INVARIANTS OF POLYNOMIAL VECTOR FIELDS

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ABSTRACT. We characterize the existence of first integrals and invariants (first integrals depending on the time) for the polynomial vector fields which are invariant under an involution.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

Let X be a polynomial vector field in $\mathbb{R}^m$ of degree $d + 1$. Given a polynomial $F(x, y)$, the algebraic hypersurface $F(x, y) = 0$ is an *invariant algebraic hypersurface* of X if there exists a polynomial $K(q, p)$ called *the cofactor* such that $XF = KF$. In this paper we will assume that
$$K(x, y) = \alpha + \bar{K}(x, y),$$
being $\alpha \in \mathbb{R}$ and $\bar{K}(x, y)$ a polynomial without constant terms. It can be proved that K has degree at most d.

A function $G = G(x, y)$ of the form $G = \exp(g/h)$ with $g = g(x, y)$ and $h = h(x, y)$ coprime polynomials is an *exponential factor* of the polynomial vector field X if there exists a polynomial $L(x, y)$ called *the cofactor* such that $XG = LG$. The exponential factors appear when some invariant algebraic hypersurface has multiplicity larger than one, for more details see [5, 12].

In this paper we assume that
$$L(x, y) = \beta + \bar{L}(x, y),$$
being $\beta \in \mathbb{R}$ and $\bar{L}(x, y)$ a polynomial without constant terms.

Given an invariant algebraic hypersurface $F(x, y) = 0$, the polynomial F is called a *Darboux polynomial*. The notion of Darboux polynomial was introduced by Darboux in [6] in the plane to study the existence of first integrals in polynomial systems. Since its appearance in 1878 the so-called Darboux theory has been intensively studied by many authors from different points of view, see for instance [1, 2, 3, 4, 8, 9, 13, 14, 15, 16, 17, 18]. In particular it has been developed an extension of the Darboux theory for polynomial differential systems in $\mathbb{R}^m$, in hypersurfaces, and so on. For a good survey on the Darboux theory we refer the reader to [10].

Let U be an open subset of $\mathbb{R}^m$ such that its closure is $\mathbb{R}^m$. We say that a function $H : U \to \mathbb{R}^m$ is a *first integral* of the polynomial vector field X if $XH = 0$ on U. We

say that the function $I\colon U \times \mathbb{R} \to \mathbb{R}^m$ is an *invariant* of the polynomial vector field X if $XI + \partial_t I = 0$ on $U \times \mathbb{R}$. Note that a first integral is an invariant which does not depend on t.

Let $\tau\colon \mathbb{R}^m \to \mathbb{R}^m$ be an involution, i.e., $\tau \circ \tau = \mathrm{Id}$, being Id the identity map. We say that the vector field X is τ-*reversible* if $\tau_*(X) = -X$, and is τ-*equivariant* if $\tau_*(X) = X$, where τ_* is the push-forward associated to τ.

The first main result of the paper is the following one concerning with the existence of first integrals and invariants in function of the existence of Darboux polynomials.

Theorem 1. *Consider a polynomial vector field X and let τ be an involution. Let $F = 0$ be an invariant algebraic hypersurface of X with cofactor K. Then the following statements hold.*

  (i) *If X is τ-reversible, then $F(F \circ \tau)$ is a polynomial first integral of the vector field X.*
 (ii) *If X is τ-reversible, $\alpha \neq 0$, $K \circ \tau = \alpha + \bar{K} \circ \tau = \alpha - \bar{K}$ and $F = 0$ is different from $F \circ \tau = 0$, then $F/(F \circ \tau)e^{-2\alpha t}$ is an invariant of the vector field X.*
(iii) *If X is τ-equivariant, $K \circ \tau = K$ and $F = 0$ is different from $F \circ \tau = 0$, then $F/(F \circ \tau)$ is a rational first integral of the vector field X.*
 (iv) *If X is τ-equivariant, $\alpha \neq 0$ and $K \circ \tau = \alpha + \bar{K} \circ \tau = \alpha - \bar{K}$ then $F(F \circ \tau)e^{-2\alpha t}$ is an invariant of the vector field X.*

The proof of Theorem 1 is given in section 4. Statements (i) and (iii) of Theorem 1 for the particular case of X being a Hamiltonian vector field and the involution τ is either $\tau(x, y) = (x, -y)$ or $\tau(x, y) = (-x, y)$ was proved, respectively in, Theorem 2.1 and Theorem 3.1 in [11]

Let $\tau\colon \mathbb{R}^m \to \mathbb{R}^m$ be an involution. A polynomial vector field X is called *time-reversible with respect to τ* if it is σ-reversible with $\sigma\colon \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}$ defined by $\sigma(x, y, t) = (\tau(x, y), -t)$. A polynomial vector field X is called *time-equivariant with respect to τ* if it is σ-equivariant with $\sigma\colon \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}$ defined by $\sigma(x, y, t) = (\tau(x, y), -t)$.

With these two notions we can state the second main result of this paper which is a similar result as Theorem 2 in the case of time-reversibility and time-equivariance.

Theorem 2. *Consider a polynomial vector field X and let τ be an involution. Let $F = 0$ be an invariant algebraic hypersurface of X with cofactor K. Then the following statements hold.*

  (i) *If X is time-reversible with respect to τ, $K \circ \tau = K$ and $F = 0$ is different from $F \circ \tau = 0$, then $F(F \circ \tau)$ is a polynomial first integral of the vector field X.*
 (ii) *If X is time-reversible with respect to τ, $\alpha \neq 0$ and $K \circ \tau = \alpha + \bar{K} \circ \tau = \alpha - \bar{K}$, then $F/(F \circ \tau)e^{-2\alpha t}$ is an invariant of the vector field X.*
(iii) *If X is time-equivariant with respect to τ, $K \circ \tau = K$ and $F = 0$ is different from $F \circ \tau = 0$, then $F/(F \circ \tau)$ is a rational first integral of the vector field X.*
 (iv) *If X is time-equivariant with respect to τ, $\alpha \neq 0$ and $K \circ \tau = \alpha + \bar{K} \circ \tau = \alpha - \bar{K}$, then $F(F \circ \tau)e^{-2\alpha t}$ is an invariant of the vector field X.*

The proof of Theorem 2 is very similar to the proof of Theorem 1 and so we omit it. We recall that statements (i) and (iii) of Theorem 2 for the particular case of X being a Hamiltonian vector field and the involution τ is either $\tau(x, y) = (x, -y)$ or $\tau(x, y) = (-x, y)$ was proved, respectively in, Theorem 2.4 and Theorem 3.3 in [11].

The third main result of this paper is a similar result to Theorem 1 in the case of existence of exponential factors instead of Darboux polynomials.

Theorem 3. *Consider a polynomial vector field X and let τ be an involution. Let G be an exponential factor of X with cofactor L. Then the following statements hold.*

  (i) *If X is τ-reversible, $L \circ \tau = L$ and $G(G \circ \tau) \notin \mathbb{R}$, then $\log(G(G \circ \tau))$ is a rational first integral of the vector field X.*
 (ii) *If X is τ-reversible, $\beta \neq 0$, $L \circ \tau = \beta + \bar{L} \circ \tau = \beta - \bar{L}$ and $G/(G \circ \tau) \notin \mathbb{R}$, then $\log(G/(G \circ \tau)) - 2\beta t$ is an invariant of the vector field X.*
(iii) *If X is τ-equivariant, $L \circ \tau = L$ and $G/(G \circ \tau) \notin \mathbb{R}$, then $\log(G/(G \circ \tau))$ is a rational first integral of the vector field X.*
 (iv) *If X is τ-equivariant, $\beta \neq 0$, $L \circ \tau = \beta + \bar{L} \circ \tau = \beta - \bar{L}$ and $G(G \circ \tau) \notin \mathbb{R}$, then $\log(G(G \circ \tau)) - 2\beta t$ is an invariant of the vector field X.*

The proof of Theorem 3 is given in section 4. Finally in the case of time-reversibility and time-equivariance we have the following result also using exponential factors. The proof of such result is very similar to the proof of Theorem 3 and so we omit it.

Theorem 4. *Consider a polynomial vector field X and let τ be an involution. Let G be an exponential factor of X with cofactor L. Then the following statements hold.*

  (i) *If X is time-reversible with respect to τ, $L \circ \tau = L$ and $G(G \circ \tau) \notin \mathbb{R}$, then $\log(G(G \circ \tau))$ is a rational first integral of the vector field X.*
 (ii) *If X is time-reversible with respect to τ, $\beta \neq 0$, $L \circ \tau = \beta + \bar{L} \circ \tau = \beta - \bar{L}$ and $G/(G \circ \tau) \notin \mathbb{R}$, then $\log(G/(G \circ \tau)) - 2\beta t$ is an invariant of the vector field X.*
(iii) *If X is time-equivariant with respect to τ, $L \circ \tau = L$ and $G/(G \circ \tau) \notin \mathbb{R}$, then $\log(G/(G \circ \tau))$ is a rational first integral of the vector field X.*
 (iv) *If X is time-equivariant with respect to τ, $\beta \neq 0$, $L \circ \tau = \beta + \bar{L} \circ \tau = \beta - \bar{L}$ and $G(G \circ \tau) \notin \mathbb{R}$, then $\log(G(G \circ \tau)) - 2\beta t$ is an invariant of the vector field X.*

We can also combine the main theorems and obtain results in the case of polynomial vector fields with Darboux polynomials and exponential factors at the same time. This is the content of the last main result of the paper.

Theorem 5. *Let τ be an involution and consider a polynomial vector field X that is either τ-reversible or τ-equivariant. Let $F = 0$ and G be an invariant algebraic hypersurface and an exponential factor of X with cofactors K and L, respectively. If*

$$\delta_1 K + \delta_2(K \circ \tau) + \delta_3 L + \delta_4(L \circ \tau) = s, \quad s \in \mathbb{R},$$

with $\delta_i \in \{-1, 1\}$ for $i = 1, 2, 3, 4$ and

$$F^{\delta_1}(F \circ \tau)^{\delta_2} G^{\delta_3}(G \circ \tau)^{\delta_4} \notin \mathbb{R},$$

then

$$I = F^{\delta_1}(F \circ \tau)^{\delta_2} G^{\delta_3}(G \circ \tau)^{\delta_4} e^{-st}$$

is an invariant if $s \neq 0$ and a first integral if $s = 0$.

We provide examples of statements (i)–(iv) in Theorems 1 and 3 in section 2. In this way we are showing that these theorems are not empty. Examples of statements (i)–(iv) in Theorems 2 and 4 can be obtained in a similar way.

We want to stress that the main theorems Theorem 1–5 can be extended without any modification to polynomial vector fields of $\mathbb{C}^m$ instead of $\mathbb{R}^m$.

2. EXAMPLES

Consider the polynomial vector field

$$\dot{x} = 2y^3, \quad \dot{y} = x.$$

Note that this vector field is τ-reversible with $\tau(x, y) = (-x, y)$. Moreover $F(x, y) = x - y^2 = 0$ is an invariant algebraic curve with cofactor $K(x, y) = -2y$. Note that $K \circ \tau = K$. Then in view of Theorem 1(i) we have that $H = (x - y^2)(x + y^2)$ is a first integral of the polynomial vector field.

Consider now the polynomial vector field

$$\dot{x} = x^2 - 1, \quad \dot{y} = 1.$$

Note that this vector field is τ-reversible with $\tau(x, y) = (-x, -y)$. Moreover $F(x, y) = x - 1 = 0$ is an invariant algebraic curve with cofactor $K(x, y) = 1 + x$. Note that $\alpha = 1$, $K \circ \tau = 1 - x$ and $F = x - 1 = 0$ is different from $F \circ \tau = 1 + x = 0$. Then in view of Theorem 1(ii) we have that $I = (x - 1)e^{-2t}/(x + 1)$ is an invariant of the polynomial vector field.

Take the polynomial vector field

$$\dot{x} = 4x, \quad \dot{y} = 2y.$$

Note that this vector field is τ-equivariant with $\tau(x, y) = (-x, y)$. Moreover $F(x, y) = x - y^2 = 0$ is an invariant algebraic curve with cofactor $K(x, y) = 4$. Note that $K \circ \tau = K$ and that $F = x - y^2 = 0$ is different from $F \circ \tau = x + y^2 = 0$. Then in view of Theorem 1(iii) we have that $H = (x - y^2)/(x + y^2)$ is a first integral of the polynomial vector field.

Consider the polynomial vector field

$$\dot{x} = x, \quad \dot{y} = 1.$$

Note that this vector field is τ-equivariant with $\tau(x, y) = (-x, y)$. Moreover $F(x, y) = x = 0$ is an invariant algebraic curve of the polynomial vector field with cofactor $K(x, y) = 1$. Note that $\alpha = 1$ and $K \circ \tau = 1$. Then in view of Theorem 1(iv) we have that $I = x^2 e^{-2t}$ is an invariant of the polynomial vector field.

Take the polynomial vector field

$$\dot{x} = -2y^3, \quad \dot{y} = -x.$$

Note that this vector field is τ-reversible with $\tau(x, y) = (-x, y)$. Moreover $G(x, y) = e^{x-x^2+y^4}$ is an exponential factor with cofactor $L(x, y) = -2y^3$. Note that $L \circ \tau = L$ and $G(G \circ \tau) = e^{2(y^4-x^2)} \notin \mathbb{R}$. Then in view of Theorem 3(i) we have that $H = e^{2(y^4-x^2)}$ is a first integral of the polynomial vector field.

Now consider the polynomial vector field

$$\dot{x} = x^2, \quad \dot{y} = 1.$$

Note that this vector field is τ-reversible with $\tau(x, y) = (-x, -y)$. Moreover $G(x, y) = e^{1/x}$ is an exponential factor with cofactor $L(x, y) = -1$. Note that $\beta = -1$, $L \circ \tau = -1$ and $G/(G \circ \tau) = e^{2/x} \notin \mathbb{R}$. Then in view of Theorem 3(ii) we have that $I = \frac{2}{x} + 2t$ is an invariant of the polynomial vector field.

Take the polynomial vector field

$$\dot{x} = 2xy, \quad \dot{y} = 1 - y^2.$$

Note that this vector field is τ-equivariant with $\tau(x, y) = (-x, y)$. Moreover $G(x, y) = e^{x-xy^2+y^3}$ is an exponential factor with cofactor $L(x, y) = -y^2(y^2 - 1)$. Note that $L \circ \tau = L$ and $G/(G \circ \tau) = e^{2x-2xy^2} \notin \mathbb{R}$ Then in view of Theorem 3(iii) we have that $H = e^{2x(1-y^2)}$ is a first integral of the polynomial vector field.

Finally consider the polynomial vector field

$$\dot{x} = x, \quad \dot{y} = 1.$$

Note that this vector field is τ-equivariant with $\tau(x, y) = (-x, y)$. Note that $G(x, y) = e^y$ is an exponential factor with cofactor $L(x, y) = 1$. Moreover $\beta = 1$, $L \circ \tau = 1$ and $G(G \circ \tau) = e^{2y} \notin \mathbb{R}$. In view of Theorem 3(iv) we have that $I = 2y - 2t$ is an invariant of the polynomial vector field.

3. Preliminaries

In this section we state and prove some auxiliary results that will be used in the proof of Theorems 1 and 2. The statement of the next proposition is well-known, for a proof see [7, Theorem 8.7].

Proposition 6. *Consider a polynomial vector field X and let $F_i = 0$, G_j for $i = 1, \ldots, r$ and $j = 1, \ldots, s$ be invariant algebraic hypersurfaces of X with cofactors K_i and exponential factors of X with cofactors L_j, respectively. Assume that*

$$\sum_{i=1}^{r} \ell_i K_i + \sum_{j=1}^{s} l_j L_j = \lambda, \quad \lambda \in \mathbb{R}.$$

Then $F_1^{\ell_1} \cdots F_r^{\ell_r} G_1^{l_1} \cdots G_s^{l_s} e^{-\lambda t}$ is an invariant of the polynomial vector field X.

Proposition 7. *Consider a polynomial vector field X and let τ be an involution. Let $F = 0$ be an invariant algebraic hypersurface of X with cofactor K. The following statements hold:*

(i) *If X is τ-reversible then $F \circ \tau$ is a Darboux polynomial with cofactor $-K \circ \tau$.*

(ii) *If X is τ-equivariant then $F \circ \tau$ is a Darboux polynomial with cofactor $K \circ \tau$.*

Proof. We will prove both statements together. Taking into account that $XF = KF$, we have

$$\tag{1} \tau_*(XF) = \tau_*(KF)$$

Using that $\tau^{-1} = \tau$, the right-hand side of (1) is

$$\tag{2} \tau_*(KF) = (KF) \circ \tau^{-1} = (KF) \circ \tau = (K \circ \tau)(F \circ \tau).$$

On the other hand, using again that $\tau^{-1} = \tau$ and that $\tau_*(X) = -X$ if X is τ-reversible, and $\tau_*(X) = X$ if X is τ-equivariant, we have

$$\tag{3} \tau_*(XF) = \pm(XF) \circ \tau^{-1} = \pm(XF) \circ \tau = \pm X(F \circ \tau),$$

where $+$ stands for τ-equivariance and $-$ stands for τ-reversibility. Using (2) and (3) we have

$$X(F \circ \tau) = \pm(K \circ \tau)(F \circ \tau).$$

So $F \circ \tau = 0$ is an invariant algebraic hypersurface with cofactor $\pm K \circ \tau$. This concludes the proof of the proposition. $\square$

With the same proof we can also prove the following proposition (the proof is omitted).

Proposition 8. *Consider a polynomial vector field X and let τ be an involution. Let G be an exponential factor of X with cofactor L. The following statements hold:*

(i) *If X is τ-reversible then $G \circ \tau$ is an exponential factor with cofactor $-L \circ \tau$.*
(ii) *If X is τ-equivariant then $G \circ \tau$ is an exponential factor with cofactor $L \circ \tau$.*

4. PROOF OF THE MAIN RESULTS

In this section we prove Theorems 1 and 3.

4.1. **Proof of Theorem 1.** Let $F = 0$ be an invariant algebraic hypersurface of a polynomial vector field X which is τ-reversible. In view of Proposition 7 and statement (i) of Theorem 1 we have

$$X(F \circ \tau) = -(K \circ \tau)(F \circ \tau) = -K(F \circ \tau).$$

Therefore

$$\begin{aligned} X(F(F \circ \tau)) &= XF(F \circ \tau) + FX(F \circ \tau) \\ &= KF(F \circ \tau) - FK(F \circ \tau) = 0, \end{aligned}$$

and so $F(F \circ \tau)$ is a polynomial first integral. Statement (i) is proved.

On the other hand, in view of Proposition 7 and statement (ii) of Theorem 1 we have

$$X(F \circ \tau) = -(K \circ \tau)(F \circ \tau) = (-\alpha + \bar{K})(F \circ \tau).$$

Therefore

$$X\Big(\frac{F}{F\circ\tau}\Big)e^{-2\alpha t} + \Big(\frac{F}{F\circ\tau}\Big)\partial_t e^{-2\alpha t}$$

$$= \frac{1}{(F\circ\tau)^2}\big(XF(F\circ\tau)e^{-2\alpha t} - FX(F\circ\tau)e^{-2\alpha t} - 2\alpha F(F\circ\tau)e^{-2\alpha t}\big)$$

$$= \frac{e^{-2\alpha t}}{(F\circ\tau)^2}\big((\alpha+\bar{K})F(F\circ\tau) + (\alpha-\bar{K})F(F\circ\tau) - 2\alpha F(F\circ\tau)\big)$$

$$= (\alpha + \bar{K} + \alpha - \bar{K} - 2\alpha)\frac{F(F\circ\tau)e^{-2\alpha t}}{(F\circ\tau)^2} = 0,$$

and so $Fe^{-2\alpha t}/(F\circ\tau)$ is an invariant of X. This concludes the proof of statement (ii).

Let $F = 0$ be an invariant algebraic hypersurface of a polynomial vector field X which is τ-equivariant. In view of Proposition 7 and statement (iii) of Theorem 1 we have

$$X(F\circ\tau) = (K\circ\tau)(F\circ\tau) = K(F\circ\tau).$$

Therefore

$$X\Big(\frac{F}{F\circ\tau}\Big) = \frac{1}{(F\circ\tau)^2}\big(XF(F\circ\tau) - FX(F\circ\tau)\big)$$

$$= \frac{1}{(F\circ\tau)^2}\big(KF(F\circ\tau) - KF(F\circ\tau)\big) = 0,$$

and so $F/(F\circ\tau)$ is a rational first integral of the polynomial vector field X. Statement (iii) is proved.

Finally in view of Proposition 7 and statement(iv) in Theorem 1 we have

$$X(F\circ\tau) = (K\circ\tau)(F\circ\tau) = (\alpha-\bar{K})(F\circ\tau).$$

Therefore

$$XF(F\circ\tau)e^{-2\alpha t} + F(F\circ\tau)\partial_t e^{-2\alpha t}$$

$$= XF(F\circ\tau)e^{-2\alpha t} + FX(F\circ\tau)e^{-2\alpha t} - 2\alpha F(F\circ\tau)e^{-2\alpha t}$$

$$= e^{-2\alpha t}\big((\alpha+\bar{K})F(F\circ\tau) + (\alpha-\bar{K})F(F\circ\tau) - 2\alpha F(F\circ\tau)\big)$$

$$= (\alpha + \bar{K} + \alpha - \bar{K} - 2\alpha)F(F\circ\tau)e^{-2\alpha t} = 0,$$

and so $F(F\circ\tau)e^{-2\alpha t}$ is an invariant of X. This concludes the proof of statement (iv) and concludes the proof of the theorem.

4.2. **Proof of Theorem 3.** Proceeding as in the proof of Theorem 1(i) using G instead of F we have that $X(G(G\circ\tau)) = 0$ and so

$$\exp\Big(\frac{f}{g}\Big)\exp\Big(\frac{f\circ\tau}{g\circ\tau}\Big)$$

is a first integral. Taking logarithms we conclude that

$$\log\big(G(G\circ\tau)\big) = \frac{f}{g} + \frac{f\circ\tau}{g\circ\tau}$$

is a rational first integral. This proves statement (i).

Proceeding as in the proof of Theorem 1(ii) using G instead of F we have that $X\big(G/(G\circ\tau)\big)e^{-2\beta t}=0$, and so

$$\exp\Big(\frac{g}{h}\Big)\exp\Big(-\frac{g\circ\tau}{h\circ\tau}\Big)\exp(-2\beta t)$$

is an invariant. Taking logarithms we conclude that

$$\log\big((G/(G\circ\tau))\exp(-2\beta t)\big)=\frac{g}{h}-\frac{g\circ\tau}{h\circ\tau}-2\beta t$$

is an invariant. This proves statement (ii).

Proceeding as in the proof of Theorem 1(iii) using G instead of F we have that $X\big(G/(G\circ\tau)\big)=0$, and so

$$\exp\Big(\frac{g}{h}\Big)\exp\Big(-\frac{g\circ\tau}{h\circ\tau}\Big)$$

is a first integral. Taking logarithms we conclude that

$$\log\big(G/(G\circ\tau)\big)=\frac{g}{h}-\frac{g\circ\tau}{h\circ\tau}$$

is a rational first integral. This proves statement (iii).

Finally proceeding as in the proof of Theorem 1(iv) using G instead of F we have that $X\big(G(G\circ\tau)\big)e^{-2\beta t}=0$, and so

$$\exp\Big(\frac{g}{h}\Big)\exp\Big(\frac{g\circ\tau}{h\circ\tau}\Big)\exp(-2\beta t)$$

is an invariant. Taking logarithms we conclude that

$$\log\big((G(G\circ\tau))\exp(-2\beta t)\big)=\frac{g}{h}+\frac{g\circ\tau}{h\circ\tau}-2\beta t$$

is an invariant. This proves statement (iv) and the proof of Theorem 3 is completed.

Acknowledgements

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