THE 16TH HILBERT PROBLEM FOR DISCONTINUOUS PIECEWISE LINEAR DIFFERENTIAL SYSTEMS SEPARATED BY THE ALGEBRAIC CURVE $y = x^n$

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Abstract. We consider planar piecewise discontinuous differential systems formed by either linear centers or linear Hamiltonian saddles and separated by the algebraic curve $y = x^n$ with $n \geq 2$. We provide in a very short way an upper bound of the number of limit cycles that these differential systems can have in terms of n, proving the extended 16th Hilbert problem in this case. In particular, we show that for $n = 2$ this bound can be reached.

1. Introduction and statement of the main result

The dynamics of piecewise linear differential systems started with Andronov et al [1] in 1930. Since then and taking into account that they are used to model many phenomena in mechanics, electronics, economy, neuroscience, ..., see for more details the papers [3, 18, 20, 21, 36, 38, 40, 43, 42] and the references therein.

One of the main problems in the qualitative theory of differential equations is to bound the number of the *limit cycles*, which are periodic orbits isolated in the set of all periodic orbits of the differential system. This problem on the number of limit cycles for a class of given differential systems restricted to polynomial differential systems is the famous 16th Hilbert's problem, see for more details in [14, 17, 22]. Although this problem was formulated originally for smooth differential systems in the last years, many authors have studied this problem for different classes of discontinuous piecewise linear differential systems in $\mathbb{R}^2$.

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The simplest piecewise linear differential systems in $\mathbb{R}^2$ are the ones having only two pieces separated by a curve, and when this curve is a straight line. For this class of discontinuous piecewise differential systems many results have been obtained (see [2, 4, 6, 7, 9, 10, 11, 12, 13, 15, 16, 23, 26, 27, 28, 31, 34, 39]) and up to now there are examples of planar discontinuous piecewise linear differential systems separated by a straight line having 3 crossing limit cycles (see below for a precise definition), but it is an open problem to know if 3 is in fact the maximum number of crossing limit cycles that these systems can have.

When the curve of discontinuity between the two pieces is not a straight line the number of crossing limit cycles can change and this number can increase arbitrarily with the number of oscillations of that curve (see for details [5, 19, 24, 25, 29, 37, 44]). We recall however that the discontinuity curves considered in these three papers are not algebraic and we want to study the case of algebraic curves. More precisely we want to study the maximum number of crossing limit cycles of discontinuous piecewise linear differential systems with only two pieces in function of the degree of the discontinuity curve when this curve is algebraic. However, since there are many computations involved and too many linear systems we will restrict our study to differential systems formed by two linear differential systems having only centers or Hamiltonian saddles and such that the algebraic curve is of the form $y = x^n$ where $n \geq 2$. When $n = 1$ and so the discontinuity curve is a straight line and both systems are either linear centers or linear Hamiltonian saddles, it was proved in [30, 32, 33] that they do not have limit cycles. So we restrict to the cases $n \geq 2$.

For a discontinuous piecewise differential system we follow the Filippov's convention to define the vector fields on its discontinuous boundary (see [8]). Thus given a discontinuous piecewise differential system $\dot{x} = F_{\pm}(x)$, $x \in \Sigma_{\pm} := \mathbb{R}^2 \setminus \Sigma$, where Σ is the line of discontinuity, $\Sigma_{\pm}$ are open sets and $F_{\pm}$ are continuous functions in $\Sigma_{\pm}$ respectively. The vector field at each point of $\Sigma_{\pm}$ is defined by $F_{\pm}$, respectively. We call a point $x_0 \in \Sigma$ a *crossing point* if $F_{\pm}(x_0)$ point into Σ_+ or Σ_- simultaneously. In this case an orbit of the system near x_0 is a concatenation of the orbits of the two subsystems. The collection of all crossing points forms the crossing region. If a closed curve is formed by concatenating the orbits of the two subsystems and it intersects with Σ only at crossing points then we call the closed curve a *crossing periodic orbit*. The so-called *crossing limit cycle* is an isolated crossing periodic orbit in the set of all crossing periodic orbits of the system.

Our main theorem is the following.

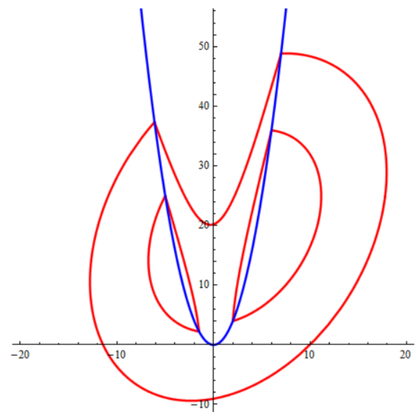

FIGURE 1. The three limit cycles of the discontinuous piecewise linear differential system formed by a linear Hamiltonian saddle and a linear center and separated by the parabola $y = x^2$.

Theorem 1. *Let $n \geq 2$ be a positive integer. Consider discontinuous piecewise differential systems in $\mathbb{R}^2$ separated by the algebraic curve $y = x^n$ and formed by linear centers or linear Hamiltonian saddles. For these systems we denote by $U(n)$ the upper bound for the maximum number of crossing limit cycles that intersect the line of discontinuity in two points.*

(a) *For $n > 2$, we have*

$$U(n) = \begin{cases} \dfrac{2n^2 - n - 1}{2} & \text{if } n \text{ is odd,} \\ \dfrac{2n^2 - n}{2} & \text{if } n \text{ is even.} \end{cases}$$

(b) *When $n = 2$ there are discontinuous piecewise linear differential systems which reach the upper bound of 3 limit cycles, see Figure 1.*

Theorem 1 is proved in section 2. We recall that in the particular case in which the two linear systems are centers the upper bound $U(n)$ for $n \geq 3$ was already obtained in [35] with the additional assumption that one of the linear centers is in a given particular normal form. We point out that this is not the general case of a discontinuous piecewise system with two linear centers separated by the algebraic curve $y = x^n$ since not all of them can be brought into a canonical normal form without destroying the curve $y = x^n$. Also in [35] the example for $n = 2$ that they provide only has two limit cycles instead of three as claimed by the authors and so it is not a rigorous proof that the upper bound provided by the main theorem in this case is reached. Also in [35] the proof of the main theorem requires more than 12 pages and is very complicated needing to preform a big amount of computations to obtain the degree of a resultant which makes it at some point hard to follow. In the present paper we generalize the study done in [35] in several directions: we study the case in which the two linear systems can be either centers or linear Hamiltonian saddles and thus not only we provide a rigorous proof in the case of both systems being centers (since we treat the general case without the assumption that one of them is in normal form) but we also consider all the other situations in which one of the linear systems can be a linear Hamiltonian saddle or both linear differential systems are Hamiltonian saddles. Moreover we provide a rigorous example of a system with three crossing limit cycles and so we provide a rigorous proof that when $n = 2$ the upper bound given in the main theorem is indeed reached. Finally we stress that our proof is very short.

We remark that from Theorem 1 it is an open question if for $n \geq 3$ the upper bounds provided are reached.

The tools used in this paper for providing upper bound on the number of limit cycles of discontinuous piecewise differential systems in the plane can be also used for all the discontinuous piecewise differential systems in dimension two or higher, if we know first integrals for the differential systems forming the piecewise differential system.

2. Proof of Theorem 1

Through the proof of Theorem 1 we will use the following results which provide a normal form for a general linear differential Hamiltonian saddle (for a proof see [32]) and for a general linear center (for a proof see [30]).

Lemma 2. *Any linear differential system having a Hamiltonian saddle can be written as*

$$\dot{x} = -bx - \delta y + d, \quad \dot{y} = \alpha x + by + c,$$

with $\alpha \in \{0, 1\}$, $b, \delta, c, d \in \mathbb{R}$. *Moreover, if* $\alpha = 1$ *then* $\delta = b^2 - \omega$ *with* $\omega > 0$ *and if* $\alpha = 0$ *then* $b = 1$. *A first integral of this system is*

$$H(x, y) = -\frac{\alpha}{2}x^2 - bxy - \frac{\delta}{2}y^2 - cx + dy. \tag{1}$$

Proposition 3. *Any linear differential system having a center can be written as*

$$\dot{x} = -\overline{b}x - \overline{\delta}y + \overline{d}, \quad \dot{y} = x + \overline{b}y + \overline{c}, \tag{2}$$

where $\overline{\delta} = \overline{b}^2 + \overline{\omega}$ *with* $\overline{\omega} > 0$. *A first integral of system* (2) *is*

$$F(x,y) = -\frac{1}{2}x^2 - \overline{b}xy - \frac{\overline{\delta}}{2}y^2 - \overline{c}x + \overline{d}y. \tag{3}$$

Note that any of the Hamiltonians (1) and (3) can be written as

$$G(x,y) = -\frac{A}{2}x^2 - Bxy - \frac{\Delta}{2}y^2 - Cx + Dy,$$

where $A = 1$ and $\Delta = B^2 + \omega$ with $\omega > 0$ if we have a linear center, and in the case we have a linear Hamiltonian saddle then $A \in \{0, 1\}$, so that if $A = 1$ then $\Delta = B^2 - \omega$ with $\omega > 0$, and if $A = 0$ then $B = 1$ and $\Delta \in \mathbb{R}$. Moreover we will consider the regions $\mathcal{R}_1$ and $\mathcal{R}_2$ where

$$\mathcal{R}_1 = \{(x,y) \in \mathbb{R}^2 : y \geq x^n\}, \quad \mathcal{R}_2 = \{(x,y) \in \mathbb{R}^2 : y \leq x^n\}.$$

Proof of statement (*a*) *Theorem 1.* In order that the piecewise linear differential system formed by two linear differential systems with Hamiltonian G_1 in the region $\mathcal{R}_1$ and Hamiltonian G_2 in the region $\mathcal{R}_2$ has a crossing limit cycle Γ, it must intersect the discontinuous curve $y = x^n$ in two points. Let (x_1, x_1^n) and (x_2, x_2^n) be these two intersecting points. Then, taking into account that G_1 and G_2 are first integrals these two points must satisfy the equations

$$\begin{aligned} e_1 &:= G_1(x_1, x_1^n) - G_1(x_2, x_2^n) = 0, \\ e_2 &:= G_2(x_1, x_1^n) - G_2(x_2, x_2^n) = 0, \end{aligned} \tag{4}$$

where $G_i = -\frac{A_i}{2}x^2 - B_ixy - \frac{\Delta_i}{2}y^2 - C_ix + D_iy$ for $i = 1, 2$.

Note that e_1 and e_2 are two polynomials of degree $2n$ in the variables x_1 and x_2. Clearly $x_1 - x_2$ is a factor of both e_1 and e_2. Note that

$$e_1 = \Delta_1(x_2^{2n} - x_1^{2n}) + 2B_1(x_2^{n+1} - x_1^{n+1}) + 2D_1(x_1^n - x_2^n) + A_1(x_2^2 - x_1^2) + 2C_1(x_2 - x_1),$$

and

$$e_2 = \Delta_2(x_2^{2n} - x_1 2n) + 2B_2(x_2^{n+1} - x_1^{n+1}) + 2D_2(x_1^n - x_2^n) + A_2(x_2^2 - x_1^2) + 2C_2(x_2 - x_1).$$

Some direct calculations show that $E_1 = e_1/(x_2-x_1)$ and $E_2 = e_2/(x_2-x_1)$ are given by

$$
\begin{aligned}
E_1 = & \Delta_1 \left(x_1^{2n-1} + x_2 x_1^{2n-2} + \ldots + x_1 x_2^{2n-2} + x_2^{2n-1}\right) \\
& + 2B_1 \left(x_1^n + x_2 x_1^{n-1} + \ldots + x_1 x_2^{n-1} + x_2^n\right) \\
& - 2D_1 \left(x_1^{n-1} + x_2 x_1^{n-2} + \ldots + x_1 x_2^{n-2} + x_2^{n-1}\right) \\
& + A_1(x_1+x_2) + 2C_1
\end{aligned}
$$

and

$$
\begin{aligned}
E_2 = & \Delta_2 \left(x_2^{2n-1} + x_1 x_2^{2n-2} + \ldots + x_2 x_1^{2n-2} + x_1^{2n-1}\right) \\
& + 2B_2 \left(x_2^n + x_1 x_2^{n-1} + \ldots + x_2 x_1^{n-1} + x_1^n\right) \\
& - 2D_2 \left(x_2^{n-1} + x_1 x_2^{n-2} + \ldots + x_2 x_1^{n-2} + x_1^{n-1}\right) \\
& + A_2(x_2+x_1) + 2C_2.
\end{aligned}
$$

Note that

$$
\begin{aligned}
E_1 - \frac{\Delta_1}{\Delta_2} E_2 = & \Delta_1 \big(2(B_1 - B_2) \left(x_1^n + x_2 x_1^{n-1} + \ldots + x_1 x_2^{n-1} + x_2^n\right) \\
& - 2(D_1 - D_2) \left(x_1^{n-1} + x_2 x_1^{n-2} + \ldots + x_1 x_2^{n-2} + x_2^{n-1}\right) \\
& + (A_1 - A_2)(x_1 + x_2) + 2(C_1 - C_2)\big),
\end{aligned}
$$

which is a polynomial of degree n.

In order to provide an upper bound for the number of limit cycles of our planar discontinuous piecewise differential systems, we note that for each solution (x_1, x_2) with $x_1 \neq x_2$ of the polynomial system $E_1 = E_2 = 0$, we can have a crossing limit cycle of the discontinuous piecewise linear differential system which intersects the curve $y = x^n$ at the two points (x_1, x_1^n) and (x_2, x_2^n). If (x_1, x_2) is a solution of the polynomial system $E_1 = E_2 = 0$, with $x_1 \neq x_2$, then it is also a solution of the polynomial system $E_1 - \Delta_1 E_2/\Delta_2 = E_2 = 0$. But system $E_1 = 0$ and $E_2 = 0$ have the same roots because these two polynomials are the same (we pass from one to the other interchanging x_1 and x_2) and so the possible values of x_1 and x_2 are the same. This implies that the number of crossing limit cycles that intersect the curve $y = x^n$ is at most the number of solutions of the system $E_1 - \Delta_1 E_2/\Delta_2 = E_2 = 0$ divided by two. Since E_1 has degree $2n-1$ and $E_1 - \Delta_1 E_2/\Delta_2$ has degree n it follows from Bézout's theorem that the system $E_1 = E_2 = 0$ has at most $2n^2 - n$ isolated solutions, and by the observation above the number of crossing limit cycles that intersect the curve $y = x^n$ is at most the integer part of $(2n^2-2)/2$, that is $(2n^2-n)/2$ if n is even and

$(2n^2 - n - 1)/2$ if n is odd. This completes the proof of the statement (a) of theorem 1 $\square$

Proof of statement (b) *Theorem 1.* We provide an example showing that the upper bound $U(2) = 3$ is reached. In the region R_1 we consider the linear center

$$\dot{x} = \frac{21145}{522} + \frac{4}{3}x - \frac{20}{9}y, \quad \dot{y} = \frac{508}{87} + 8x - \frac{4}{3}y,$$

with the first integral

$$G_1(x, y) = \frac{1}{522}(3048x + 2088x^2 - 21145y - 696xy + 580y^2).$$

In the region R_2 we consider the linear Hamiltonian saddle

$$\dot{x} = 2x - \frac{2464}{663}y, \quad \dot{y} = -\frac{81322}{663}x - 2y$$

with the first integral

$$G_2(x, y) = -\frac{40661}{663}x^2 - 2xy + \frac{1232}{663}y^2.$$

This discontinuous piecewise differential system has three crossing limit cycles, because the unique real solutions (x_1, x_2) of system (4) (with $n = 2$) are $(x_1^1, x_2^1) = (6, 2)$, $(x_1^2, x_2^2) = (-5, -3/2)$ and (x_1^3, x_2^3) where

$$x_1^3 = \frac{1}{116}(51 - \sqrt{577921}) \quad \text{and} \quad x_2^3 = \frac{1}{116}(51 + \sqrt{577921}).$$

See these three crossing limit cycles in Figure 1, which are travelled in couterclockwise sense. $\square$

Data availability statement

This manuscript has no associated data.

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