ON THE TRAVELING PULSES OF THE FITZHUGH-NAGUMO EQUATIONS

JAUME LLIBRE[1] AND CLAUDIA VALLS[2]

Abstract. In this paper we characterize the phase portraits of the polynomial differential systems

$$\dot{x} = y, \quad \dot{y} = -cy + x(x-a)(x-1) + w,$$

with $a, c, w \in \mathbb{R}$. We give the complete description of their phase portraits in the Poincaré disc (i.e. in the compactification of $\mathbb{R}^2$ adding the circle $\mathbb{S}^1$ of the infinity) modulo topological equivalence. When $w = 0$ the homoclinic orbits of the origin of coordinates are related with the travelling pulses of the FitzHugh-Nagumo equations.

1. Introduction and statement of the main results

Numerous problems of applied mathematics, or in physics, chemistry, economics, ... are modeled by polynomial differential systems. Excluding linear differential systems the quadratic polynomial differential systems are the ones with the lowest degree of complexity, and the large bibliography on them proves their relevance, see the books [1, 12, 13] and the surveys [2, 3]. After the quadratic polynomial differential systems come the cubic ones, which also have many applications.

This paper deals with the travelling waves of the FitzHugh-Nagumo equations

$$u_t = u_{xx} + f(u) - w, \quad w_t = \varepsilon(u - \gamma w), \tag{1}$$

being $f(u) = u(u-a)(1-u)$, where $a < 1/2$, ε and γ are positive real numbers. Here we are interested in the case $\varepsilon << 1$ and $\gamma << 1$.

The FitzHugh-Nagumo equations are a simplification of the Hodgking-Hexley equations for nerve conduction, see FitzHugh [5] and Nagumo et al. [10].

A solution of system (1) of the form $(u(s), w(s))$ with $s = x - ct$ with $c \neq 0$ is a *travelling wave* and it satisfies the differential system

$$-cu' = u'' + f(u) - w, \quad -cw' = \varepsilon(u - \gamma w), \tag{2}$$

where the prime denotes derivative with respect to the variable s.

A travelling way satisfying $(u, w) \to (0, 0)$ when $s \to \pm\infty$ is a *travelling pulse.*

We write the differential system (2) as the differential system of first order

$$u' = v, \quad v' = -cv - f(u) + w, \quad w' = -(\varepsilon/c)(u - \gamma w). \tag{3}$$

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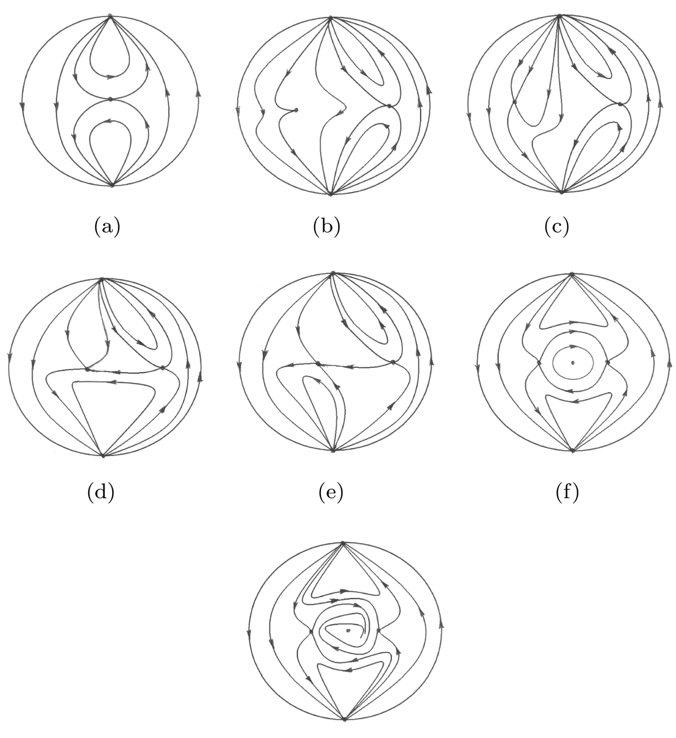
System (3) has $\mathbb{R}^3$ as its phase space and it has a origin $(0,0,0)$ as a singular point, and consequently a travelling pulse is a homoclinic orbit to the origin. If $\varepsilon = 0$ it follows that each plane $w =$ constant is invariant, i.e. if an orbit of system (3) has a point in a such plane the whole orbit is contained in that plane.

The objective of this paper is to study the phase portraits of system (3) with $\varepsilon = 0$ that we shall write as

$$x' = y, \quad y' = -cy - x(x-a)(1-x) + w \tag{4}$$

with $a, c, w \in \mathbb{R}$ and the prime means derivative with respect to $s \in \mathbb{R}$. We are specially interested in the homoclinic orbits at the origin when $c > 0$, $a < 1/2$ and $w = 0$. For more details in the travelling pulses see [6, 7, 8, 9].

Doing the change of variables $(x, y, t, c) \to (x, -y, -t, c)$ in what follows we assume that $c \geq 0$.



(a) (b) (c) (d) (e) (f) (g)

Figure 1. The seven phase portraits in the Poincaré disc of Theorem 1.

More precisely, first we classify the phase portraits of the polynomial differential systems (4) in the Poincaré disc modulo topological equivalence. As any polynomial differential system, system (4) can be extended to an analytic system on a closed disc of radius one, whose interior is diffeomorphic to $\mathbb{R}^2$ and its boundary, the circle $\mathbb{S}^1$, plays the role of the infinity. This closed disc is denoted by $\mathbb{D}^2$ and called the *Poincaré disc*, because the technique for doing such an extension is the *Poincaré compactification* for a polynomial differential system in $\mathbb{R}^2$, which is described in details in Chapter 5 of [4]. In this paper we shall use the notation of that chapter. By using this compactification technique the dynamics of system (4) in a neighborhood of the infinity can be studied.

Theorem 1. *The phase portraits of the polynomial differential system* (4) *in the Poincaré disc are topologically equivalent to one of the seven phase portraits presented in Figure 1.*

2. Infinite singular points

The following lemma summarizes the information at the infinite singular points of system (4).

Lemma 2. *There is a pair of infinite singular points located at the origin of the local charts $U_2 \cup V_2$ and are formed by two elliptic sectors separated by parabolic sectors, and the infinity straight line crosses the two parabolic sectors leaving in each side an elliptic sector, see Figure* 2.

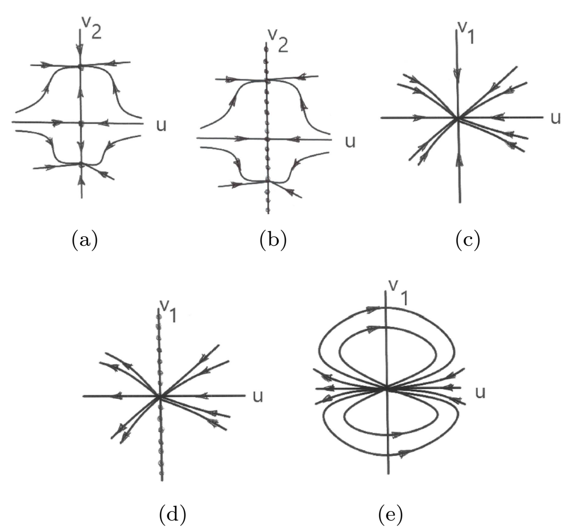


Figure 2. The local phase portraits of the blow ups of Lemma 2.

Proof. First we determine the local phase portrait of the infinite singular points in the local chart U_1. The expression of system (4) in this chart is

$$\begin{aligned}\dot{u} &= 1-(a+1)v+av^2-cuv^2+wv^3-u^2v^2,\\ \dot{v} &= -uv^3.\end{aligned}$$

There are no infinite singular points in the local chart U_1.

Now we analyze the phase portrait in the local chart U_2, we only need to study the origin of U_2, the other infinite singular points have been studied in the local chart U_1. The expression of the system in this chart is

$$\begin{aligned} \dot{u} &= v^2 + cuv^2 - u^4 + (a+1)u^3v - au^2v^2 - wuv^3, \\ \dot{v} &= -v(-cv^2 + u^3 - (a+1)u^2v + auv^2 + wv^3). \end{aligned} \tag{5}$$

Note that the origin of the local chart U_2 is a singular point whose linear part is identically zero. We need to do a blow-up procedure to study its local phase portrait. We first introduce the coordinates (u, v_1) where $v_1 = v/u$. In these new coordinates we obtain

$$\begin{aligned} \dot{u} &= -u^2(u^2 - v_1^2 - (a+1)u^2v_1 - cuv_1^2 + au^2v_1^2 + wu^2v_1^3), \\ \dot{v}_1 &= -uv_1^3. \end{aligned}$$

We introduce the reparameterization of time $ds = udt$ and we get

$$\begin{aligned} \dot{u} &= -u(u^2 - v_1^2 - (a+1)u^2v_1 - cuv_1^2 + au^2v_1^2 + wu^2v_1^3), \\ \dot{v}_1 &= -v_1^3, \end{aligned} \tag{6}$$

where now the dot means derivative in the new time s. Only the origin is on the straight line $u = 0$ and again its linear part is identically zero. We apply a second blow-up introducing the new coordinates (u, v_2) where $v_2 = v_1/u$. In these new coordinates we obtain the system

$$\begin{aligned} \dot{u} &= -u^3(1 - (a+1)uv_2 - v_2^2 - cuv_2^2 + au^2v_2^2 + wu^3v_2^3), \\ \dot{v}_2 &= u^2v_2(1 - (a+1)uv_2 - 2v_2^2 - cuv_2^2 + au^2v_2^2 + wu^3v_2^3). \end{aligned}$$

We introduce the reparameterization of time $dr = u^2ds$ and we get

$$\begin{aligned} \dot{u} &= -u(1 - (a+1)uv_2 - v_2^2 - cuv_2^2 + au^2v_2^2 + wu^3v_2^3), \\ \dot{v}_2 &= v_2(1 - (a+1)uv_2 - 2v_2^2 - cuv_2^2 + au^2v_2^2 + wu^3v_2^3). \end{aligned}$$

where now the dot means derivative in the new time r. There are three singular points in $u = 0$ which are $(0, 0)$, $(0, -1/\sqrt{2})$ and $(0, 1/\sqrt{2})$. Computing the eigenvalues of the Jacobian matrix at these singular points we get that $(0, 0)$ is a saddle (the two eigenvalues are $-1, 1$) and the singular points $(0, \pm 1/\sqrt{2})$ are both stable nodes (the two eigenvalues are in both cases $-2, -1/2$). Therefore the local phase portrait near the straight line $u = 0$ is topologically equivalent to the one of Figure 2(a).

Undoing the rescaling $dr = u^2ds$ we get the phase portrait of Figure 2(b). Going back through the changes of variables from the phase portrait of Figure 2(b), we obtain the local phase portrait at the origin of system (6) which is topologically equivalent to the one of Figure 2(c). Again undoing the rescaling $ds = udt$ we obtain the phase portrait of Figure 2(d), and going back through the changes of variables from the phase portrait of Figure 2(d), we obtain the local phase portrait at the origin of system (5) which is topologically equivalent to the one of Figure 2(e). Hence the origin of the local chart U_2 is formed by two elliptic sectors separated by parabolic sectors, and the infinite straight line crosses the two parabolic sectors leaving in each side an elliptic sector. □

3. Finite singular points

We first obtain the following result concerning the non-existence of limit cycles.

Proposition 3. *System* (4) *has no limit cycles.*

Proof. The divergence of system (4) is constant equal to $-c$. Therefore if $c > 0$ it follows from the Bendixson Theorem (see [4, Theorem 7.10]) that there are no limit cycles. On the other hand if $c = 0$ system (4) has the polynomial first integral

$$H(x,y) = \frac{y^2}{2} - \frac{x^4}{4} + \frac{(a+1)x^3}{3} - \frac{ax^2}{2} - wx, \tag{7}$$

and so when $c = 0$ there are no limit cycles. □

Proposition 4. *The vector field*

$$(P(x,y), Q(x,y)) = (y, -cy - x(x-a)(1-x) + w),$$

associated to system (4) *is a generalized rotated vector field with respect the parameter* c*.*

Proof. Since

$$\left| \begin{pmatrix} P & Q \\ \dfrac{\partial P}{\partial c} & \dfrac{\partial Q}{\partial c} \end{pmatrix} \right| = -y^2 \leq 0,$$

the proposition follows. See for more details [4, Chapter 7], [11] and [14, Chapter 4]. □

The finite singular points of system (4) are the real solutions of $\dot{x} = \dot{y} = 0$. Computing such solutions we obtain $y = 0$ and $x = x^*$ where x^* is any real solution of

$$F(x) = x(x-a)(x-1) + w = 0.$$

We have several possibilities.

3.1. **Three real roots for $F(x)$.** In this case we have that $F(x)$ is of the form $(x - r_1)(x - r_2)(x - r_3)$ with $r_1, r_2, r_3 \in \mathbb{R}$ and $r_1 < r_2 < r_3$ if

(i) the parameters of $F(x)$ when $r_3 \neq 1 - r_2$. are

$$w = \frac{(r_2-1)r_2(r_3-1)r_3}{r_2+r_3-1}, \quad a = \frac{-r_2 + r_2^2 - r_3 + r_2 r_3 + r_3^2}{r_2+r_3-1},$$

with
either $r_2 < 0$, $r_2 + r_3 < 1$ and $(r_2-1)^2 + 2r_2r_3 < r_3$;
or $0 < 2r_2 < 1$, $r_2 + r_3 > 1$ and $(r_2-1)^2 + 2r_2r_3 > r_3$;
or $2r_2 \geq 1$ and $r_3 > r_2$;

(ii) or the parameters of $F(x)$ when $r_3 = 1 - r_2$, are $w = 0$, $a = r_1$, $r_2 = 0$, and so $r_3 = 1$.

Now we study the local behavior of each singular point $(r_i, 0)$ with $i = 1, 2, 3$.

The Jacobian matrix at each singular point $(r_i, 0)$ is of the form

$$\begin{pmatrix} 0 & 1 \\ \prod_{j\in\{1,2,3\}, j\neq i}(r_i - r_j) & -c \end{pmatrix},$$

whose eigenvalues are

$$\lambda = \frac{-c \pm \sqrt{c^2 + 4\prod_{j\in\{1,2,3\},j\neq i}(r_i - r_j)}}{2}.$$

Therefore, since $r_1 < r_2 < r_3$ we get that $(r_1, 0)$ is a saddle. Moreover, $(r_2, 0)$ is a stable node if $c > 0$ and $c^2 + 4(r_2 - r_1)(r_2 - r_3) \geq 0$, it is a stable focus if $c > 0$ and $c^2 + 4(r_2 - r_1)(r_2 - r_3) < 0$ and it is a center if $c = 0$ (by the proof of Proposition 3 it has a polynomial first integral). Finally, $(r_3, 0)$ is a saddle.

3.2. **Two real roots for** $F(x)$**.** In this case we can write $F(x) = (x - r_1)^2(x - r_2)$ with $r_1, r_2 \in \mathbb{R}$ and $r_1 \neq r_2$ whenever

$$w = \frac{(r_1 - 1)^2 r_1^2}{2r_1 - 1}, \quad a = \frac{r_1(3r_1 - 2)}{2r_1 - 1}, \quad 2r_1 - 1 \neq 0, \quad r_1 \neq 1 - r_2 \pm \sqrt{r_2(r_2 - 1)}. \tag{8}$$

Now we study the local behavior of each singular point $(r_1, 0)$ and $(r_2, 0)$.

We first study the local behavior of $(r_2, 0)$. The Jacobian matrix at the singular point $(r_2, 0)$ is

$$\begin{pmatrix} 0 & 1 \\ (r_2 - r_1)^2 & -c \end{pmatrix},$$

whose eigenvalues are

$$\lambda = \frac{-c \pm \sqrt{c^2 + 4(r_2 - r_1)^2}}{2}$$

and so $(r_2, 0)$ is a saddle.

The Jacobian matrix at each singular point $(r_1, 0)$ is of the form

$$\begin{pmatrix} 0 & 1 \\ 0 & -c \end{pmatrix}$$

and so it is semihyperbolic if $c \neq 0$ and nilpotent if $c = 0$. In the first case using [4, Proposition 2.19] we get that it is a saddle-node and in the second case, using [4, Proposition 3.5] we get that it is a cusp.

3.3. **One real root for** $F(x)$**.** In this case we have that either $F(x) = (x - r_1)^3$ with $r_1 \in \mathbb{R}$, or $F(x) = (x - r_1)(x^2 - 2\alpha x + (\alpha^2 + \beta^2))$ with $r_1, \alpha, \beta \in \mathbb{R}$.

The first case is not possible because the equation $(x - r_1)^3 = x(x - a)(x - 1) + w$ has no real solutions. The second case is achieved whenever

$$w = \frac{\alpha^2 - 2\alpha^3 + \alpha^4 + \beta^2 - 2\alpha\beta^2 + 2\alpha^2\beta^2 + \beta^4}{2\alpha - 1}, \; a = \frac{3\alpha^2 - \beta^2 - 2\alpha}{2\alpha - 1}, \; 2\alpha - 1 \neq 0.$$

The Jacobian matrix at the singular point $(r_1, 0)$ is

$$\begin{pmatrix} 0 & 1 \\ \beta^2 + (r_1 - \alpha)^2 & -c \end{pmatrix},$$

whose eigenvalues are

$$\lambda = \frac{-c \pm \sqrt{c^2 + 4(\beta^2 + (r_1 - \alpha)^2)}}{2}$$

Therefore, $(r_1, 0)$ is a saddle because $-\beta^2 - (r_1 - \alpha)^2 < 0$.

4. Phase portraits in the Poincaré disc

Here we shall study the phase portraits in the Poincaré disc using the local phase portraits of the finite and infinite singular points, together with Proposition 3, which states that system (4) has no limit cycles.

For all values of the parameters a, c and w system (4) at infinity has a unique pair of singular points, the origins of U_2 and V_2, which in the Poincaré disc has one elliptic sector and two parabolic sectors in its sides, see Lemma 2.

4.1. **One real root for $F(x)$.** In this case system (4) has a unique finite singular point which is a saddle, see subsection 3.3. Taking into account the singular points at infinity the unique possible global phase portrait in the Poincaré disc is given in Figure 1(a).

4.2. **Two real roots for $F(x)$.** *Case* $c = 0$. Then system (4) is a Hamiltonian system with the Hamiltonian $H(x, y)$ given in (7). This system has two finite singular points, a saddle and a cusp, see subsection 3.2.

The unique separatrices in the interior of the Poincaré disc are the six separatrices, the four of the saddle and the two of the cusp. We claim that there is no connections between the saddle $(r_2, 0)$ and the cusp $(r_1, 0)$. Indeed, if a connection exists we must have $H(r_1, 0) = H(r_2, 0)$, and taking into account (8) this equality, after some computations, is equivalent to

$$\frac{3(r_1 - 1)r_1 + 1}{2r_1 - 1} = 0.$$

Since this equation has no real roots, the claim is proved.

Taking into account the local phase portraits at the finite and infinite singular points, and that there are no connections between the separatrices the unique possible phase portrait in the Poincaré disc is 1(b).

Case $c > 0$. Now system (4) has two finite singular points, a saddle at $(r_2, 0)$ and a saddle-node at $(r_1, 0)$. We consider that $r_1 < r_2$, the case $r_1 > r_2$ can be studied in a similar way.

Computing the eigenvalues of the linear part of system (4) at the singular points $(r_1, 0)$ and $(r_2, 0)$ we obtain that the local separatrices near these two singular points have the orientation with respect to the straight line $y = 0$ shown in Figure 3.

Note that system (4) has seven separatrices in the interior of the Poincaré disc, we must determine where they born and where they die. Now taking into account the flow of system (4) on $y = 0$ and the local phase portraits of the finite and infinite singular points we get that the α- and ω-limit of the separatrices drawn in Figure 3 are determined. It remains to determine the α-limit of the separatrix s_1 of $(r_1, 0)$ and the ω-limit of the separatrix s_2 of $(r_2, 0)$.

The ω-limit of the unstable separatrix of $(r_2, 0)$ that we must determine has three possibilities: first to be the origin of the local chart V_2, second to connect with the free stable separatrix of $(r_1, 0)$, and third to reach the saddle-node $(r_1, 0)$ through its parabolic sector. The first case can be realized for the value $c = 1$, $r_1 = 0$ and

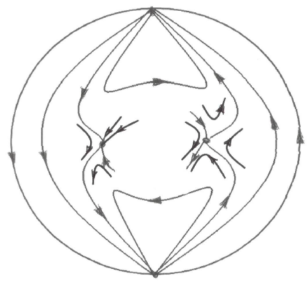


Figure 3. The phase portrait with a finite saddle and a finite saddle-node.

$r_2 = 1$ in Figure 1(c), the third case is realized by $c = 2$, $r_1 = 0$, $r_2 = 1$ in Figure 1(e). So by continuity moving c between the values 1 and 2 we obtain the phase portrait of Figure 1(d).

4.3. **Three real roots for** $F(x)$**.** *Case* $c = 0$. Then system (4) is a Hamiltonian system with the Hamiltonian $H(x, y)$ given in (7). This system has three finite singular points, two saddles at $(r_1, 0)$ and $(r_3, 0)$, and a center at $(r_2, 0)$, see subsection 3.3.

The unique separatrices in the interior of the Poincaré disc are the eight separatrices of the two saddles. We claim that there are two connections between the saddles $(r_1, 0)$ and $(r_3, 0)$ forming a heteroclinic loop. Indeed, if such connections exist we must have $H(r_1, 0) = H(r_3, 0)$, and this is possible whenever $r_1 - 2r_2 + r_3 = 0$. Taking into account the local phase portraits of the finite and infinite singular points, and that there is a unique heteroclinic loop with the two saddles, it follows the existence of a unique topologically equivalent phase portrait in the Poincaré disc given in Figure 1(f).

Case $c > 0$. Now system (4) has three finite singular points, a saddle at $(r_1, 0)$ a stable node (or a stable focus) at $(r_2, 0)$ and a saddle at $(r_3, 0)$. We consider fixed $r_1 < r_2 < r_3$ and we vary c from 0 to ∞. Note that system (4) has eight separatrices in the interior of the Poincaré disc and we must determine where they born and where they die. Now taking into account the flow of system (4) on $y = 0$ and the local phase portraits of the finite and infinite singular points we get that the α- and ω-limit of the separatrices drawn in Figure 4 are determined. It remains to determine the α-limit and ω-limit of the four separatrices which for $c = 0$ form the heteroclinic loop.

It follows from Proposition 4 that the vector field associated to system (4) is a generalized rotated vector field with respect to the parameter c. So, the heteroclinic loop that exists for $c = 0$ breaks down (see [14, Chapter 4]) and since for $c > 0$ sufficiently small the singular point $(r_2, 0)$ is a stable focus, the ω-limit of the separatrices s_1 and s_2 of Figure 4 is the stable focus $(r_2, 0)$ and consequently the α-limit of the separatrices s_3 and s_4 of Figure 4 are determined provinding the phase portrait 1(g). Note that when c increases the stability of the stable focus, and after

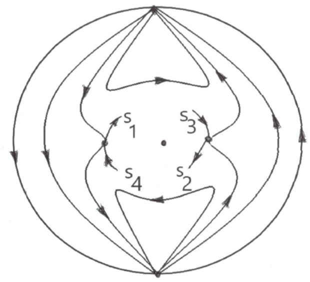


Figure 4. The phase portrait with two finite saddles and a finite anti-saddle, i.e. either a center, or a focus, or a node.

of the stable node, increases and consequently the ω-limit of the separatrices s_1 and s_2 is always the stable singular point $(r_2, 0)$.

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[1] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

Email address: jllibre@mat.uab.cat

[2] Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais 1049–001, Lisboa, Portugal

Email address: cvalls@math.tecnico.ulisboa.pt