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The phase portrait of all polynomial Liénard isochronous centers

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A B S T R A C T

We prove that there is a unique topologically equivalent phase portrait for all Liénard polynomial differential systems of arbitrary degree having an isochronous center.

1. Introduction and statement of the main results

In this paper we deal with the Liénard real polynomial differential equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{1}$$

where f and g are non-zero polynomials such that $g(x) = x + g_2(x)$ with $g_2(0) = 0$ and $g_2'(0) = 0$. Here one dot denotes the first derivative with respect to the time t, and two dots the second derivative with respect to the time t.

The differential equations (1) have been studied by many authors. Thus for instance in MathSciNet if we put in four different "Anywhere" the four words Lienard, polynomial, differential, equation or system, it appears now 155 references.

Of course the differential equation (1) of second order can be written as the following differential system of first order

$$\dot{x} = -y, \qquad \dot{y} = x + g_2(x) - f(x)y. \tag{2}$$

Clearly the origin $O = (0,0)$ is a singular point of the differential system (2). We say that O is a *center* if there exists a neighborhood U of O such that $U\setminus\{O\}$ is filled of periodic orbits. The maximal connected set of periodic orbits surrounding the center O and having O in its boundary is called the *period annulus* of the center O. Moreover we say that O is an *isochronous center* if O is a center and all the periodic orbits of its period annulus have the same period.

The notion of a center appeared already in the works of Poincaré [1] in 1881 and Dulac [2] in 1908. Informally the notion of an isochronous center goes back to Huygens, who in 1656 invented the pendulum clock, see [3,4].

The next conjecture was stated by Algaba et al. in [5].

Conjecture 1. *The origin O of the Liénard polynomial differential system (2) with the non-zero polynomials $f(x)$ and $g_2(x)$ is an isochronous center if and only if the polynomial $f(x)$ is an odd function of degree $d \geq 3$ and the polynomial*

$$g_2(x) = \frac{1}{x^3}\left(\int_0^x s f(s) ds\right)^2. \tag{3}$$

Doing the change of coordinates $(x,y) \to (X,Y)$ where

$$X = x, \qquad Y = y - xh(x), \quad \text{with} \quad h(x) = \frac{1}{x^2}\int_0^x s f(s) ds, \tag{4}$$

system (2) becomes

$$\dot{X} = -Y - Xh(X), \qquad \dot{Y} = X - Yh(X) + g_2(X) - Xh(X)^2. \tag{5}$$

Under condition (3) system (5) writes

$$\dot{x} = -y - xh(x), \qquad \dot{y} = x - yh(x), \tag{6}$$

where instead of writing (X,Y) we have written (x,y). It is well known (see for instance [6]) that the differential systems (6) have a uniform isochronous center at the origin of coordinates, thus the origin is the unique singular point.

The objective of this paper to classify all phase portraits of the Liénard differential systems (2) having an isochronous center, or equivalent of the differential systems (6). In fact, we shall classify the phase portraits of system (6) in the Poincaré disc.

We recall that the *Poincaré disc* $\mathbb{D}^2$ is the unit closed disc at the origin of coordinates with interior diffeomorphic to $\mathbb{R}^2$ and whose boundary $\mathbb{S}^1$ is identified with the infinity of $\mathbb{R}^2$. Any polynomial differential system, as the systems (6), can be extended analytically to $\mathbb{D}^2$. This

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