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Crossing limit cycles for discontinuous piecewise differential systems formed by linear Hamiltonian saddles or linear centers separated by a conic

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ABSTRACT

The extension of the 16th Hilbert problem to discontinuous piecewise linear differential systems asks for an upper bound for the maximum number of crossing limit cycles that such systems can exhibit. The study of this problem is being very active, especially for discontinuous piecewise linear differential systems defined in two zones and separated by one straight line. In the case that the differential systems in these zones are formed either by linear centers or linear Hamiltonian saddles it is known that there are no crossing limit cycles. However it is also known that the number of crossing limit cycles can change if we change the shape of the discontinuity curve. In this paper we study the maximum number of crossing limit cycles of discontinuous piecewise differential systems formed by either linear Hamiltonian saddles or linear centers and separated by a conic which intersect the conic in two points. For this class of discontinuous piecewise differential systems we solve the extended 16th Hilbert problem.

1. Introduction and statement of the main results

Poincaré [22,23] was the first in introducing the notion of *limit cycle* of a differential system, i.e. a periodic orbit isolated in the set of all periodic orbits of the differential system. After the limit cycles became of great importance because they model many real world phenomena. This caused that the study of their existence, their number and their properties became very active, see for instance [3,5,12,19–21,26,27].

In general the problem of finding the limit cycles of a given class of differential systems is very difficult, in especial to provide an upper bound on the maximal number of limit cycles that a given class differential systems can exhibit. One of these classes is the class of discontinuous piecewise linear differential systems. Such systems were studied by first time by Andronov, Vitt and Khaikin in [1], and after their appearance it became clear that they have many applications in different areas, modeling real phenomena in a quite accurate way (see for instance [5, 25]). So now there is a great activity in studying these systems.

A *discontinuous piecewise differential system* on $\mathbb{R}^2$ is a pair of C^r (with $r \geq 1$) differential systems in $\mathbb{R}^2$ separated by a smooth codimension one manifold Σ. The *line of discontinuity* Σ of the discontinuous piecewise differential system is defined by $\Sigma = h^{-1}(0)$, where $h : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function having 0 as a regular value. Note that Σ is the separating boundary of the regions $\Sigma^+ = \{(x,y) \in \mathbb{R}^2 | h(x,y) > 0\}$ and $\Sigma^- = \{(x,y) \in \mathbb{R}^2 | h(x,y) < 0\}$. So the piecewise C^r vector field associated to a piecewise differential system with line of discontinuity Σ is

$$Z(x,y) = \begin{cases} X(x,y), & \text{if } h(x,y) \geq 0, \\ Y(x,y), & \text{if } h(x,y) \leq 0. \end{cases} \tag{1}$$

As usual the vector field associated to system (1) is denoted by $Z = (X,Y,\Sigma)$ or simply by $Z = (X,Y)$, when the separation line Σ is well understood. In order to establish a definition for the trajectories of Z and investigate its behavior, we need a criterion for the transition of the orbits between Σ^+ and Σ^- across Σ. The contact between the vector field X (or Y) and the line of discontinuity Σ is characterized by the derivative of h in the direction of the vector field X, i.e.

$$Xh(p) = \langle \nabla h(p), X(p) \rangle,$$

where $\langle .,. \rangle$ is the usual inner product in $\mathbb{R}^2$. The basic results of the discontinuous piecewise differential systems in this context were stated by Filippov [7]. We can divide the line of discontinuity Σ in the following sets:

(a) *Crossing set*: $\Sigma^c : \{p \in \Sigma : Xh(\mathbf{x}) \cdot Yh(\mathbf{x}) > 0\}$.
(b) *Escaping set*: $\Sigma^e : \{p \in \Sigma : Xh(\mathbf{x}) > 0 \text{ and } Yh(\mathbf{x}) < 0\}$.
(c) *Sliding set*: $\Sigma^s : \{p \in \Sigma : Xh(\mathbf{x}) < 0 \text{ and } Yh(\mathbf{x}) > 0\}$.

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