GLOBAL PHASE PORTRAITS OF THE GENERALIZED VAN DER POL SYSTEMS

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ABSTRACT. We consider the generalized van der Pol systems

$$\dot{x} = y, \quad \dot{y} = -x + (1 - x^2)f(y),$$

where $f \in \mathbb{R}[y]$. The classical van der Pol systems have $f(y) = y$. We first characterize when the origin of the generalized van der Pol systems is a center, and second we provide the global phase portraits in the Poincaré disc of the generalized van der Pol when $f(y) = a_1y + a_2y^2$ for all $a_1, a_2 \in \mathbb{R}$.

1. Introduction and statement of the main results

In this paper we deal with the generalized van der Pol systems

$$\dot{x} = y, \quad \dot{y} = -x + (1 - x^2)f(y), \tag{1}$$

where $f(y)$ is the polynomial $\sum_{i=1}^{n} a_i y^i$ with $n \geq 1$. The classical van der Pol equation has $f(y) = a_1y$. There is no doubt about the importance of this differential system and this is one of the reasons why it has been studied for so many authors. For instance, if one enters the four words van der Pol, differential, equation or system in MathSciNet, one would receive 768 articles at the time that paper is being written. For instance some four recent papers on variations on the van der Pol system are [3, 7, 8, 9].

The main two theorems of this paper are the following ones.

Theorem 1. *System (1) has a center at the origin if and only if the polynomial $f(y)$ is even.*

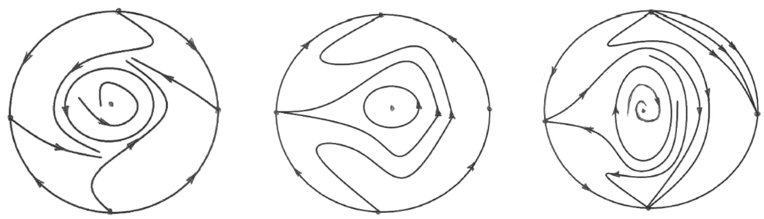
The proof of Theorem 1 will be given in section 3.

Theorem 2. *The global phase portrait of system (1) with $f(y) = a_1y + a_2y^2$ with $a_1, a_2 \in \mathbb{R}$ is topologically equivalent to the one of :*

(i) *Figure 1(a) if $a_2 = 0$ and $a_1 \neq 0$;*
(ii) *Figure 1(b) if $a_2 \neq 0$ and $a_1 = 0$.*

Moreover, there are systems (1) with $a_2a_1 \neq 0$ whose global phase portrait is topologically equivalent to the one of Figure 1(c).

2010 *Mathematics Subject Classification.* 34C05.
Key words and phrases. van der Pol systems, center, Lyapunov constant, Poincaré compactification, blow ups.



(a) $a_2 = 0$, $a_1 \neq 0$ (b) $a_2 \neq 0$, $a_1 = 0$ (c) $a_2 a_1 \neq 0$

FIGURE 1. The three phase portraits in the Poincaré disc of Theorem 2.

The proof of Theorem 2 will be given in section 4. It was proved in [6, 10] that system (1) with $a_2 a_1 = 0$ has always a unique limit cycle. Based in numerical evidence it seems that system (1) with $a_2 a_1 \neq 0$ has also a unique limit cycle, so we conjecture that system (1) has always a unique limit cycle which allows us to state the following conjecture on the global phase portraits of system (1) with $a_2 a_1 \neq 0$.

Conjecture 3. *The global phase portrait of system* (1) *with* $f(y) = a_1 y + a_2 y^2$ *and* $a_2 a_1 \neq 0$ *is topologically equivalent to the one of Figure* 1(*c*).

Before proving Theorems 1 and 2 we have a preliminary section, section 2, where we have the notions and results for proving such theorems.

2. Preliminary results

2.1. **Singular points.** Consider a differential system of the form

$$\dot{x} = p(x, y), \quad \dot{y} = q(x, y) \tag{2}$$

being $p, q \in \mathbb{R}[x, y]$.

The point (a, b) is a *singular point* of the differential system (2) if $p(a, b) = q(a, b) = 0$.

The singular point (a, b) is *hyperbolic* if the eigenvalues of the Jacobian matrix of the function (p, q) evaluated at (a, b) have non-zero real part. The classification of the local phase portraits of the hyperbolic singular points is well known, see for instance [4, Theorem 2.15]. In this paper when we characterize the local phase portrait of a hyperbolic singular point we will use that theorem.

The singular point (a, b) is *semi-hyperbolic* if one and only one of the eigenvalues of the Jacobian matrix of the function (p, q) evaluated at (a, b) is zero. Also the classification of the local phase portraits of the semi-hyperbolic singular points is well known, see for instance [4, Theorem 2.19]. Again in this paper when we characterize the local phase portrait of a semi-hyperbolic singular point we will use that theorem.

Consider the differential system

$$\dot{x} = \sum_{i=1}^{\infty} p_i(x, y), \qquad \dot{y} = \sum_{i=1}^{\infty} q_i(x, y), \tag{3}$$

where p_i and q_i are homogeneous polynomials of degree i, for $i \geq 1$. The *characteristic directions* of the singular point localized at the origin of coordinates of system (3) are given by the straight lines trough the origin defined by the real linear factors of the homogeneous polynomial $p_k(x, y)y - q_k(x, y)x$, where k is the minimum i for which the polynomials p_i or q_i are non-zero. It is known that the orbits which end or start at the origin of coordinates must arrive or exit tangent to these straight lines. For more details on the characteristic directions see for example [2].

When the Jacobian matrix of the function (p, q) evaluated at the singular point (a, b) of the differential system (2) is identically zero, then the singular point is called *linearly zero* and the local phase portrait at this singular point can be studied doing special changes of variables called blow ups, see for instance [1]. Here we use vertical blow ups and when the vertical direction is a characteristic direction we twist it to another direction in order that in the new variables the vertical axis is not a characteristic direction.

Let Φ_t be a smooth flow on a manifold M and let C be a submanifold of M consisting entirely of singular points of the flow. C is called *normally hyperbolic* if the tangent bundle to M over C splits into three subbundles TC, E^s and E^u invariant under the differential $d\Phi_t$ and satisfying

(i) $d\Phi_t$ contracts E^s exponentially,
(ii) $d\Phi_t$ expandsE^u exponentially,
(iii) TC is the tangent bundle of C.

For normally hyperbolic submanifolds one has the usual existence of smooth stable and unstable manifolds together with the persistence of these invariant manifolds under small perturbations. More precisely, we have the following theorem, for a proof see [5].

Theorem 4. *Let C be a normally hyperbolic submanifold of singular points for the flow Φ_t. Then there exist smooth stable and unstable manifolds tangent along C to $E^s \oplus TC$ and $E^u \oplus TC$, respectively. Moreover, both C and the stable and unstable manifolds are persistent under small perturbations of the flow.*

2.2. **The Poincaré compactification.** Roughly speaking the Poincaré compactification consists in identifying the plane $\mathbb{R}^2$ with the interior of a closed unit disc centered at the origin of coordinates, called the *Poincaré disc*. Then the boundary of this disc (the unit circle centered at the origin) is identified with the infinity of $\mathbb{R}^2$. Note that in $\mathbb{R}^2$ we can go or come from the infinity in as many as directions as points has that circle.

In order to classify the global dynamics of a polynomial differential system one of the main steps is to characterize the local phase portraits of its finite and infinite singular points in the Poincaré disc. For doing this we need the equations of our polynomial differential systems initially in $\mathbb{R}^2$ in the Poincaré disc.

Consider the differential system (2) in $\mathbb{R}^2$, where p and q are real polynomials in the variables x and y of degrees d_1 and d_2, respectively. Then the degree of the polynomial differential system (2) is $d = \max\{d_1, d_2\}$.

Denote by $T_p\mathbb{S}^2$ be the tangent space to the 2-dimensional sphere

$$\mathbb{S}^2 = \{\mathbf{s} = (s_1, s_2, s_3) \in \mathbb{R}^3 : s_1^2 + s_2^2 + s_3^2 = 1\}$$

at the point p, we call this sphere the *Poincaré sphere*. We consider that the polynomial differential system (2) is defined in the tangent plane to $\mathbb{S}^2$ at the point $(0,0,1)$, i.e. we have identified $\mathbb{R}^2$ with $T_{(0,0,1)}\mathbb{S}^2$. The central projection $f\colon T_{(0,0,1)}\mathbb{S}^2 \to \mathbb{S}^2$ send each point p of $T_{(0,0,1)}\mathbb{S}^2$ to two points of $\mathbb{S}^2$, one in the northern hemisphere and the other in the southern hemisphere. These two points are the intersection of the straight line through p and the origin of coordinates (the center of the sphere). So the map f defines two copies of the polynomial differential system (2) on the sphere, one in the open northern hemisphere and the other in the open southern hemisphere.

If $X = (p, q)$ is the vector field associated to the polynomial differential system (2), we denote by X' the vector field $Df \circ X$ defined on $\mathbb{S}^2$ except on its equator $\mathbb{S}^1 = \{s \in \mathbb{S}^2 : s_3 = 0\}$. Clearly $\mathbb{S}^1$ can be identified with the infinity of $\mathbb{R}^2$. If the degree of the polynomial vector fiedl X is d, then $p(X)$ is the only analytic extension of $s_3^{d-1}X'$ to $\mathbb{S}^2$. The vector field $p(X)$ on $\mathbb{S}^2$ is called the *Poincaré compactification* of the vector field X, for more details see [4, chapter 5].

On the Poincaré sphere $\mathbb{S}^2$ we use the following six local charts, which are given by $U_i = \{\mathbf{s} \in \mathbb{S}^2 : \ s_i > 0\}$ and $V_i = \{\mathbf{s} \in \mathbb{S}^2 : \ s_i < 0\}$, for $i = 1, 2, 3$, with the corresponding diffeomorphisms

$$\varphi_i : \ U_i \to \mathbb{R}^2, \qquad \psi_i : \ V_i \to \mathbb{R}^2,$$

defined by $\varphi_i(\mathbf{s}) = \psi_i(\mathbf{s}) = (s_m/s_i, s_n/s_i) = (u, v)$ for $m < n$ and $m, n \neq i$. Thus the coordinates (u, v) will play different roles in the distinct local charts. The expressions of the vector field $p(X)$ are

$$(\dot u, \dot v) = \left(v^d \left(Q\left(\frac{1}{v}, \frac{u}{v}\right) - uP\left(\frac{1}{v}, \frac{u}{v}\right)\right), -v^{d+1}P\left(\frac{1}{v}, \frac{u}{v}\right)\right) \qquad \text{in } U_1,$$

$$(\dot u, \dot v) = \left(v^d \left(P\left(\frac{u}{v}, \frac{1}{v}\right) - uQ\left(\frac{u}{v}, \frac{1}{v}\right)\right), -v^{d+1}Q\left(\frac{u}{v}, \frac{1}{v}\right)\right) \qquad \text{in } U_2,$$

$$(\dot u, \dot v) = (P(u, v), Q(u, v)) \qquad \text{in } U_3.$$

We note that the expressions of the vector field $p(X)$ in the local chart (V_i, ψ_i) is equal to the expression in the local chart (U_i, ϕ_i) multiplied by $(-1)^{d-1}$ for $i = 1, 2, 3$.

The orthogonal projection under $\pi(y_1, y_2, y_3) = (y_1, y_2)$ of the closed northern hemisphere of $\mathbb{S}^2$ onto the plane $s_3 = 0$ is a closed disc $\mathcal{D}^2$ of radius one centered at the origin of coordinates called the *Poincaré disc*. Since a copy of the vector field X on the plane $\mathbb{R}^2$ is in the open northern hemisphere of $\mathbb{S}^2$, the interior of the Poincaré disc $\mathcal{D}^2$ is identified with $\mathbb{R}^2$ and the boundary of $\mathcal{D}^2$, the equator $\mathbb{S}^1$ of $\mathbb{S}^2$, is identified with the infinity of $\mathbb{R}^2$. Consequently the phase portrait of the vector field X extended to the infinity corresponds to the projection of the phase portrait of the vector field $p(X)$ on the Poincaré disc $\mathcal{D}^2$.

The singular points of $p(X)$ in the Poincaré disc lying on $\mathbb{S}^1$ are the *infinite singular points* of the vector field X. The singular points of $p(X)$ in the interior of the Poincaré disc, i.e. on $\mathcal{D}^2 \setminus \mathbb{S}^1$, are the *finite singular points*. We note that in the local charts U_1, U_2, V_1 and V_2 the infinite singular points have their coordinate $v = 0$.

For a polynomial differential system (2) if $s \in \mathbb{S}^1$ is an infinite singular point, then $-s \in \mathbb{S}^1$ is another infinite singular point. Thus the number of infinite singular points is even.

2.3. **The Poincaré-Bendixson Theorem.** In what follows we are going to assume that Δ is an open subset of $\mathbb{R}^2$ and X is a vector field of class C^r with $r \geq 1$. Also, in Δ, γ_p^+ denotes a positive semi–orbit passing through the point p.

Let $\varphi(t) = \varphi(t, p) = \varphi_p(t)$ be the integral curve of X passing through the point p, defined on its maximal interval $I_p = (\alpha, \omega)$. If $\omega = \infty$ we define the set

$$\omega(p) = \{q \in \Delta : \text{ there exist } \{t_n\} \text{ with } t_n \to \infty \text{ and } \varphi(t_n) \to q \text{ when } n \to \infty\}.$$

In the same way, if $\alpha = -\infty$ we define the set

$$\alpha(p) = \{q \in \Delta : \text{ there exist } \{t_n\} \text{ with } t_n \to -\infty \text{ and } \varphi(t_n) \to q \text{ when } n \to \infty\}.$$

The sets $\omega(p)$ and $\alpha(p)$ are called the *ω–limit set* and the *α–limit set* of p, respectively.

Theorem 5 (Poincaré–Bendixson Theorem)**.** *Let $\varphi(t) = \varphi(t, p)$ be an orbit of X defined for all $t \geq 0$, such that γ_p^+ is contained in a compact set $K \subset \Delta$. Assume that the vector field X has at most a finite number of singularities in K. Then one of the following statements holds.*

(i) *If $\omega(p)$ contains only regular points, then $\omega(p)$ is a periodic orbit.*
(ii) *If $\omega(p)$ contains both regular and singular points, then $\omega(p)$ is formed by a set of orbits, every one of which tends to one of the singular points in $\omega(p)$ as $t \to \pm\infty$.*
(iii) *If $\omega(p)$ does not contain regular points, then $\omega(p)$ is a unique singular point.*

For a proof of the Poincaré–Bendixson Theorem see for instance [4, Theorem 1.25].

2.4. **Uniqueness of limit cycles.** The next result is proved in [6], see also [10, Theorem 4.1].

Theorem 6. *Consider the differential system*

$$\dot{x} = y, \quad \dot{y} = -g(x) - f(x)y. \tag{4}$$

Let $F(x) = \int_0^x f(s)ds$. *Assume that the following conditions hold.*

(i) *$f : \mathbb{R} \to \mathbb{R}$ is continuous and there exist $a < 0 < b$ such that $f(x) < 0$ if $x \in (a, b)$, and $f(x) > 0$ if $x < a$ and $x > b$;*
(ii) *there exists $c > 0$ such that $F(c) = F(-c) = 0$;*
(iii) *$F(+\infty) = +\infty$, or $F(-\infty) = -\infty$.*

Then the differential system (4) has a unique limit cycles which is stable.

3. Proof of Theorem 1

Assume that the polynomial $f(y)$ is even. Then the differential system (1) is invariant under the change of variables $(x, y, t) \mapsto (x, -y, -t)$. Therefore, since the origin is monodromic in a neighborhood of it, the origin must be a center.

Now assume that the polynomial $f(y)$ is not even, so we can write it as $f(y) = f_e(y)+f_o(y)$, where $f_e(y)$ and $f_o(y)$ are the even and the odd part of the polynomial $f(y)$, respectively. Consider the determinant

$$\begin{vmatrix} y & -x + f_e(y)(1-x^2) \\ y & -x + (f_e(y) + f_o(y))(1-x^2) \end{vmatrix} = y f_o(y)(1-x^2).$$

Since $yf_o(y)(1-x^2)$ has a constant sign in a convenient sufficiently small neighborhood U of the origen, at any point of U the oriented angle between the vector field $(y, -x + f_e(y)(1-x^2))$ having a center in U and the vector field $(y, -x + (f_e(y) + f_o(y))(1-x^2))$ has the same sign. Consequently the vector field $(y, -x + (f_e(y) + f_o(y))(1-x^2))$ has an unstable focus at the origin if $yf_o(y)(1-x^2) < 0$ in U, and it has a stable focus at the origin if $yf_o(y)(1-x^2) > 0$. Consequently the vector field has a center at the origin if and only if $f_o(y) = 0$. This completes the proof of Theorem 1.

4. Proof of Theorem 2

The unique finite singular point of system (1) is the origin. If $a_1 \in (-2, 2)$ it is a strong focus if $a_1 \neq 0$, and if $a_1 = 0$ in view of Theorem 1 it is a center. Moreover if $|a_2| > 2$ then the origin is a node. More precisely, the origin is a stable node if $a_1 \in (-\infty, 2]$, a stable focus if $a_1 \in (-2, 0)$, a center if $a_1 = 0$, an unstable focus if $a_1 \in (0, 2)$ and an unstable node if $a_1 \in [2, +\infty)$.

Now we study the infinite singular points for each of the cases in the statements of the theorem.

4.1. The infinite singular points of the local charts U_1 and U_2 when $a_2 = 0$ and $a_1 \neq 0$.

Now we shall study the infinite singular points of the differential system (1) in the local charts U_1 and U_2. We note that in this case doing the change of variables $(x, y, t) \to (x, -y, -t)$ we can assume without loss of generality that $a_1 > 0$.

4.1.1. *The local chart U_1.* System (1) in the chart U_1 writes

$$\dot{u} = -a_1 u - v^2 + a_1 u v^2 - u^2 v^2, \qquad \dot{v} = -uv^3. \tag{5}$$

The unique infinite singular point in this chart is the origin. The eigenvalues of the linear part of system (5) at the singular point $(0, 0)$ are 0 and $-a_1$ and so it is semi-hyperbolic singular point. Applying [4, Theorem 2.19] we obtain that it is a semi-hyperbolic saddle.

4.1.2. *The local chart U_2.* System (1) in the chart U_2 writes

$$\dot u = v^2 + a_1u^3 - a_1uv^2 + u^2v^2, \qquad \dot v = v(a_1u^2 - a_1v^2 + uv^2). \tag{6}$$

The origin is a singular point. The linear part of system (6) at the origin is identically zero, so in order to determine its local phase portrait we must do blow ups. The characteristic directions at the origin of U_2 are given by the real linear factors of v^3. Since $u = 0$ is not a characteristic direction we do the vertical blow up $(u, v) \to (u_1, u_1v_1)$ and system (6) writes in the new variables

$$\begin{aligned} \dot u_1 = &\ -u_1^2(-a_1u_1 - v_1^2 + a_1u_1v_1^2 - u_1^2v_1^2),\\ \dot v_1 = &\ -u_1v_1^3. \end{aligned}$$

Doing a rescaling of the independent variable we eliminate the common factor u_1 between $\dot u_1$ and $\dot v_1$ and we obtain the system

$$\begin{aligned} \dot u_1 = &\ -u_1(-a_1u_1 - v_1^2 + a_1u_1v_1^2 - u_1^2v_1^2),\\ \dot v_1 = &\ -v_1^3. \end{aligned} \tag{7}$$

The unique singular point of system (7) on the straight line $u_1 = 0$ is $(0, 0)$. The linear part of system (7) evaluated at $(0, 0)$ is linearly zero so we must do another blow up. Then the vertical axis $u_1 = 0$ is a characteristic direction at the origin of the local chart U_2. Therefore before doing a vertical blow up we translate the direction $u_1 = 0$ to $u_1 = v_1$ doing the change of variables $(u_1, v_1) = (u_2 - v_2, v_2)$. Then system (7) becomes

(8)
$$\begin{aligned} \dot u_2 = &\ a_1u_2^2 - 2a_1u_2v_2 + a_1v_2^2 + u_2v_2^2 - a_1u_2^2v_2^2 + u_2^3v_2^2 - 2v_2^3 + 2a_1u_2v_2^3 - 3u_2^2v_2^3\\ &\ -a_1v_2^4 + 3u_2v_2^4 - v_2^5,\\ \dot v_2 = &\ -v_2^3. \end{aligned}$$

Now we do the vertical blow up $(u_2, v_2) \to (u_3, u_3v_3)$ and system (8) writes in the new variables

$$\begin{aligned} \dot u_3 = &\ -u_3^2(-a_1 + 2a_1v_3 - a_1v_3^2 - u_3v_3^2 + a_1u_3^2v_3^2 - u_3^3v_3^2 + 2u_3v_3^3 - 2a_1u_3^2v_3^3\\ &\ +3u_3^3v_3^3 + a_1u_3^2v_3^4 - 3u_3^3v_3^4 + u_3^3v_3^5),\\ \dot v_3 = &\ u_3(v_3 - 1)v_3(a_1 - a_1v_3 + 2u_3v_3^2 - a_1u_3^2v_3^2 + u_3^3v_3^2 + a_1u_3^2v_3^3 - 2u_3^3v_3^3 + u_3^3v_3^4). \end{aligned}$$

Doing a rescaling of the time we eliminate the common factor u_3 between $(\dot u_3, \dot v_3)$ and we get the system

(9)
$$\begin{aligned} \dot u_3 = &\ -u_3(-a_1 + 2a_1v_3 - a_1v_3^2 - u_3v_3^2 + a_1u_3^2v_3^2 - u_3^3v_3^2 + 2u_3v_3^3 - 2a_1u_3^2v_3^3\\ &\ +3u_3^3v_3^3 + a_1u_3^2v_3^4 - 3u_3^3v_3^4 + u_3^3v_3^5),\\ \dot v_3 = &\ (v_3 - 1)v_3(a_1 - a_1v_3 + 2u_3v_3^2 - a_1u_3^2v_3^2 + u_3^3v_3^2 + a_1u_3^2v_3^3 - 2u_3^3v_3^3\\ &\ +u_3^3v_3^4). \end{aligned}$$

The singular points of system (9) on the straight line $u_3 = 0$ are $(0, 0)$ and $(0, 1)$. The eigenvalues of the linear part of system (9) evaluated at the singular point $(0, 0)$ are $-a_1$ and a_1, and so it is a saddle. The linear part of system (9) evaluated at $(0, 1)$ is linearly zero so we must do another blow-up. First we translate the singular point $(0, 1)$ to the origin of coordinates in order to study its local phase portrait doing the change $(u_3, v_3) = (u_4, 1 + v_4)$. Therefore system (9) in the variables (u_4, v_4) becomes

(10)
$$\begin{aligned}\dot u_4 = &\ -u_4(u_4 + 4u_4v_4 - a_1v_4^2 + 5u_4v_4^2 + a_1u_4^2v_4^2 + 2u_4v_4^3 + 2a_1u_4^2v_4^3 + u_4^3v_4^3\\ &+a_1u_4^2v_4^4 + 2u_4^3v_4^4 + u_4^3v_4^5),\\ \dot v_4 = &\ 2u_4v_4 - a_1v_4^2 + 6u_4v_4^2 + a_1u_4^2v_4^2 - a_1v_4^3 + 6u_4v_4^3 + 3a_1u_4^2v_4^3 + u_4^3v_4^3 + 2u_4v_4^4\\ &+3a_1u_4^2v_4^4 + 3u_4^3v_4^4 + a_1u_4^2v_4^5 + 3u_4^3v_4^5 + u_4^3v_4^6.\end{aligned}$$

The characteristic directions at the origin are the real linear factors of $u_4v_4(a_1v_4 - 3u_4) = 0$. Then the vertical axis $u_4 = 0$ is a characteristic direction at the origin of system (10). Before doing a vertical blow up we translate the direction $u_4 = 0$ to $u_4 = v_4$ doing the change of variables $(u_4, v_4) = (u_5 - v_5, v_5)$. Then system (10) becomes

(11)
$$\begin{aligned}\dot u_5 = &\ -u_5^2 + 4u_5v_5 - 4u_5^2v_5 - 3v_5^2 - a_1v_5^2 + 14u_5v_5^2 + a_1u_5v_5^2 - 5u_5^2v_5^2 + a_1u_5^2v_5^2\\ &-a_1u_5^3v_5^2 - 10v_5^3 - 2a_1v_5^3 + 16u_5v_5^3 - 2a_1u_5v_5^3 - 2u_5^2v_5^3 + 6a_1u_5^2v_5^3 + u_5^3v_5^3\\ &-2a_1u_5^3v_5^3 - u_5^4v_5^3 - 11v_5^4 + a_1v_5^4 + 6u_5v_5^4 - 9a_1u_5v_5^4 - 3u_5^2v_5^4 + 9a_1u_5^2v_5^4\\ &+7u_5^3v_5^4 - a_1u_5^3v_5^4 - 2u_5^4v_5^4 - 4v_5^5 + 4a_1v_5^5 + 3u_5v_5^5 - 12a_1u_5v_5^5 - 15u_5^2v_5^5\\ &+4a_1u_5^2v_5^5 + 11u_5^3v_5^5 - u_5^4v_5^5 - v_5^6 + 5a_1v_5^6 + 13u_5v_5^6 - 5a_1u_5v_5^6 - 21u_5^2v_5^6\\ &+5u_5^3v_5^6 - 4v_5^7 + 2a_1v_5^7 + 17u_5v_5^7 - 9u_5^2v_5^7 - 5v_5^8 + 7u_5v_5^8 - 2v_5^9,\\ \dot v_5 = &\ 2u_5v_5 - 2v_5^2 - a_1v_5^2 + 6u_5v_5^2 + a_1u_5^2v_5^2 - 6v_5^3 - a_1v_5^3 + 6u_5v_5^3 - 2a_1u_5v_5^3\\ &+3a_1u_5^2v_5^3 + u_5^3v_5^3 - 6v_5^4 + a_1v_5^4 + 2u_5v_5^4 - 6a_1u_5v_5^4 - 3u_5^2v_5^4 + 3a_1u_5^2v_5^4\\ &+3u_5^3v_5^4 - 2v_5^5 + 3a_1v_5^5 + 3u_5v_5^5 - 6a_1u_5v_5^5 - 9u_5^2v_5^5 + a_1u_5^2v_5^5 + 3u_5^3v_5^5 - v_5^6\\ &+3a_1v_5^6 + 9u_5v_5^6 - 2a_1u_5v_5^6 - 9u_5^2v_5^6 + u_5^3v_5^6 - 3v_5^7 + a_1v_5^7 + 9u_5v_5^7 - 3u_5^2v_5^7\\ &-3v_5^8 + 3u_5v_5^8 - v_5^9.\end{aligned}$$

Now we do the vertical blow up $(u_5, v_5) \to (u_6, u_6v_6)$ and system (11) writes

$$\begin{aligned}\dot u_6 = &\ -u_6^2\big(1 - 4v_6 + 4u_6v_6 + 3v_6^2 + a_1v_6^2 - 14u_6v_6^2 - a_1u_6v_6^2 + 5u_6^2v_6^2 - a_1u_6^2v_6^2\\ &+a_1u_6^3v_6^2 + 10u_6v_6^3 + 2a_1u_6v_6^3 - 16u_6^2v_6^3 + 2a_1u_6^2v_6^3 + 2u_6^3v_6^3 - 6a_1u_6^3v_6^3 - u_6^4v_6^3\\ &+2a_1u_6^4v_6^3 + u_6^5v_6^3 + 11u_6^2v_6^4 - a_1u_6^2v_6^4 - 6u_6^3v_6^4 + 9a_1u_6^3v_6^4 + 3u_6^4v_6^4 - 9a_1u_6^4v_6^4\\ &-7u_6^5v_6^4 + a_1u_6^5v_6^4 + 2u_6^6v_6^4 + 4u_6^3v_6^5 - 4a_1u_6^3v_6^5 - 3u_6^4v_6^5 + 12a_1u_6^4v_6^5 + 15u_6^5v_6^5\\ &-4a_1u_6^5v_6^5 - 11u_6^6v_6^5 + u_6^7v_6^5 + u_6^4v_6^6 - 5a_1u_6^4v_6^6 - 13u_6^5v_6^6 + 5a_1u_6^5v_6^6 + 21u_6^6v_6^6\\ &-5u_6^7v_6^6 + 4u_6^5v_6^7 - 2a_1u_6^5v_6^7 - 17u_6^6v_6^7 + 9u_6^7v_6^7 + 5u_6^6v_6^8 - 7u_6^7v_6^8 + 2u_6^7v_6^9\big),\\ \dot v_6 = &\ u_6(v_6 - 1)v_6\big(- 3 + 3v_6 + a_1v_6 - 10u_6v_6 - a_1u_6^2v_6 + 10u_6v_6^2 + 2a_1u_6v_6^2\\ &-11u_6^2v_6^2 + 2a_1u_6^2v_6^2 - 4a_1u_6^3v_6^2 - u_6^4v_6^2 + 11u_6^2v_6^3 - a_1u_6^2v_6^3 - 4u_6^3v_6^3 + 8a_1u_6^3v_6^3\\ &+3u_6^4v_6^3 - 5a_1u_6^4v_6^3 - 4u_6^5v_6^3 + 4u_6^3v_6^4 - 4a_1u_6^3v_6^4 - 3u_6^4v_6^4 + 10a_1u_6^4v_6^4 + 12u_6^5v_6^4\\ &-2a_1u_6^5v_6^4 - 5u_6^6v_6^4 + u_6^4v_6^5 - 5a_1u_6^4v_6^5 - 12u_6^5v_6^5 + 4a_1u_6^5v_6^5 + 15u_6^6v_6^5 - 2u_6^7v_6^5\\ &+4u_6^5v_6^6 - 2a_1u_6^5v_6^6 - 15u_6^6v_6^6 + 6u_6^7v_6^6 + 5u_6^6v_6^7 - 6u_6^7v_6^7 + 2u_6^7v_6^8\big).\end{aligned}$$

Eliminating the common factor u_6 between $\dot{u}_6$ and $\dot{v}_6$ rescaling the time we obtain the system

(12)

$$
\begin{aligned}
\dot{u}_6 = \;& -u_6\big(1 - 4v_6 + 4u_6v_6 + 3v_6^2 + a_1v_6^2 - 14u_6v_6^2 - a_1u_6v_6^2 + 5u_6^2v_6^2 - a_1u_6^2v_6^2 \\
& + a_1u_6^3v_6^2 + 10u_6v_6^3 + 2a_1u_6v_6^3 - 16u_6^2v_6^3 + 2a_1u_6^2v_6^3 + 2u_6^3v_6^3 - 6a_1u_6^3v_6^3 - u_6^4v_6^3 \\
& + 2a_1u_6^4v_6^3 + u_6^5v_6^3 + 11u_6^2v_6^4 - a_1u_6^2v_6^4 - 6u_6^3v_6^4 + 9a_1u_6^3v_6^4 + 3u_6^4v_6^4 - 9a_1u_6^4v_6^4 \\
& - 7u_6^5v_6^4 + a_1u_6^5v_6^4 + 2u_6^6v_6^4 + 4u_6^3v_6^5 - 4a_1u_6^3v_6^5 - 3u_6^4v_6^5 + 12a_1u_6^4v_6^5 + 15u_6^5v_6^5 \\
& - 4a_1u_6^5v_6^5 - 11u_6^6v_6^5 + u_6^7v_6^5 + u_6^4v_6^6 - 5a_1u_6^4v_6^6 - 13u_6^5v_6^6 + 5a_1u_6^5v_6^6 + 21u_6^6v_6^6 \\
& - 5u_6^7v_6^6 + 4u_6^5v_6^7 - 2a_1u_6^5v_6^7 - 17u_6^6v_6^7 + 9u_6^7v_6^7 + 5u_6^6v_6^8 - 7u_6^7v_6^8 + 2u_6^7v_6^9\big), \\
\dot{v}_6 = \;& (v_6 - 1)v_6\big(- 3 + 3v_6 + a_1v_6 - 10u_6v_6 - a_1u_6^2v_6 + 10u_6v_6^2 + 2a_1u_6v_6^2 \\
& - 11u_6^2v_6^2 + 2a_1u_6^2v_6^2 - 4a_1u_6^3v_6^2 - u_6^4v_6^2 + 11u_6^2v_6^3 - a_1u_6^2v_6^3 - 4u_6^3v_6^3 + 8a_1u_6^3v_6^3 \\
& + 3u_6^4v_6^3 - 5a_1u_6^4v_6^3 - 4u_6^5v_6^3 + 4u_6^3v_6^4 - 4a_1u_6^3v_6^4 - 3u_6^4v_6^4 + 10a_1u_6^4v_6^4 + 12u_6^5v_6^4 \\
& - 2a_1u_6^5v_6^4 - 5u_6^6v_6^4 + u_6^4v_6^5 - 5a_1u_6^4v_6^5 - 12u_6^5v_6^5 + 4a_1u_6^5v_6^5 + 15u_6^6v_6^5 - 2u_6^7v_6^5 \\
& + 4u_6^5v_6^6 - 2a_1u_6^5v_6^6 - 15u_6^6v_6^6 + 6u_6^7v_6^6 + 5u_6^6v_6^7 - 6u_6^7v_6^7 + 2u_6^7v_6^8\big).
\end{aligned}
$$

The unique singular points of system (12) on the straight line $u_6 = 0$ are $(0,0)$, $(0,1)$ and $(0,3/(a_1+3))$. The eigenvalues of the linear part of system (12) evaluated at the singular point $(0,0)$ are 3 and -1, and so it is a hyperbolic saddle. Moreover the eigenvalues of the linear part of system (12) evaluated at the singular point $(0,1)$ are $-a_1$ and a_1, and so it is also a hyperbolic saddle. Finally the eigenvalues of the linear part of system (12) evaluated at the singular point $(0,3/(a_1+3))$ are $-3a_1/(3+a_1)$ and $-a_1/(3+a_1)$ and so it is a stable node. Therefore the local phase portrait near the straight line $u_6 = 0$ for system (12) is topologically equivalent to the one of Figure 2(a).

Undoing the rescaling $dt_3 = u_6dt_2$ we get the phase portrait of Figure 2(b). Going back through the changes of variables from the phase portrait of Figure 2(b), we obtain the local phase portrait at the origin of system (11) which is topologically equivalent to the one of Figure 2(c).

Going back through the changes of variables from the phase portrait of Figure 2(c), we obtain the local phase portrait at the origin of system (10) which is topologically equivalent to the one of Figure 2(d).

Going back through the changes of variables from the phase portrait of Figure 2(d), we obtain the local phase portrait around the straight line $u_3 = 0$ which is topologically equivalent to the one of Figure 2(e).

Again undoing the rescaling $dt_2 = u_3dt_1$ we obtain the phase portrait of Figure 2(f). Going back through the changes of variables from the phase portrait of Figure 2(f), we obtain the local phase portrait at he origin of system (8) which is topologically equivalent to the one of Figure 2(g).

Going back through the changes of variables from the phase portrait of Figure 2(g), we obtain the local phase portrait at he origin of system (7) which is topologically equivalent to the one of Figure 2(h).

Undoing the rescaling $dt_1 = u_1 dt$ we get the phase portrait of Figure 2(i).

Finally going back through the changes of variables from the phase portrait of Figure 2(i), we obtain the local phase portrait at the origin of system (6) which is topologically equivalent to the one of Figure 2(j). Hence the origin of the local chart U_2 has a local phase portrait which is topologically equivalent to an unstable node.

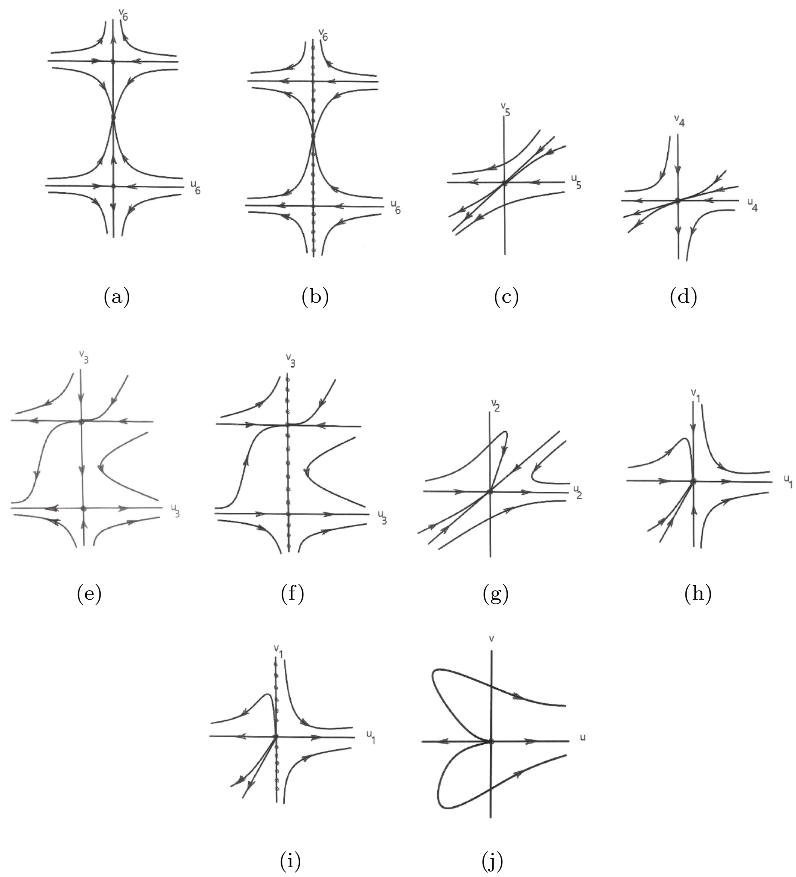


FIGURE 2. Figures of the blow up of the singular point located at the origin of the local chart U_2 of system (5).

4.2. **The infinite singular points of the local charts U_1 and U_2 when $a_2 \neq 0$ and $a_1 = 0$.** We shall study the infinite singular points of the differential system (1) in the local charts U_1 and U_2. Note that in this case doing the change of variables $(x, y) \to (-x, -y)$ we can assume, without loss of generality, that $a_2 > 0$.

4.2.1. *The local chart U_1.* System (1) in the chart U_1 writes

$$\dot{u} = -v^3 + u^2(-a_2 + a_2 v^2 - v^3), \quad \dot{v} = -uv^4 \tag{13}$$

The unique infinite singular point in this chart is the origin. The linear part of system (13) at the origin is identically zero, so in order to determine its local phase portrait we must do blow ups. The characteristic directions at the origin of U_2 are given by the real linear factors of $u^2v = 0$. Since $u = 0$ is a characteristic direction before doing a vertical blow up we translate the direction $u = 0$ to $u = v$ doing the change of variables $(u, v) = (u_1 - v_1, v_1)$. Then system (13) becomes

(14)

$$\begin{aligned} \dot{u}_1 = &\quad -a_2u_1^2 + 2a_2u_1v_1 - a_2v_1^2 + a_2u_1^2v_1^2 - v_1^3 - 2a_2u_1v_1^3 - u_1^2v_1^3 + a_2v_1^4 + u_1v_1^4, \\ \dot{v}_1 = &\quad -u_1v_1^4 + v_1^5. \end{aligned}$$

Now we do the vertical blow up $(u_1, v_1) \to (u_2, u_2v_2)$ and system (14) writes in the new variables

$$\begin{aligned} \dot{u}_2 = &\quad u_2^2(-a_2 + 2a_2v_2 - a_2v_2^2 + a_2u_2^2v_2^2 - u_2v_2^3 - 2a_2u_2^2v_2^3 - u_2^3v_2^3 + a_2u_2^2v_2^4 + u_2^3v_2^4), \\ \dot{v}_2 = &\quad -u_2v_2(-a_2 + 2a_2v_2 - a_2v_2^2 + a_2u_2^2v_2^2 - u_2v_2^3 - 2a_2u_2^2v_2^3 + a_2u_2^2v_2^4). \end{aligned}$$

Eliminating the common factor u_2 between $\dot{u}_2$ and $\dot{v}_2$ rescaling the time we obtain the system

(15)

$$\begin{aligned} \dot{u}_2 = &\quad u_2(-a_2 + 2a_2v_2 - a_2v_2^2 + a_2u_2^2v_2^2 - u_2v_2^3 - 2a_2u_2^2v_2^3 - u_2^3v_2^3 + a_2u_2^2v_2^4 \\ &\quad + u_2^3v_2^4), \\ \dot{v}_2 = &\quad -v_2(-a_2 + 2a_2v_2 - a_2v_2^2 + a_2u_2^2v_2^2 - u_2v_2^3 - 2a_2u_2^2v_2^3 + a_2u_2^2v_2^4). \end{aligned}$$

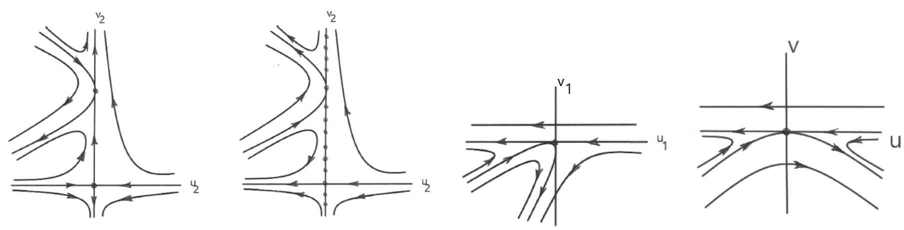
The singular points of system (15) on the straight line $u_2 = 0$ are $(0, 0)$ and $(0, 1)$. The eigenvalues of the linear part of system (15) evaluated at the singular point $(0, 0)$ are $-a_2$ and a_2, and so it is a hyperbolic saddle. The eigenvalues of the linear part of system (15) evaluated at the singular point $(0, 1)$ are both 0 but the linear part of system (15) evaluated at $(0, 1)$ is not linearly zero, so it is a nilpotent singular point.

Before studying it we translate the point $(0, 1)$ to the origin by doing the change $(u_2, v_2) = (u_3, 1 + v_3)$. Therefore system (15) in the variables (u_3, v_3) becomes

$$\begin{aligned} \dot{u}_3 = &\quad -u_3^2 - 3u_3^2v_3 + u_3^4v_3 - a_2u_3v_3^2 - 3u_3^2v_3^2 + a_2u_3^3v_3^2 + 3u_3^4v_3^2 - u_3^2v_3^3 + 2a_2u_3^3v_3^3 \\ &\quad + 3u_3^4v_3^3 + a_2u_3^3v_3^4 + u_3^4v_3^4, \\ \dot{v}_3 = &\quad u_3 + 4u_3v_3 + a_2v_3^2 + 6u_3v_3^2 - a_2u_3^2v_3^2 + a_2v_3^3 + 4u_3v_3^3 - 3a_2u_3^2v_3^3 + u_3v_3^4 \\ &\quad -3a_2u_3^2v_3^4 - a_2u_3^2v_3^5. \end{aligned}$$

Applying [4, Theorem 3.5] we get that it is a nilpotent saddle. Therefore the local phase portrait near the straight line $u_2 = 0$ for system (15) is topologically equivalent to the one of Figure 3(a).

Undoing the rescaling $dt_1 = u_2dt$ we get the local phase portrait of Figure 3(b). Going back through the changes of variables from the phase portrait of Figure 3(b), we obtain the local phase portrait at the origin of system (14) which is topologically equivalent to the one of Figure 3(c). Again going back through the changes of variables from the phase portrait of Figure 3(c), we obtain the local phase portrait around the origin of system (13) which is topologically equivalent to the one of Figure 3(d). Hence the origin of the local chart U_1 has a nilpotent saddle having two separatrices (one stable and one unstable) on the infinite circle and two separatrices (one stable and one unstable) in $v < 0$.



(a) (b) (c) (d)

FIGURE 3. Figures of the blow up of the singular point located at the origin of the local chart U_1 of system (13).

4.2.2. *The local chart U_2.* System (1) in the chart U_2 writes

$$\dot{u} = a_2u^3 - a_2uv^2 + v^3 + u^2v^3, \quad \dot{v} = v(a_2u^2 - a_2v^2 + uv^3). \tag{16}$$

The origin is an infinite singular point in this chart. The linear part of system (16) at the origin is identically zero, so in order to determine its local phase portrait we must do blow ups. The characteristic directions at the origin of U_2 are given by the real linear factors of $v^4 = 0$. Since $u = 0$ is not a characteristic direction, we do the vertical blow up $(u, v) \to (u_1, u_1v_1)$, and system (16) writes in the new variables

$$\dot{u}_1 = -u_1^3(-a_2 + a_2v_1^2 - v_1^3 - u_1^2v_1^3), \quad \dot{v}_1 = -u_1^2v_1^4.$$

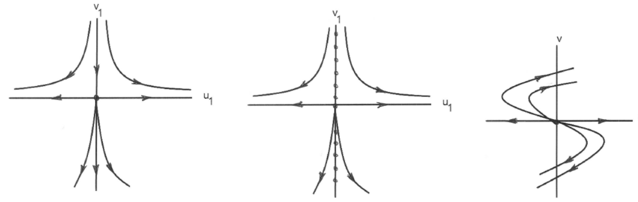
Eliminating the common factor u_1^2 between $\dot{u}_1$ and $\dot{v}_1$ rescaling the time we obtain the system

$$\dot{u}_1 = -u_1(-a_2 + a_2v_1^2 - v_1^3 - u_1^2v_1^3), \quad \dot{v}_1 = -v_1^4. \tag{17}$$

The unique singular point of system (17) on the straight line $u_1 = 0$ is $(0, 0)$.

The eigenvalues of the linear part of system (17) at the singular point $(0, 0)$ are 0 and a_2 and so it is semi-hyperbolic. Applying [4, Theorem 2.19] we obtain that it is a saddle-node. Therefore the local phase portrait near the straight line $u_1 = 0$ for system (17) is topologically equivalent to the one of Figure 4(a). Undoing the rescaling $dt_1 = u_1^2dt$ we get the phase portrait of Figure 4(b). Going back through the changes of variables from the phase portrait of Figure 4(b), we obtain the local phase portrait at the origin of system (16) which is topologically equivalent to the one of Figure 4(c). Hence the origin of the local chart U_2 is topologically equivalent to an unstable node.

4.3. **The infinite singular points of the local charts U_1 and U_2 when $a_2a_1 \neq 0$.** We shall study the infinite singular points of the differential system (1) in the local charts U_1 and U_2. We note that in this case doing the change of variables $(x, y) \to (-x, -y)$ we can assume, without loss of generality, that $a_2a_1 < 0$.



(a) (b) (c)

FIGURE 4. Figures of the blow up of the singular point located at the origin of the local chart U_2 of system (16).

4.3.1. *The local chart* U_1. System (1) in the chart U_1 writes

$$\dot{u} = -a_2u^2 - a_1uv + a_2u^2v^2 - v^3 + a_1uv^3 - u^2v^3, \quad \dot{v} = -uv^4. \tag{18}$$

The unique infinite singular point in this chart is the origin. The linear part of system (18) at the origin is identically zero, so in order to determine its local phase portrait we must do blow ups. The characteristic directions at the origin are the real linear factors of $uv(a_2u + a_1v) = 0$. Since $u = 0$ is a characteristic direction before doing a vertical blow up we translate the direction $u = 0$ to $u = v$ doing the change of variables $(u, v) = (u_1 - v_1, v_1)$. Then system (18) becomes

$$\begin{aligned} \dot{u}_1 = &\ -a_2u_1^2 - a_1u_1v_1 + 2a_2u_1v_1 + a_1v_1^2 - a_2v_1^2 + a_2u_1^2v_1^2 - v_1^3 + a_1u_1v_1^3 \\ &-2a_2u_1v_1^3 - u_1^2v_1^3 - a_1v_1^4 + a_2v_1^4 + u_1v_1^4, \\ \dot{v}_1 = &\ -u_1v_1^4 + v_1^5. \end{aligned} \tag{19}$$

Now we do the vertical blow up $(u_1, v_1) \to (u_2, u_2v_2)$ and system (19) writes in the new variables

$$\begin{aligned} \dot{u}_2 = &\ u_2^2(-a_2 - a_1v_2 + 2a_2v_2 + a_1v_2^2 - a_2v_2^2 + a_2u_2^2v_2^2 - u_2v_2^3 + a_1u_2^2v_2^3 - 2a_2u_2^2v_2^3 \\ &-u_2^3v_2^3 - a_1u_2^2v_2^4 + a_2u_2^2v_2^4 + u_2^3v_2^4), \\ \dot{v}_2 = &\ -u_2v_2(-a_2 - a_1v_2 + 2a_2v_2 + a_1v_2^2 - a_2v_2^2 + a_2u_2^2v_2^2 - u_2v_2^3 + a_1u_2^2v_2^3 - 2a_2u_2^2v_2^3 \\ &-a_1u_2^2v_2^4 + a_2u_2^2v_2^4). \end{aligned}$$

Eliminating the common factor u_2 between $\dot{u}_2$ and $\dot{v}_2$ rescaling the time we obtain the system

$$\begin{aligned} \dot{u}_2 = &\ u_2(-a_2 - a_1v_2 + 2a_2v_2 + a_1v_2^2 - a_2v_2^2 + a_2u_2^2v_2^2 - u_2v_2^3 + a_1u_2^2v_2^3 \\ &-2a_2u_2^2v_2^3 - u_2^3v_2^3 - a_1u_2^2v_2^4 + a_2u_2^2v_2^4 + u_2^3v_2^4), \\ \dot{v}_2 = &\ -v_2(-a_2 - a_1v_2 + 2a_2v_2 + a_1v_2^2 - a_2v_2^2 + a_2u_2^2v_2^2 - u_2v_2^3 + a_1u_2^2v_2^3 \\ &-2a_2u_2^2v_2^3 - a_1u_2^2v_2^4 + a_2u_2^2v_2^4). \end{aligned} \tag{20}$$

The singular points of system (20) on the straight line $u_2 = 0$ are $(0, 0)$, $(0, 1)$ and $(0, a_2/(a_2 - a_1))$. The eigenvalues of the linear part of system (20) evaluated at the singular point $(0, 0)$ are $-a_2$ and a_2, and so it is a hyperbolic saddle. The eigenvalues of the linear part of system (20) evaluated at the singular point $(0, 1)$ are 0 and $-a_1$ and so it is semi-hyperbolic. Before studying it we translate the point $(0,1)$ to the origin by doing the change $(u_2,v_2)=(u_3,1+v_3)$. Therefore system (20) in the variables (u_3,v_3) becomes

$$\begin{aligned}
\dot u_3 = \;& -u_3^2+a_1u_3v_3-3u_3^2v_3-a_1u_3^3v_3+u_3^4v_3+a_1u_3v_3^2-a_2u_3v_3^2-3u_3^2v_3^2-3a_1u_3^3v_3^2\\
&+a_2u_3^3v_3^2+3u_3^4v_3^2-u_3^2v_3^3-3a_1u_3^3v_3^3+2a_2u_3^3v_3^3+3u_3^4v_3^3-a_1u_3^3v_3^4+a_2u_3^3v_3^4+u_3^4v_3^4,\\
\dot v_3 = \;& u_3-a_1v_3+4u_3v_3+a_1u_3^2v_3-2a_1v_3^2+a_2v_3^2+6u_3v_3^2+4a_1u_3^2v_3^2-a_2u_3^2v_3^2-a_1v_3^3\\
&+a_2v_3^3+4u_3v_3^3+6a_1u_3^2v_3^3-3a_2u_3^2v_3^3+u_3v_3^4+4a_1u_3^2v_3^4-3a_2u_3^2v_3^4+a_1u_3^2v_3^5\\
&-a_2u_3^2v_3^5.
\end{aligned}$$

Now applying [4, Theorem 2.19] we get that it is a semi-hyperbolic saddle. The eigenvalues of the linear part of system (20) evaluated at the singular point $(0,a_2/(a_2-a_1))$ are 0 and $a_1a_2/(a_2-a_1)$, and so it is semi-hyperbolic. Before studying it we translate that point to the origin by doing the change $(u_2,v_2)=(u_3,\frac{a_2}{a_2-a_1}+v_3)$. Therefore system (20) in the variables (u_3,v_3) after multiplied by $(a_2-a_1)^4$ doing a convenient rescaling, we obtain

$$\begin{aligned}
\dot u_3 = \;& u_3\big(a_1a_2^3u_3-a_2^4u_3+a_1a_2^3u_3^3-a_1^5v_3+4a_1^4a_2v_3-6a_1^3a_2^2v_3+4a_1^2a_2^3v_3-a_1a_2^4v_3\\
&-3a_1^2a_2^2u_3v_3+6a_1a_2^3u_3v_3-3a_2^4u_3v_3+a_1^3a_2^2u_3^2v_3-2a_1^2a_2^3u_3^2v_3+a_1a_2^4u_3^2v_3\\
&-3a_1^2a_2^2u_3^3v_3+2a_1a_2^3u_3^3v_3+a_2^4u_3^3v_3+a_1^5v_3^2-5a_1^4a_2v_3^2+10a_1^3a_2^2v_3^2-10a_1^2a_2^3v_3^2\\
&+5a_1a_2^4v_3^2-a_2^5v_3^2+3a_1^3a_2u_3v_3^2-9a_1^2a_2^2u_3v_3^2+9a_1a_2^3u_3v_3^2-3a_2^4u_3v_3^2-2a_1^4a_2u_3^2v_3^2\\
&+5a_1^3a_2^2u_3^2v_3^2-3a_1^2a_2^3u_3^2v_3^2-a_1a_2^4u_3^2v_3^2+a_2^5u_3^2v_3^2+3a_1^3a_2u_3^3v_3^2-3a_1^2a_2^2u_3^3v_3^2\\
&-3a_1a_2^3u_3^3v_3^2+3a_2^4u_3^3v_3^2-a_1^4u_3v_3^3+4a_1^3a_2u_3v_3^3-6a_1^2a_2^2u_3v_3^3+4a_1a_2^3u_3v_3^3-a_2^4u_3v_3^3\\
&+a_1^5u_3^2v_3^3-2a_1^4a_2u_3^2v_3^3-2a_1^3a_2^2u_3^2v_3^3+8a_1^2a_2^3u_3^2v_3^3-7a_1a_2^4u_3^2v_3^3+2a_2^5u_3^2v_3^3-a_1^4u_3^3v_3^3\\
&+6a_1^2a_2^2u_3^3v_3^3-8a_1a_2^3u_3^3v_3^3+3a_2^4u_3^3v_3^3-a_1^5u_3^2v_3^4+5a_1^4a_2u_3^2v_3^4-10a_1^3a_2^2u_3^2v_3^4\\
&+10a_1^2a_2^3u_3^2v_3^4-5a_1a_2^4u_3^2v_3^4+a_2^5u_3^2v_3^4+a_1^4u_3^3v_3^4-4a_1^3a_2u_3^3v_3^4+6a_1^2a_2^2u_3^3v_3^4\\
&-4a_1a_2^3u_3^3v_3^4+a_2^4u_3^3v_3^4\big),\\
\dot v_3 = \;& (a_2+(a_2-a_1)v_3)\big(-a_2^3u_3+a_1^4v_3-3a_1^3a_2v_3+3a_1^2a_2^2v_3-a_1a_2^3v_3+3a_1a_2^2u_3v_3\\
&-3a_2^3u_3v_3-a_1^2a_2^2u_3^2v_3+a_1a_2^3u_3^2v_3-a_1^4v_3^2+4a_1^3a_2v_3^2-6a_1^2a_2^2v_3^2+4a_1a_2^3v_3^2-a_2^4v_3^2\\
&-3a_1^2a_2u_3v_3^2+6a_1a_2^2u_3v_3^2-3a_2^3u_3v_3^2+2a_1^3a_2u_3^2v_3^2-3a_1^2a_2^2u_3^2v_3^2+a_2^4u_3^2v_3^2+a_1^3u_3v_3^3\\
&-3a_1^2a_2u_3v_3^3+3a_1a_2^2u_3v_3^3-a_2^3u_3v_3^3-a_1^4u_3^2v_3^3+a_1^3a_2u_3^2v_3^3+3a_1^2a_2^2u_3^2v_3^3-5a_1a_2^3u_3^2v_3^3\\
&+2a_2^4u_3^2v_3^3+a_1^4u_3^2v_3^4-4a_1^3a_2u_3^2v_3^4+6a_1^2a_2^2u_3^2v_3^4-4a_1a_2^3u_3^2v_3^4+a_2^4u_3^2v_3^4\big).
\end{aligned}$$

Now applying [4, Theorem 2.19] we get that it is a saddle-node. In this case the local phase portrait near the straight line $u_2=0$ for system (20) is topologically equivalent to the one of Figure 5(a). Undoing the rescaling $dt_1=u_2dt$ we get the phase portrait of Figure 5(b). Going back through the changes of variables from the phase portrait of Figure 5(b), we obtain the local phase portrait of system (19) in Figure 5(c). Finally, going back through the changes of variables from the local phase of Figure 5(c), we obtain the local phase portrait at the origin of system (18) which is topologically equivalent to the one of Figure 5(d). Hence the origin of the local chart U_1 is formed by four hyperbolic sectors (two stable and two unstable) separated by one parabolic sector.

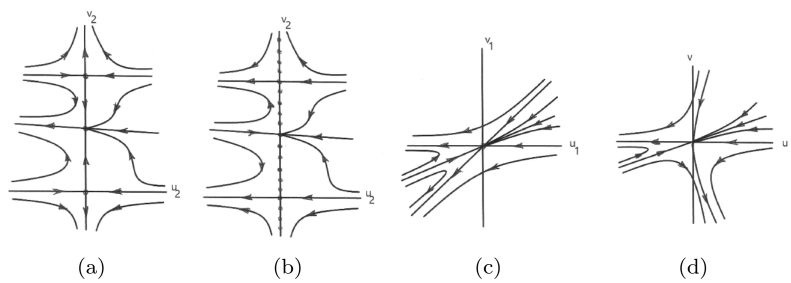


FIGURE 5. Figures of the blow up of the singular point located at the origin of the local chart U_1 of system (18).

4.3.2. *The local chart U_2.* System (1) in the chart U_2 writes

(21)
$$\dot{u} = a_2u^3+a_1u^3v-a_2uv^2+v^3-a_1uv^3+u^2v^3, \quad \dot{v} = a_2u^2v+a_1u^2v^2-a_2v^3-a_1v^4+uv^4.$$

The origin is an infinite singular point in this chart. The linear part of system (21) at the origin is identically zero, so in order to determine its local phase portrait we must do blow ups. The characteristic directions at the origin of U_2 are given by the real linear factors of $v^4 = 0$. Since $u = 0$ is not a characteristic direction, we do the vertical blow up $(u, v) \to (u_1, u_1v_1)$ and system (21) writes in the new variables

$$\dot{u}_1 = -u_1^3(-a_2 - a_1u_1v_1 + a_2v_1^2 - v_1^3 + a_1u_1v_1^3 - u_1^2v_1^3), \quad \dot{v}_1 = -u_1^2v_1^4.$$

Eliminating the common factor u_1^2 between $\dot{u}_1$ and $\dot{v}_1$ rescaling the time we obtain the system

$$\dot{u}_1 = -u_1(-a_2 - a_1u_1v_1 + a_2v_1^2 - v_1^3 + a_1u_1v_1^3 - u_1^2v_1^3), \quad \dot{v}_1 = -v_1^4 \tag{22}$$

The unique singular point of system (22) on the straight line $u_1 = 0$ is $(0, 0)$.

The eigenvalues of the linear part of system (22) at the singular point $(0, 0)$ are 0 and a_2 and so it is semi-hyperbolic. Applying [4, Theorem 2.19] we obtain that it is a saddle-node. Going back through the changes of variables and reascaling of the time we get that the origin of the local chart U_2 is topologically equivalent to the one provided in Figure 4(c) when $a_2 > 0$ and Figure 4(c) reversing the orientation of the orbits when $a_2 < 0$.

Proof of Theorem 2. When $a_2 = 0$ and $a_1 > 0$ the two stable separatrices of the origins of U_1 and V_1 are at infinity, and their unstable separatrices go to the finite part of the Poincaré disc. Moreover the origins of the local charts U_2 and V_2 are unstable nodes. Since the origin is an unstable focus if $a_1 \in (0, 2)$ and an unstable node if $a_1 \geq 2$ the Poincaré Bendixson Theorem forces the existence of at least one limit cycle. In view of Theorem 6 with $f(x) = a_1(x^2 - 1)$, $g(x) = x$, $F(x) = a_1(x^3/3 - x)$, $a = -1$, $b = 1$ and $c = \sqrt{3}$, we get that the limit cycle is unique. Combining all this information we obtain that the global phase portrait of system (1) when $a_2 = 0$ is topologically equivalent to the one of Figure 1(a) and so statement (i) is proved.

When $a_1 = 0$ and $a_2 > 0$ taking into account the local phase portraits at the origin of coordinates (which is a center), at the origins of U_1 and V_1 that are formed by the four hyperbolic sectors of the nilpotent saddle of Figure 3(d), the unstable node at the origin of U_2, the stable node at the origin of V_2, and that the global phase portrait is symmetric with respect to the x-axis (because the differential system (1) is invariant under the change $(x, y, t) \to (x, -y, -t)$) we get that the phase portrait in the Poincaré disc is topologically equivalent to the one of Figure 1(b). Hence statement (ii) is proved.

Finally, when $a_1 a_2 \neq 0$ and $a_1 a_2 < 0$ with $a_2 > 0$, taking into account the local phase portraits at the origin of coordinates (which is a either a stable focus or a stable node), at the origins of U_1 and V_1 that are formed by the four hyperbolic sectors separated by parabolic sectors of Figure 5(d), the unstable node at the origin of U_2 and the stable node at the origin of V_2 we get that a possible phase portrait in the Poincaré disc is topologically equivalent to the one of Figure 1(c) and we see that there are numerical values such that this phase portrait is realizable. Note that in this phase portrait there is a limit cycle. Hence the theorem is proved. In the case of $a_2 < 0$ we get the same global phase portrait reverting the orientation of the orbits. There are topologically two more possible global phase portraits but numerical evidence seem to show that they are not realizable. □

Acknowledgements

The first author is supported by the Agencia Estatal de Investigación grant PID2019-104658GB-I00 and the H2020 European Research Council grant MSCA-RISE-2017-777911. The second author is partially supported by FCT/Portugal through CAMGSD, IST-ID, projects UIDB/04459/2020 and UIDP/04459/2020.

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