient of the monomial x^{n-1} must be zero. So $\mu_n = 0$. Now the coefficient of the monomial x^{n-2} must be zero, so $\mu_{n-1} = 0$. Continuing in this way, at the end we obtain $\mu_n = \mu_{n-1} = \cdots = \mu_2 = 0$. The linear combination now becomes

$$\lambda a(-1+xy^{n-1}) - \mu_1 a(y-1) = 0,$$

and so $\lambda = \mu_1 = 0$.

For $n = 1$, the linear combination becomes $0 = \lambda K + \mu_1 L_1 = \lambda a(-1+x) + \mu_1 a(y-1)$ and therefore $\lambda = \mu_1 = 0$.

Now we consider the linear combination of the cofactors and the divergence of the vector field. We have

$$0 = \lambda K + \sum_{i=1}^{n} \mu_i L_i - \operatorname{div}(X) = \lambda a(-1+xy^{n-1}) - \sum_{i=1}^{n} \mu_i i a(y-1)(ax+y)^{i-1} + anxy^{n-1} - y^n - a.$$

For $n \geq 2$, similar arguments yields to $\mu_n = \mu_{n-1} = \cdots = \mu_2 = 0$ and so the linear combination now becomes

$$\lambda a(-1+xy^{n-1}) - \mu_1 a(y-1) + anxy^{n-1} - y^n - a = 0,$$

and for $n \geq 2$ cannot be satisfied. For $n = 1$, the linear combination becomes

$$0 = \lambda K + \mu_1 L_1 + \text{div}(X) = \lambda a(-1 + x) - \mu_1 a(y - 1) + ax - y - a$$

and cannot be satisfied.

In summary, system (1) is not Liouvillian integrable. □

3. Proof of Theorem 2

In order to prove Theorem 2 we need to study the behaviour of the finite and infinite equilibrium points, and the existence of periodic orbits and limit cycles.

3.1. **The finite equilibrium point, periodic orbits and limit cycles.** System (1) has only one finite equilibrium point, namely $P = (1, 1)$. The Jacobian matrix of the system at P is

$$\begin{pmatrix} -1 & -n \\ a & a\,(n-1) \end{pmatrix}.$$

The eigenvalues of the Jacobian matrix are

$$\frac{a\,(n-1) - 1 \pm \sqrt{(a(n-1)-1)^2 - 4a}}{2}.$$

For $n = 1$ the point P is a stable node when $a \leq 1/4$ and is a stable focus when $a > 1/4$.

Now we consider that $n > 1$. We set $R(a) = (a\,(n-1) - 1)\,/2$ and $D = (a(n-1)-1)^2 - 4a)$. Then $R(a) = 0$ gives $a = 1/(n-1)$ and relation $D = 0$ gives

$$a = \left(\frac{\sqrt{n} - 1}{n - 1}\right)^2.$$

Hence for $n > 1$ the equilibrium point $P = (1, 1)$ is (see also [20, 3, 15])

- a stable hyperbolic node if $a \in \left(0,\ \left(\frac{\sqrt{n}-1}{n-1}\right)^2\right]$.
- a stable hyperbolic focus if $a \in \left(\left(\frac{\sqrt{n}-1}{n-1}\right)^2, \frac{1}{n-1}\right)$.
- a stable weak focus if $a = \frac{1}{n-1}$ and there is a Hopf bifurcation.
- an unstable hyperbolic focus if $a \in \left(\frac{1}{n-1},\ \left(\frac{\sqrt{n}+1}{n-1}\right)^2\right)$.
- an unstable hyperbolic node if $a \geq \left(\frac{\sqrt{n}+1}{n-1}\right)^2$.

Note that $R'(a) = (n-1)/2 > 0$ for $n > 1$. In particular, $R'(a) > 0$ for $a = 1/(n-1)$. Then from Theorem 3.4.2 of [12] appears a Hopf bifurcation. So there is a periodic orbit. In order to study the stability of the periodic orbit we need to calculate the first Lyapunov constant at P. By relation (3.4.11) of [12] the first Lyapunov coefficient is equal to $-1/16$. So, there is a supercritical bifurcation for $a \geq 1/(n-1)$ and close to $1/(n-1)$. So the unique bifurcated limit cycle in the Hopf bifurcation must be stable.

The following theorem characterize the existence of periodic orbits and limit cycles, for a proof see [5, 6]. We recall that here we consider $a > 0$.

Theorem 5. *For every positive integer $n \geq 2$, there exists a unique constant $a^* \in \left(\frac{1}{n-1},\ \frac{2^n-1}{2^n-2}\right)$ such that system* (1) *has no periodic orbits when $a \in (-\infty, 1/(n-1)] \bigcup [a^*,\ +\infty)$ and has a unique limit cycle when $a \in (1/(n-1),\ a^*)$, which is stable and hyperbolic. Moreover, when the limit cycle exists, its amplitude increases with a.*

3.2. **The infinite equilibrium points.** In order to understand the behaviour of the infinite equilibrium points of system (1) we need to study the corresponding compactified vector field on the local charts of the Poincaré disc, see subsection 4.2 of the Appendix.

The expression of system (1) in the local chart (U_1, F_1) is

$$\begin{aligned} \dot{z}_1 &= -{z_2}^{n+1} z_1 + {z_1}^{n+1} - {z_2}^n a z_1 + a {z_1}^n, \\ \dot{z}_2 &= -{z_2}^{n+2} + z_2\, {z_1}^n, \end{aligned} \tag{21}$$

and there are two infinite equilibrium points $(0, 0)$ and $(-a, 0)$. For $n = 1$ the origin is a semi–hyperbolic saddle and the point $(-a, 0)$ is a hyperbolic stable node.

For $n > 1$ the Jacobian matrix at the origin is linearly zero and the blow up technique is applied (see [3] and [15]). For n odd the origin of the chart (U_1, F_1) in the Poincaré sphere is the union of one parabolic and four hyperbolic sectors, the line of infinity separates the four hyperbolic sectors two in each side, see the origin of the local chart U_1 in Figure 1. Whereas for n even is the union of two hyperbolic and two parabolic sectors separated by the line of the infinity as it is indicated in Figure 2.

For $n > 1$ the Jacobian matrix at the point $(-a, 0)$ is

$$\begin{pmatrix} (-a)^n & 0 \\ 0 & (-a)^n \end{pmatrix}.$$

So the point $(-a, 0)$ is a hyperbolic node. For n even is unstable whereas for n odd is stable.

In the local chart (U_2, F_2) system (1) is written

$$\begin{aligned} \dot{z}_1 &= -z_1\ (a z_1 + 1) + {z_2}^n a z_1 + {z_2}^{n+1}, \\ \dot{z}_2 &= a z_2\ ({z_2}^n - z_1), \end{aligned} \tag{22}$$

and the origin has eigenvalues 0 and -1. Since the origin is a semi–hyperbolic equilibrium point we apply Theorem 2.19 of [11] and for $n = 1$ is a saddle–node. For $n > 1$ and n odd the origin of the chart (U_2, F_2) is a saddle node whereas for $n > 1$ and n even it is a saddle.

Combine all these previous results we are ready to prove Theorem 2.

For $n = 1$ and $a \in (0,\ 1/4]$ the point P is a stable node and the phase portrait is given in Figure 1(e). Note that for $a > 1/4$ the point P is a stable focus and in this case the phase portrait is equivalent to Figure 1(e).

Now consider that $n > 1$ and n is odd. Then for $a \in (0,\ (\sqrt{n} - 1)^2/(n - 1)^2]$ the finite equilibrium point is a stable node. Since the stable node is topological equivalent with a stable focus we obtain the same phase portrait as the one for $a \in \left((\sqrt{n} - 1)^2/(n - 1)^2,\ 1/(n - 1)\right)$ where the finite equilibrium point is now a stable focus. This corresponds to Figure 1(a). Moreover, the separatrix configuration in Figure 1(a) is different from the separatrix configuration in Figure 1(d). Hence these two phase portraits are not topological equivalent. In addition, the numbers R and S are distinct in the rest of the cases, so we obtain five different phase portraits for n odd, see Figure 1.

For n even using similar arguments we obtain four distinct phase portraits, see Figure 2. Moreover, these new four phase portraits have different separatrix configuration than the ones for n odd because of the different behaviour of the origin of the chart (U_1, F_1). This completes the proof of Theorem 2.

4. Appendix

This Appendix has three subsections.

4.1. **Invariant curves, multiplicity and Liouvillian integrability.** Consider the polynomial differential system

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y). \tag{23}$$

We say that this system has *degree* d if d is the maximum of the degrees of the polynomials P and Q. The associated vector field to system (23) is

$$X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y},$$

and we denote by div(X) the divergence of X, namely, $\mathrm{div(X)} = \partial P/\partial x + \partial Q/\partial y$.

Let $f = f(x, y)$ be a polynomial in the variables x and y. The algebraic curve $f(x, y) = 0$ is an *invariant algebraic curve* of system (23) if

$$P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf,$$

for some polynomial $K = K(x, y)$ called the *cofactor* of the invariant algebraic curve $f = 0$. The polynomial structure of system (23) forces that the degree of the cofactor is at most $d - 1$. Note that the curve $f = 0$ is formed by orbits of the differential system (23), and consequently it is *invariant* under the flow of this system.

The *$m-th$ extactic curve* of X, $\mathcal{E}_m(X)$ is given by the polynomial equation

$$\mathcal{E}_m(X) = \det\begin{pmatrix} v_1 & v_2 & \cdots & v_\ell \\ X(v_1) & X(v_2) & \cdots & X(v_\ell) \\ \vdots & \vdots & \cdots & \vdots \\ X^{\ell-1}(v_1) & X^{\ell-1}(v_2) & \cdots & X^{\ell-1}(v_\ell) \end{pmatrix} = 0,$$

where $v_1, \cdots, v_\ell$ is a basis of $\mathbb{C}_m[x, y]$ (the $\mathbb{C}$ vector space of polynomials in $\mathbb{C}[x, y]$ of degree at most m) and consequently $\ell = (m + 1)(m + 2)/2$.

Proposition 6. *Every algebraic curve of degree m invariant by the vector field X is a factor of $\mathcal{E}_m(X)$.*

An invariant algebraic curve f of degree m for the vector field X has *algebraic multiplicity* k when k is the greatest positive integer such that f^k divides $\mathcal{E}_m(X)$. For more details about the multiplicity of an invariant curve and other properties of the extactic curve see [8].

The algebraic multiplicity of a curve is connected with an object called *exponential factor*. Exponential factors also provide cofactors and appear when invariant algebraic curves collide, i.e. when they have multiplicity larger than one. Let $F(x, y) = \exp(g(x, y)/f(x, y))$ where f, g polynomials. Then F is an *exponential factor* of system (23) if

$$P\frac{\partial F}{\partial x} + Q\frac{\partial F}{\partial y} = LF,$$

for some polynomial L of degree at most $d-1$ called the *cofactor* of the exponential factor F. The next result is proved in [8].

Proposition 7. *Let $f=0$ be an irreducible invariant algebraic curve of degree m of the polynomial vector field X with cofactor K. Then the algebraic multiplicity of the curve $f=0$ is k if and only if X has $k-1$ exponential factors of the form $\exp(g_i/f^i)$ for $i=1,\cdots,k-1$ and the degree of g_i is at most im.*

A *Darboux* function of a vector field X is a function of the form

$$D=\prod f_i^{\lambda_i}F_j^{\mu_j}, \tag{24}$$

where the $f_i=0$ are invariant algebraic curves and F_j are exponential factors of X.

The first integrals given by functions (24) are called *Darboux first integrals* and the integrating factors given by (24) are called *Darboux integrating factors.*

By definition a *Liouvillian function* is an element in the Liouvillian field extension of the field of rational functions $\mathbb{C}(x,y)$. For a good review about Darboux and Liouvillian integrability see chapter 3 of [24], and chapter 8 of [11]. In 1992 Singer [21] and later on in 1999 Christopher [7] proved that for a planar polynomial differential system, the existence of Liouvillian first integrals is equivalent to the existence of a Darboux integrating factor. So now we can say that system (1) is *Liouvillian integrable* if has a first integral or an integrating factor given by a Darboux function. The following result started with Darboux [9], for the present version see for instance [11].

Theorem 8. *Suppose that a polynomial system* (23) *admits p irreducible invariant algebraic curves $f_i=0$ with cofactors K_i and q exponential factors F_j with cofactors L_j. Then the following statements hold.*

(a) *There exist λ_i's and μ_j's in $\mathbb{C}$ not all zero such that $\sum\limits_{i=1}^{p}\lambda_iK_i+\sum\limits_{j=1}^{q}\mu_jL_j=0$, if and only if the Darboux function* (24) *is a first integral of system* (23)*.*

(b) *There exist λ_i's and μ_j's in $\mathbb{C}$ not all zero such that $\sum\limits_{i=1}^{p}\lambda_iK_i+\sum\limits_{j=1}^{q}\mu_jL_j=-div(X)$, if and only if the Darboux function* (24) *is an integrating factor of system* (23)*.*

4.2. **Poincaré compactification.** Consider the polynomial differential system (23) of degree d and its corresponding vector field $\mathcal{X}$. We also consider the *Poincaré compactification* of system (23) in order to control the orbits that come or escape at infinity. For more details on this compactification see Chapter 5 of [11].

Let $\mathbb{R}^2$ be the plane in $\mathbb{R}^3$ defined by $(y_1,y_2,y_3)=(x_1,x_2,1)$. We define the *Poincaré sphere* $\mathbb{S}^2=\{y=(y_1,y_2,y_3)\in\mathbb{R}^3: y_1^2+y_2^2+y_3^2=1\}$ and $T_{(0,0,1)}\mathbb{S}^2$ is the tangent space to $\mathbb{S}^2$ at the point $(0,0,1)$ (see [19]). We consider now the central projection $f^{\pm}:\mathbb{R}^2\to\mathbb{S}^2$ with $f^{\pm}(x_1,x_2,1)=\pm(x_1,x_2,1)/\sqrt{x_1^2+x_2^2+1}$. We see that $f^{\pm}$ defines two copies of $\mathcal{X}$, one in the northern hemisphere $\{y\in\mathbb{S}^2: y_3>0\}$ and the other in the southern hemisphere. Denote by $\hat{\mathcal{X}}=Df\circ\mathcal{X}$ and $\hat{\mathcal{X}}$ is defined on $\mathbb{S}^2$ except on its equator $\mathbb{S}^1$. Note that the points at infinity of $\mathbb{R}^2$ are in bijective correspondence with the equator of $\mathbb{S}^2$, namely $\mathbb{S}^1=\{y\in\mathbb{S}^2: y_3=0\}$. Thus, $\mathbb{S}^1$ is identified with the infinity of $\mathbb{R}^2$. Then the *Poincaré compactified vector field* $p(\mathcal{X})$ of $\mathcal{X}$ will be the analytic vector field induced on $\mathbb{S}^2$ as follows: We multiply $\hat{\mathcal{X}}$ by the factor y_3^d and so the vector field $y_3^d\hat{\mathcal{X}}$ is defined in the whole $\mathbb{S}^2$.

Now on $\mathbb{S}^2\setminus\mathbb{S}^1$ there are two symmetric copies of $\mathcal{X}$. So the behavior of $p(\mathcal{X})$ around $\mathbb{S}^1$ describes the behavior of $\mathcal{X}$ near the infinity. The *Poincaré disc* $\mathbb{D}$ is the projection of the closed northern hemisphere of $\mathbb{S}^2$ on $y_3 = 0$ under $(y_1, y_2, y_3) \longmapsto (y_1, y_2)$. Additionally, $\mathbb{S}^1$ is invariant under the flow of $p(\mathcal{X})$.

Two polynomial vector fields $\mathcal{X}$ and $\mathcal{Y}$ on $\mathbb{R}^2$ are *topologically equivalent* if there exists a homeomorphism on $\mathbb{S}^2$ preserving the infinity $\mathbb{S}^1$ carrying orbits of the flow induced by $p(\mathcal{X})$ into orbits of the flow induced by $p(\mathcal{Y})$. Moreover, the homeomorphism should preserve or reverse simultaneously the sense of all orbits of the two compactified vector fields $p(\mathcal{X})$ and $p(\mathcal{Y})$.

Since $\mathbb{S}^2$ is a differentiable manifold we consider the six local charts $U_i = \{y \in \mathbb{S}^2 : y_i > 0\}$, and $V_i = \{y \in \mathbb{S}^2 : y_i < 0\}$ for $i = 1, 2, 3$ with the diffeomorphisms $F_i : V_i \longrightarrow \mathbb{R}^2$ and $G_i : V_i \longrightarrow \mathbb{R}^2$, which are the inverses of the central projections from the planes tangent at the points $(1, 0, 0), (-1, 0, 0), (0, 1, 0), (0, -1, 0), (0, 0, 1)$ and $(0, 0, -1)$, respectively. Let $z = (z_1, z_2)$ be the value of $F_i(y)$ or $G_i(y)$ for any $i = 1, 2, 3$. Then the expressions of the compactified vector field $p(\mathcal{X})$ of $\mathcal{X}$ are

$$
\begin{aligned}
z_2^d\Delta(z)\left(Q\Big(\frac{1}{z_2},\frac{z_1}{z_2}\Big)-z_1P\Big(\frac{1}{z_2},\frac{z_1}{z_2}\Big),\ -z_2P\Big(\frac{1}{z_2},\frac{z_1}{z_2}\Big)\right) \qquad &\text{in} \qquad U_1,\\
z_2^d\Delta(z)\left(P\Big(\frac{z_1}{z_2},\frac{1}{z_2}\Big)-z_1Q\Big(\frac{z_1}{z_2},\frac{1}{z_2}\Big),\ -z_2Q\Big(\frac{z_1}{z_2},\frac{1}{z_2}\Big)\right) \qquad &\text{in} \qquad U_2,\\
\Delta(z)\big(P(z_1,z_2),Q(z_1,z_2)\big) \qquad &\text{in} \qquad U_3,
\end{aligned}
$$

with $\Delta(z) = (z_1^2 + z_2^2 + 1)^{-\frac{1}{2(d-1)}}$. Note that the expressions of the vector field $p(\mathcal{X})$ in the local chart V_i is the same as in the chart U_i multiplying by the factor $(-1)^{d-1}$. In these coordinates, $z_2 = 0$ denotes the points of $\mathbb{S}^1$. We can omit the factor $\Delta(z)$ by rescaling the vector field $p(\mathcal{X})$, and therefore we obtain a polynomial vector field in each local chart. Note that the infinity $\mathbb{S}^1$ is invariant with $p(\mathcal{X})$.

4.3. **Separatrix configuration.** Let $p(\mathcal{X})$ be the Poincaré compactification in $\mathbb{S}^2$ of a polynomial vector field $\mathcal{X}$ in $\mathbb{R}^2$.

The *separatrices* of the vector field $p(\mathcal{X})$ in the Poincaré disc $\mathbb{D}$ are

- (i) all the orbits of $p(\mathcal{X})$ which are in the boundary $\mathbb{S}^1$ of the Poincaré disc (recall that $\mathbb{S}^1$ is the infinity of $\mathbb{R}^2$);
- (ii) all the finite singular points of $p(\mathcal{X})$;
- (iii) all the limit cycles of $p(\mathcal{X})$; and
- (iv) all the separatrices of the hyperbolic sectors of the finite and infinite singular points of $p(\mathcal{X})$.

We also consider the definition of parallel flows given by Markus [16] and Neumann in [17]. Let ϕ be a $\mathcal{C}^\omega$ local flow on the two dimensional manifold $\mathbb{R}^2$ or $\mathbb{R}^2 \setminus \{0\}$. The flow (M, ϕ) is $\mathcal{C}^k$ *parallel* if it is $\mathcal{C}^\omega$-equivalent to one of the following ones:

1. *strip*: $(\mathbb{R}^2, \phi)$ with the flow ϕ defined by $\dot{x} = 1, \dot{y} = 0$;
2. *annular*: $(\mathbb{R}^2 \setminus \{0\}, \phi)$ with the flow ϕ defined (in polar coordinates) by $\dot{r} = 0, \dot{\theta} = 1$;
3. *spiral*: $(\mathbb{R}^2 \setminus \{0\}, \phi)$ with the flow ϕ defined by $\dot{r} = r, \dot{\theta} = 1$.

We denote by Σ the union of all separatrices of the flow $(\mathbb{D}, \phi)$ defined by the compactified vector field $p(\mathcal{X})$ in the Poincaré disc $\mathbb{D}$. Then Σ is a closed invariant subset of $\mathbb{D}$. Every connected component of $\mathbb{D} \setminus \Sigma$, with the restricted flow, is called a *canonical region* of ϕ.

For a proof of the following result see [14] and [17].

Theorem 9. *Let ϕ be a $\mathcal{C}^\omega$ flow in the Poincaré disc with finitely many separatrices, and let Σ be the union of all its separatrices. Then the flow restricted to every canonical region is $\mathcal{C}^\omega$ parallel.*

The *separatrix configuration* Σ_c of a flow (D, ϕ) is the union of all the separatrices Σ of the flow together with an orbit belonging to each canonical region. The separatrix configuration Σ_c of the flow (D, ϕ) is said to be *topologically equivalent* to the separatrix configuration $\tilde{\Sigma}_c$ of the flow $(D, \tilde{\phi})$ if there exists a homeomorphism from Σ_c to Σ_c which transforms orbits of Σ_c into orbits of $\tilde{\Sigma}_c$, and orbits of Σ into orbits of $\tilde{\Sigma}$.

The following theorem and its proof appears in [16, 17, 18].

Theorem 10. *Let (D, ϕ) and $(D, \tilde{\phi})$ be two compactified Poincaré flows with finitely many separatrices coming from two polynomial vector fields. Then they are topologically equivalent if and only if their separatrix configurations are topologically equivalent.*

So in order to classify the phase portraits in the Poincaré disc of a planar polynomial differential system having finitely many separatrices, it is enough to describe their separatrix configuration and an orbit inside each canonical region.

Acknowledgments

J. Llibre is partially supported by the H2020 European Research Council grant MSCA-RISE-2017-777911, AGAUR (Generalitat de Catalunya) grant 2021SGR00113, and by the Acadèmia de Ciències i Arts de Barcelona. Both authors are partially supported by the Ministerio de Ciéncia e Innovación grant (PID2019-104658GB-I00). C. Pantazi has been funded partially by the grant PID-2021-122954NB-100 funded by MCIN/AEI/ 10.13039/501100011033 and by "ERDF A way of making Europe".

References

[1] M. Abramowitz and I.A. Stegun, *Incomplete Gamma Function*, §6.5 in Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing. New York: Dover, pp.260-263, 1972.

[2] J.C. Artés, J. Llibre and C. Valls, *Dynamics of the Higgins—Selkov and Selkov systems*, Chaos Sol. Frac. **114** (2018) 145–150.

[3] P. Brechmann and A.D. Rendall, *Dynamics of the Selkov oscillator*, Math. Biosci. **306** (2018) 152–159.

[4] P. Brechmann and A.D. Rendall, *Unbounded solutions of models for glycolysis*, J. Math. Biol. **82**, (2021) 23 pages.

[5] H. Chen, J. Llibre and Y. Tang, *The limit cycles of the Higgins–Selkov systems*, J. Nonlinear Sci. **31** 85 (2021) 25 pages.

[6] H. Chen and Y. Tang, *Proof of Artés–Llibre–Valls's conjectures for the Higgins-Selkov and the Selkov systems*, J. Differential Equations **266** (2019) 7638–7657.

[7] C. Christopher, *Liouvillian first integrals of second order polynomial differential equations*, Electron. J. Differ. Equ. **49** (1999) 1—7.

[8] C. Christopher, J. Llibre and J.V. Pereira, *Multiplicity of invariant algebraic curves in polynomial vector fields*, Pac. J. Math. **229** (2007) 63—117.

[9] G. Darboux, *Mémoire sur les équations différentielles algébriques du premier ordre et du premier degré (Mélanges)*, Bull. Sci. Math. 2ème Série **2** (1878) 60–96, 123–144, 151–200.

[10] A. d'Onofrio, *Uniqueness and global attractivity of glycolytic oscillations suggested by selkov's model*, J. Math. Chem. **48** (2010) 339–346.

[11] F. Dumortier, J. Llibre and J. C. Artés, *Qualitative theory of planar polynomial systems*, Springer, New York, 2006.

[12] J. Guckenheimer and P. Holmes, *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields*, in: Applied Mathematical Sciences, vol. **42**, Springer Verlag, New York, 1986.

[13] J. Higgins, *A chemical mechanism for oscillation of glycolytic intermediates in yeast cells*, Proc. Natl. Acad. Sci. (USA) **51** (1964) 989–994.
[14] W. Li, J. Llibre, M. Nicolau & X. Zhang, *On the differentiability of first integrals of two dimensional flows*, Proc. Amer. Math. Soc. **130** (2002) 2079–2088.
[15] J. Llibre and M. Mousavi, *Phase portraits of the Higgins–Selkov system*, Discrete Contin. Dyn. Syst. Ser. B, **27** (2021) 245–256.
[16] L. Markus, *Global structure of ordinary differential equations in the plane*, Trans. Amer. Math Soc. **76** (1954) 127–148.
[17] D.A. Neumann, *Classification of continuous flows on 2–manifolds*, Proc. Amer. Math. Soc. **48** (1975) 73–81.
[18] M.M. Peixoto, *Dynamical Systems. Proccedings of a Symposium held at the University of Bahia*, 389–420, Acad. Press, New York, 1973.
[19] H. Poincaré, *Sur l'intégration des équations différentielles du premier ordre et du premier degré I*, Rend. Circ. Mat. di Palermo **5** (1891) 161–191.
[20] E.E. Selkov, *Self-oscillations in glycolysis*, I. A simple kinetic model. Eur. J. Biochem. **4** (1968) 79–86.
[21] M.F. Singer, *Liouvillian first integrals of differential equations*, Trans. Am. Math. Soc. **333** (1992) 673—688.
[22] N.J.A. Sloane, *Sequences A000079/M1129, A001147/M3002, and A084253* in "The On-Line Encyclopedia of Integer Sequences."
[23] L. Yang, *Recent advances on determing the number of real roots of parametric polynomials*, J. Symb. Comput. **28** (1999) 225–242.
[24] X. Zhang, *Integrability of Dynamical Systems: Algebra and Analysis*, Springer, 2017.

[1] Departament de Matemàtiques, Universitat Autònoma de Barcelona, Edifici C, 08193 Bellaterra, Barcelona, Catalonia, Spain

Email address: `jllibre@mat.uab.cat`

[2] Departament de Matemàtiques, Universitat Politècnica de Catalunya, (EPSEB), Av. Doctor Marañón, 44–50, 08028 Barcelona, Spain

Email address: `chara.pantazi@upc.edu`