

REVERSIBLE GLOBAL CENTERS WITH QUINTIC HOMOGENEOUS NONLINEARITIES

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ABSTRACT. A *center* of a differential system in the plane \mathbb{R}^2 is an equilibrium point \mathbf{p} having a neighbourhood U such that $U \setminus \{\mathbf{p}\}$ is filled of periodic orbits. A *global center* is a center \mathbf{p} such that $\mathbb{R}^2 \setminus \{\mathbf{p}\}$ is filled of periodic orbits. To determine when a given differential system has a center is in general a difficult problem, but to determine if a given differential system has a global center is even more difficult.

We deal with the class of polynomial differential systems of the form

$$(1) \quad \dot{x} = -y + P(x, y), \quad \dot{y} = x + Q(x, y),$$

with P and Q homogeneous polynomials of degree n . It is known that these systems only can have global centers if n is odd. The global centers when n is 1 or 3 have been characterized.

Here for $n = 5$ we classify the global centers of a four parameter family of systems (1). In particular we illustrate how to study the local phase portraits of the singular points whose linear part is identically zero using only vertical blow ups.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

The rigorous notion of a center appeared in the works of Poincaré [17] in 1881 and Dulac [6] in 1908. But informally the notion of a center can be found already in the work of Huygens in 1656 on the pendulum clock, see [12, 16].

A *polynomial differential system* in the plane \mathbb{R}^2 is a differential system

$$(2) \quad \dot{x} = p(x, y), \quad \dot{y} = q(x, y),$$

with p and q polynomials in the variables x and y with real coefficients, and the dot denotes derivate with respect to the time t . A polynomial

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differential system has *degree* n if n is the maximum of the degrees of the polynomials p and q .

If a polynomial differential system has a center at the origin, then after a linear change of variables and a rescaling of the time variable, it can be written in one of the following three forms:

$$\dot{x} = -y + X_2(x, y), \quad \dot{y} = x + Y_2(x, y),$$

called a *linear type center*,

$$\dot{x} = y + X_2(x, y), \quad \dot{y} = Y_2(x, y),$$

called a *nilpotent center*,

$$\dot{x} = X_2(x, y), \quad \dot{y} = Y_2(x, y),$$

called a *degenerate center*, where $X_2(x, y)$ and $Y_2(x, y)$ are real analytic functions without constant and linear terms, defined in a neighborhood of the origin. Here we only will work with linear type centers.

The easiest global centers are the linear differential centers, which after an affine change of variables can be written as $\dot{x} = -y$, $\dot{y} = x$.

It is known that if the degree of a polynomial differential system is even then it cannot have global centers because these differential systems always have some orbit which escape or come from the infinity, see Galeotti and Villarini [8], or [14] for two different proofs.

Conti also studied the global centers in [4, 5]. In fact he stated the following problem: *Identify all polynomial differential systems (of odd degree) having a global center*, see the Problem 14.1 of [5].

The classification of the centers of the polynomial differential systems is a very difficult problem, in fact only the centers of the polynomial differential systems of degree 2 have been classified, see Kapteyn [13] and Bautin [3]. For polynomial differential systems of degree higher than 2 there are only partial results.

We consider the particular class of polynomial differential systems of the form

$$(3) \quad \dot{x} = -y + P(x, y), \quad \dot{y} = x + Q(x, y),$$

with P and Q homogeneous polynomials of degree n .

Note that all polynomial differential systems of degree 2 can be written in the form (3).

The centers of the polynomial differential systems of degree 3 of the particular class (3) were classified by Malkin [15] and Vulpe and Sibirskii [18].

The global centers of the polynomial differential systems (3) of degree 3 have been classified recently in [9].

The objective of this paper would be to classify the global centers of the polynomial differential systems (3) of degree 5. But unfortunately at the present moment we do not have the complete classification of the centers of these class of differential systems. Such classification depends on the computation of all the independent Liapunov constants of these systems, see for more details chapter 5 of [7]. So we restrict our attention to the classification of the global centers of the subclass of polynomial differential systems (3) of degree 5 having a reversible center. More precisely, we restrict our attention to the subclass of polynomial differential systems (3) of degree 5 which are invariant under the symmetry $(x, y, t) \rightarrow (x, -y, -t)$.

A first result is the following.

Proposition 1. *Polynomial differential systems (3) of degree 5 which are invariant under the symmetry $(x, y, t) \rightarrow (x, -y, -t)$ and that can have a global center can be written as*

$$(4) \quad \dot{x} = -y + ax^4y + bx^2y^3 - r^2y^5, \quad \dot{y} = x + s^2x^5 + cx^3y^2 + dxy^4,$$

with $a, b, c, d, r, s \in \mathbb{R}$.

The classification of the reversible global centers of the polynomial differential systems (4) depending on six parameters is not possible at the present moment due to the huge computations for doing it. So in this paper we shall classify the reversible global centers of the polynomial differential systems (4) when $r = s = 0$, i.e. of the system

$$(5) \quad \dot{x} = -y + ax^4y + bx^2y^3, \quad \dot{y} = x + cx^3y^2 + dxy^4,$$

with $a, b, c, d \in \mathbb{R}$.

The following result characterize systems (5) having the origin of coordinates as the unique finite singular point.

Proposition 2. *The unique finite singular point of system (5) is the origin of coordinates if and only if one of the following two sets of conditions holds:*

- (a) $-ad(bc - ad) > 0$ and $-d(bc - ad)(2ad - bc - c^2) > 0$;
- (b) $d(bc - ad) = 0$ and either $a = 0$, or $2ad - bc - c^2 = 0$.

The global centers of systems (5) are classified in the next result.

Theorem 3. *Polynomial differential systems (5) have a global center at the origin of coordinates if and only if one of the following five sets of conditions holds:*

- (i) $a = 0, c = 0$ and $0 \leq d \leq -b$;
- (ii) $a \leq 0, b = 0, c > 0, d = 0$ and $c + a > 0$;
- (iii) $b < 0, c > 0, |c| > |a|$, and $|b| \geq |d|$;
- (iv) $b < 0, c > 0, c + a = 0$, and $|b| \geq |d|$;
- (v) $c = a > 0, b < 0, d > b$ and $b + d \leq 0$.

In section 2 we provide some definitions and preliminary results that we need for proving Theorem 3. In section 3 we prove Propositions 1 and 2, and we do the blow ups for studying the infinite singular points. Finally in section 4 we prove Theorem 3.

2. PRELIMINARY RESULTS

2.1. Singular points. The point (a, b) is a *singular point* of the differential system (2) if $p(a, b) = q(a, b) = 0$.

The singular point (a, b) is *hyperbolic* if the eigenvalues of the Jacobian matrix of the function (p, q) evaluated at (a, b) have non-zero real part. The classification of the local phase portraits of the hyperbolic singular points is well known, see for instance Theorem 2.15 of [7]. In this paper when we characterize the local phase portrait of a hyperbolic singular point we will be using that theorem.

The singular point (a, b) is *semi-hyperbolic* if one and only one of the eigenvalues of the Jacobian matrix of the function (p, q) evaluated at (a, b) is zero. Also the classification of the local phase portraits of the semi-hyperbolic singular points is well known, see for instance Theorem 2.19 of [7]. Again in this paper when we characterize the local phase portrait of a semi-hyperbolic singular point we will be using that theorem.

Consider the differential system

$$(6) \quad \dot{x} = \sum_{i=1}^{\infty} p_i(x, y), \quad \dot{y} = \sum_{i=1}^{\infty} q_i(x, y),$$

where p_i and q_i are homogeneous polynomials of degree i , for $i \geq 1$. The *characteristic directions* of the singular point localized at the origin of coordinates of system (6) are given by the straight lines trough

the origin defined by the real linear factors of the homogeneous polynomial $p_k(x, y)y - q_k(x, y)x$, where k is the minimum i for which the polynomials p_i or q_i are non-zero. It is known that the orbits which end or start at the origin of coordinates must arrive or exit tangent to these straight lines. For more details on the characteristic directions see for example [2].

When the Jacobian matrix of the function (p, q) evaluated at the singular point (a, b) of the differential system (2) is identically zero, then the local phase portrait at this singular points can be studied doing special changes of variables called blow ups, see for instance [1]. Here we only shall use vertical blow ups, and when the vertical direction is a characteristic direction we twist it to the diagonal direction.

Let Φ_t be a smooth flow on a manifold M and let C be a submanifold of M consisting entirely of singular points of the flow. C is called *normally hyperbolic* if the tangent bundle to M over C splits into three subbundles TC , E^s and E^u invariant under the differential $d\Phi_t$ and satisfying

- (i) $d\Phi_t$ contracts E^s exponentially,
- (ii) $d\Phi_t$ expands E^u exponentially,
- (iii) TC is the tangent bundle of C .

For normally hyperbolic submanifolds one has the usual existence of smooth stable and unstable manifolds together with the persistence of these invariant manifolds under small perturbations. More precisely, we have the following theorem, for a proof see [11].

Theorem 4. *Let C be a normally hyperbolic submanifold of singular points for the flow Φ_t . Then there exist smooth stable and unstable manifolds tangent along C to $E^s \oplus TC$ and $E^u \oplus TC$, respectively. Moreover, both C and the stable and unstable manifolds are permanent under small perturbations of the flow.*

2.2. The Poincaré compactification. Roughly speaking the Poincaré compactification consists in identifying the plane \mathbb{R}^2 with the interior of a closed unit disc centered at the origin of coordinates, called the *Poincaré disc*. Then the boundary of this disc (the unit circle centered at the origin) is identified with the infinity of \mathbb{R}^2 . Note that in \mathbb{R}^2 we can go or come from the infinity in as many as directions as points has that circle.

In order to classify the global dynamics of a polynomial differential system one of the main steps is to characterize the local phase portraits of its finite and infinite singular points in the Poincaré disc. For doing this we need the equations of our polynomial differential systems initially in \mathbb{R}^2 in the Poincaré disc.

Consider the differential system (2) in \mathbb{R}^2 , where p and q are real polynomials in the variables x and y of degrees d_1 and d_2 , respectively. Then the degree of the polynomial differential system (2) is $d = \max\{d_1, d_2\}$.

Denote by $T_p\mathbb{S}^2$ be the tangent space to the 2-dimensional sphere

$$\mathbb{S}^2 = \{\mathbf{s} = (s_1, s_2, s_3) \in \mathbb{R}^3 : s_1^2 + s_2^2 + s_3^2 = 1\}$$

at the point p , we call this sphere the *Poincaré sphere*. We consider that the polynomial differential system (2) is defined in the tangent plane to \mathbb{S}^2 at the point $(0, 0, 1)$, i.e. we have identified \mathbb{R}^2 with $T_{(0,0,1)}\mathbb{S}^2$. The central projection $f: T_{(0,0,1)}\mathbb{S}^2 \rightarrow \mathbb{S}^2$ send each point p of $T_{(0,0,1)}\mathbb{S}^2$ to two points of \mathbb{S}^2 , one in the northern hemisphere and the other in the southern hemisphere. These two points are the intersection of the straight line through p and the origin of coordinates (the center of the sphere). So the map f defines two copies of the polynomial differential system (2) on the sphere, one in the open northern hemisphere and the other in the open southern hemisphere.

If $X = (p, q)$ is the vector field associated to the polynomial differential system (2), we denote by X' the vector field $Df \circ X$ defined on \mathbb{S}^2 except on its equator $\mathbb{S}^1 = \{s \in \mathbb{S}^2 : s_3 = 0\}$. Clearly \mathbb{S}^1 can be identified with the infinity of \mathbb{R}^2 . If the degree of the polynomial vector field X is d , then $p(X)$ is the only analytic extension of $s_3^{d-1}X'$ to \mathbb{S}^2 . The vector field $p(X)$ on \mathbb{S}^2 is called the *Poincaré compactification* of the vector field X , for more details see [7, chapter 5].

On the Poincaré sphere \mathbb{S}^2 we use the following six local charts, which are given by $U_i = \{\mathbf{s} \in \mathbb{S}^2 : s_i > 0\}$ and $V_i = \{\mathbf{s} \in \mathbb{S}^2 : s_i < 0\}$, for $i = 1, 2, 3$, with the corresponding diffeomorphisms

$$\varphi_i : U_i \rightarrow \mathbb{R}^2, \quad \psi_i : V_i \rightarrow \mathbb{R}^2,$$

defined by $\varphi_i(\mathbf{s}) = -\psi_i(\mathbf{s}) = (s_m/s_i, s_n/s_i) = (u, v)$ for $m < n$ and $m, n \neq i$. Thus the coordinates (u, v) will play different roles in the distinct local charts. The expressions of the vector field $p(X)$ are

$$(\dot{u}, \dot{v}) = \left(v^d \left(Q \left(\frac{1}{v}, \frac{u}{v} \right) - uP \left(\frac{1}{v}, \frac{u}{v} \right) \right), -v^{d+1}P \left(\frac{1}{v}, \frac{u}{v} \right) \right) \quad \text{in } U_1,$$

$$\begin{aligned}
 (\dot{u}, \dot{v}) &= \left(v^d \left(P \left(\frac{u}{v}, \frac{1}{v} \right) - uQ \left(\frac{u}{v}, \frac{1}{v} \right) \right), -v^{d+1}Q \left(\frac{u}{v}, \frac{1}{v} \right) \right) \quad \text{in } U_2, \\
 (\dot{u}, \dot{v}) &= (P(u, v), Q(u, v)) \quad \text{in } U_3.
 \end{aligned}$$

We note that the expressions of the vector field $p(X)$ in the local chart (V_i, ψ_i) is equal to the expression in the local chart (U_i, ϕ_i) multiplied by $(-1)^{d-1}$ for $i = 1, 2, 3$.

The orthogonal projection under $\pi(y_1, y_2, y_3) = (y_1, y_2)$ of the closed northern hemisphere of \mathbb{S}^2 onto the plane $s_3 = 0$ is a closed disc \mathbb{D}^2 of radius one centered at the origin of coordinates called the *Poincaré disc*. Since a copy of the vector field X on the plane \mathbb{R}^2 is in the open northern hemisphere of \mathbb{S}^2 , the interior of the Poincaré disc \mathbb{D}^2 is identified with \mathbb{R}^2 and the boundary of \mathbb{D}^2 , the equator \mathbb{S}^1 of \mathbb{S}^2 , is identified with the infinity of \mathbb{R}^2 . Consequently the phase portrait of the vector field X extended to the infinity corresponds to the projection of the phase portrait of the vector field $p(X)$ on the Poincaré disc \mathbb{D}^2 .

The singular points of $p(X)$ in the Poincaré disc lying on \mathbb{S}^1 are the *infinite singular points* of the vector field X . The singular points of $p(X)$ in the interior of the Poincaré disc, i.e. on $\mathbb{D}^2 \setminus \mathbb{S}^1$, are the *finite singular points*. We note that in the local charts U_1, U_2, V_1 and V_2 the infinite singular points have their coordinate $v = 0$.

For a polynomial differential system (2) if $s \in \mathbb{S}^1$ is an infinite singular point, then $-s \in \mathbb{S}^1$ is another infinite singular point. Thus the number of infinite singular points is even and the local phase portrait of one is that of the other multiplied by $(-1)^{d-1}$.

3. PROOF OF PROPOSITIONS 1 AND 5 AND THE BLOW UPS

Proof of Proposition 1. We write the polynomial differential systems (3) of degree 5 as follows

$$\begin{aligned}
 (7) \quad \dot{x} &= -y + a_1x^5 + a_2x^4y + a_3x^3y^2 + a_4x^2y^3 + a_5xy^4 + a_6y^5, \\
 \dot{y} &= x + b_1x^5 + b_2x^4y + b_3x^3y^2 + b_4x^2y^3 + b_5xy^4 + b_6y^5.
 \end{aligned}$$

Since this differential system is invariant with respect to the symmetry $(x, y, t) \rightarrow (x, -y, -t)$, i.e. its phase portrait is symmetric with respect the y -axis, we obtain that $a_1 = a_3 = a_5 = b_2 = b_4 = b_6 = 0$. So system (7) reduces to

$$\begin{aligned}
 (8) \quad \dot{x} &= -y + a_2x^4y + a_4x^2y^3 + a_6y^5, \\
 \dot{y} &= x + b_1x^5 + b_3x^3y^2 + b_5xy^4.
 \end{aligned}$$

System (8) has, among others, the singular points

$$\left(0, \frac{1}{\sqrt[4]{a_6}}\right), \quad \left(\frac{1}{\sqrt[4]{-b_1}}, 0\right).$$

So in order that the origin of system (8) can be a global center it must be the unique singular point and so we need that $a_6 \leq 0$ and $b_1 \geq 0$. This completes the proof of the proposition. \square

Proof of Proposition 2. For proving the non-existence of finite singular points distinct from the origin of coordinates of the differential system (5) we need to show that the system $p = -1 + ax^4 + bx^2y^2 = 0$ and $q = 1 + cx^2y^2 + dy^4 = 0$ has no real solutions. For this we compute the Gröebner basis of the polynomials p and q with respect to the variables x and y , and we get an equivalent polynomial system to system $p = q = 0$ with five polynomial equations, one of these equations is

$$(9) \quad a + (-bc - c^2 + 2ad)y^4 - d(bc - ad)y^8 = 0.$$

Therefore if we prove that equation (10) has no real roots, then the unique finite singular point of system (5) will be the origin of coordinates. But to prove this is equivalent to prove that all the roots of the polynomial

$$(10) \quad a + (-bc - c^2 + 2ad)z - d(bc - ad)z^2 = 0,$$

are non-real or if they are real then they are non-positive.

The Routh-Hurwitz criterion for a polynomial of degree two says: The polynomial $s + rz + z^2$ has both roots in the open left half complex plane if and only if r and s are positive, for a proof see the page 14 of the book [10]. Then applying this criterion to our polynomial (10) the proposition follows easily. \square

3.1. A characterization of the global center. The next result characterizes when a polynomial differential system in \mathbb{R}^2 has a global center.

Proposition 5. *A polynomial differential system $(\dot{x}, \dot{y}) = (p(x, y), q(x, y))$ without a line of singular points at infinity, has a global center if and only if it has a unique finite singular point which is a center and all the infinite singular points in the Poincaré disc, if they exist, its local phase portrait is formed by two hyperbolic sectors having all of them both separatrices on the infinite circle.*

Proof. Assume that we have a global center. Then the exterior boundary of the period annulus of this center is the circle at infinity. Consequently, since the infinite circle is not filled up with singular points, if there is some infinite singular point this must be formed by two hyperbolic sectors having all of them both separatrices on the infinite circle.

Now assume that the polynomial differential system has a unique finite singular point which is a center, and that all the infinite singular points, if they exist, its local phase portrait is formed by two hyperbolic sectors having all of them both separatrices on the infinite circle. Then consider the period annulus of the center, its inner boundary is the center, its outer boundary γ is a curve homeomorphic to a circle. If the circle γ is contained in \mathbb{R}^2 , since the unique finite singular point is the center, it must be a periodic orbit, but we claim that this is not possible. Indeed consider a local transversal section Σ to the periodic orbit γ and the Poincaré map π defined on Σ . Then π on the part of Σ contained in the period annulus is the identity. Since π is an analytic function of one variable (because the polynomial differential system is an analytic differential system) it follows that π is also the identity on the part of Σ outside the period annulus. So γ is contained in the interior of the period annulus, a contradiction. Hence the claim is proved.

Since the boundary of the period annulus, the circle γ cannot be contained in \mathbb{R}^2 , this boundary must contain some points of the infinite circle, but since all the infinite singular points, if they exist, its local phase portrait is formed by two hyperbolic sectors having all of them both separatrices on the infinite circle, the boundary γ is the infinite circle. Hence the center is global. \square

3.2. The infinite singular points of the local chart U_1 when $c \neq 0$. Now we shall study the infinite singular points of the differential system (5) in the local chart U_1 . Thus system (5) in the chart U_1 writes

$$(11) \quad \dot{u} = (c - a)u^2 + (d - b)u^4 + v^4 + u^2v^4, \quad \dot{v} = -uv(a + bu^2 - v^4).$$

The infinite singular points in this chart are

$$(0, 0), \quad p_+ = \left(\sqrt{\frac{c - a}{b - d}}, 0 \right), \quad p_- = \left(-\sqrt{\frac{c - a}{b - d}}, 0 \right).$$

The eigenvalues of the linear part of system (11) at the singular point p_+ are

$$2\sqrt{\frac{(c-a)^3}{b-d}} \quad \text{and} \quad (bc-ad)\sqrt{\frac{(c-a)}{(b-d)^2}},$$

and the ones of the singular point p_- are the same with a change of sign.

Assume that the infinite singular points p_+ and p_- exist, i.e. that $(c-a)(b-d) > 0$. If $bc-ad \neq 0$ then p_+ and p_- are hyperbolic singular points, and since they are infinite singular points they can only be saddles or nodes, and consequently their local phase portraits are not formed by two hyperbolic sectors. Therefore system (5) cannot have a global center. If $bc-ad = 0$ then p_+ and p_- are semi-hyperbolic singular points, so they are saddles, nodes or saddle-nodes. Again system (5) cannot have a global center.

In summary the infinite singular points p_+ and p_- cannot exist if we want that system (5) has a global center, so

$$(12) \quad (c-a)(b-d) \leq 0,$$

and the unique infinite singular point in the local chart U_1 is the origin.

The linear part of system (11) at the origin is identically zero, so in order to determine its local phase portrait we must do blow ups. The characteristic directions at the origin of U_1 are given by the real linear factors of cu^2v .

Here we assume that $c \neq 0$. Then the vertical axis $u = 0$ is a characteristic direction at the origin of the local chart U_1 . Therefore before doing a vertical blow up we translate the direction $u = 0$ to $u = v$ doing the change of variables $(u, v) = (u_1 - v_1, v_1)$. Then system (11) becomes

$$(13) \quad \begin{aligned} \dot{u}_1 &= (c-a)u_1^2 + (a-2c)u_1v_1 + cv_1^2 + (d-b)u_1^4 + (3b-4d)u_1^3v_1 - \\ &\quad 3(b-2d)u_1^2v_1^2 + (b-4d)u_1v_1^3 + (1+d)v_1^4 + u_1^2v_1^4 - u_1v_1^5, \\ \dot{v}_1 &= -au_1v_1 + av_1^2 - bu_1^3v_1 + 3bu_1^2v_1^2 - 3bu_1v_1^3 + bv_1^4 + u_1v_1^5 - v_1^6. \end{aligned}$$

Now we do the vertical blow up $(u_1, v_1) \rightarrow (u_2, u_2v_2)$ and system (13) writes in the new variables

$$\begin{aligned} \dot{u}_2 &= u_2^2((a-2c)v_2 + (d-b)u_2^2 + cv_2^2 + (3b-4d)u_2^2v_2 - 3(b-2d)u_2^2v_2^2 + \\ &\quad (b-4d)u_2^2v_2^3 + (1+d)u_2^2v_2^4 + u_2^4v_2^4 - u_2^4v_2^5), \\ \dot{v}_2 &= -u_2v_2(c-2cv_2 + du_2^2 + cv_2^2 - 4du_2^2v_2 + 6du_2^2v_2^2 - 4du_2^2v_2^3 + \\ &\quad (1+d)u_2^2v_2^4). \end{aligned}$$

Doing a rescaling of the time we eliminate the common factor between (\dot{u}_2, \dot{v}_2) and we get the system

$$(14) \quad \begin{aligned} \dot{u}_2 &= u_2((a-2c)v_2 + (d-b)u_2^2 + cv_2^2 + (3b-4d)u_2^2v_2 - 3(b-2d)u_2^2v_2^2 + \\ &\quad (b-4d)u_2^2v_2^3 + (1+d)u_2^2v_2^4 + u_2^4v_2^4 - u_2^4v_2^5), \\ \dot{v}_2 &= -v_2(c-2cv_2 + du_2^2 + cv_2^2 - 4du_2^2v_2 + 6du_2^2v_2^2 - 4du_2^2v_2^3 + \\ &\quad (1+d)u_2^2v_2^4). \end{aligned}$$

The singular points of system (14) on the straight line $u_2 = 0$ are $(0, 0)$ and $(0, 1)$. The linear part of system (14) evaluated at $(0, 0)$ has eigenvalues $-c$ and $c-a$. If $-c(c-a) > 0$ then the singular point $(0, 0)$ is a hyperbolic node, and consequently going back through the changes of variables there would be orbits ending or starting at the origin of the local chart U_1 , and consequently system (5) could not be a global center. Hence we must assume that

$$(15) \quad -c(c-a) \leq 0.$$

Now we consider the following two cases because $c \neq 0$.

Case 1: $-c(c-a) < 0$. Then the singular point $(0, 0)$ is a hyperbolic saddle. We translate the singular point $(0, 1)$ to the origin of coordinates in order to study its local phase portrait doing the change $(u_2, v_2) = (u_3, 1 + v_3)$. Therefore system (14) in the variables (u_3, v_3) becomes

$$(16) \quad \begin{aligned} \dot{u}_3 &= -u_3(-av_3 - u_3^2 - cv_3^2 - 4u_3^2v_3 - 6u_3^2v_3^2 + u_3^4v_3 - (4+b)u_3^2v_3^3 + 4u_3^4v_3^2 \\ &\quad - (1+d)u_3^2v_3^4 + 6u_3^4v_3^3 + 4u_3^4v_3^4 + u_3^4v_3^5), \\ \dot{v}_3 &= -(1+v_3)(u_3^2 + cv_3^2 + 4u_3^2v_3 + 6u_3^2v_3^2 + 4u_3^2v_3^3 + (1+d)u_3^2v_3^4). \end{aligned}$$

The characteristic directions at the origin are the real linear factors of $u_3(u_3^2 + (a+c)v_3^2)$. Then the vertical axis $u_3 = 0$ is a characteristic direction at the origin of system (16). Therefore before doing a vertical blow up we translate the direction $u_3 = 0$ to $u_3 = v_3$ doing the change of variables $(u_3, v_3) = (u_4 - v_4, v_4)$. Then system (16) becomes

$$(17) \quad \begin{aligned} \dot{u}_4 &= -u_4^2 + 2u_4v_4 - v_4^2 + u_4^3 - 8u_4^2v_4 + 13u_4v_4^2 - 6v_4^3 + 4u_4^3v_4 - 22u_4^2v_4^2 + \\ &\quad 32u_4v_4^3 - 14v_4^4 + 6u_4^3v_4^2 - 28u_4^2v_4^3 + 38u_4v_4^4 - 16v_4^5 - u_4^5v_4 + 5u_4^4v_4^2 + \\ &\quad (b-6)u_4^3v_4^3 - (7+3b+d)u_4^2v_4^4 + (17+3b+2d)u_4v_4^5 - (8+b+d)v_4^6 \\ &\quad - 4u_4^5v_4^2 + 20u_4^4v_4^3 + (-39+d)u_4^3v_4^4 - 4(-9+d)u_4^2v_4^5 + 5(-3+d)u_4v_4^6 \\ &\quad - 2(-1+d)v_4^7 - 6u_4^5v_4^3 + 30u_4^4v_4^4 - 60u_4^3v_4^5 + 60u_4^2v_4^6 - 30u_4v_4^7 + 6v_4^8 - \\ &\quad 4u_4^5v_4^4 + 20u_4^4v_4^5 - 40u_4^3v_4^6 + 40u_4^2v_4^7 - 20u_4v_4^8 + 4v_4^9 - u_4^5v_4^5 + 5u_4^4v_4^6 - \\ &\quad 10u_4^3v_4^7 + 10u_4^2v_4^8 - 5u_4v_4^9 + v_4^{10}, \end{aligned}$$

$$\begin{aligned}\dot{v}_4 = & -u_4^2 + 2u_4v_4 - v_4^2 - 5u_4^2v_4 + 10u_4v_4^2 - 5v_4^3 - 10u_4^2v_4^2 + 20u_4v_4^3 - 10v_4^4 - \\ & 10u_4^2v_4^3 + 20u_4v_4^4 - 10v_4^5 - (5+d)u_4^2v_4^4 + 2(5+d)u_4v_4^5 - (5+d)v_4^6 - \\ & (1+d)u_4^2v_4^5 + 2(1+d)u_4v_4^6 - (1+d)v_4^7.\end{aligned}$$

Now we do the vertical blow up $(u_4, v_4) \rightarrow (u_5, u_5v_5)$ and system (3.2) writes

$$\begin{aligned}\dot{u}_5 = & u_5^2(-1 + u_5 + 2v_5 - 8u_5v_5 - v_5^2 + 4u_5^2v_5 + 13u_5v_5^2 - 22u_5^2v_5^2 - 6u_5v_5^3 \\ & - u_5^4v_5 + 6u_5^3v_5^2 + 32u_5^2v_5^3 + 5u_5^4v_5^2 - 28u_5^3v_5^3 - 14u_5^2v_5^4 - 4u_5^3v_5^2 + \\ & (b-6)u_5^4v_5^3 + 38u_5^3v_5^4 + 20u_5^5v_5^3 - (7+3b+d)u_5^4v_5^4 - 16u_5^3v_5^5 - 6u_5^6v_5^3 \\ & + (d-39)u_5^5v_5^4 + (17+3b+2d)u_5^4v_5^5 + 30u_5^6v_5^4 - 4(d-9)u_5^5v_5^5 - \\ & (8+b+d)u_5^4v_5^6 - 4u_5^7v_5^4 - 60u_5^6v_5^5 + 5(-3+d)u_5^5v_5^6 + 20u_5^7v_5^5 + \\ & 60u_5^6v_5^6 - 2(d-1)u_5^5v_5^7 - u_5^8v_5^5 - 40u_5^7v_5^6 - 30u_5^6v_5^7 + 5u_5^8v_5^6 + 40u_5^7v_5^7 + \\ & 6u_5^6v_5^8 - 10u_5^8v_5^7 - 20u_5^7v_5^8 + 10u_5^8v_5^8 + 4u_5^7v_5^9 - 5u_5^8v_5^9 + u_5^8v_5^{10}), \\ \dot{v}_5 = & -u_5(v_5 - 1)(-1 + 2v_5 - 6u_5v_5 - v_5^2 + 12u_5v_5^2 - 14u_5^2v_5^2 - 6u_5v_5^3 + \\ & 28u_5^2v_5^3 + u_5^4v_5^2 - 16u_5^3v_5^3 - 14u_5^2v_5^4 - 4u_5^4v_5^3 + 32u_5^3v_5^4 + 4u_5^5v_5^3 - \\ & (3+b+d)u_5^4v_5^4 - 16u_5^3v_5^5 - 16u_5^5v_5^4 + 2(7+b+d)u_5^4v_5^5 + 6u_5^6v_5^4 - \\ & 2(d-11)u_5^5v_5^5 - (8+b+d)u_5^4v_5^6 - 24u_5^6v_5^5 + 4(-3+d)u_5^5v_5^6 + 4u_5^7v_5^5 + \\ & + 36u_5^6v_5^6 - 2(-1+d)u_5^5v_5^7 - 16u_5^7v_5^6 - 24u_5^6v_5^7 + u_5^8v_5^6 + 24u_5^7v_5^7 + \\ & 6u_5^6v_5^8 - 4u_5^8v_5^7 - 16u_5^7v_5^8 + 6u_5^8v_5^8 + 4u_5^7v_5^9 - 4u_5^8v_5^9 + u_5^8v_5^{10}).\end{aligned}$$

Eliminating the common factor u_5 between \dot{u}_5 and \dot{v}_5 rescaling the time we obtain the system

(18)

$$\begin{aligned}\dot{u}_5 = & u_5(-1 + u_5 + 2v_5 - 8u_5v_5 - v_5^2 + 4u_5^2v_5 + 13u_5v_5^2 - 22u_5^2v_5^2 - 6u_5v_5^3 \\ & - u_5^4v_5 + 6u_5^3v_5^2 + 32u_5^2v_5^3 + 5u_5^4v_5^2 - 28u_5^3v_5^3 - 14u_5^2v_5^4 - 4u_5^3v_5^2 + \\ & (b-6)u_5^4v_5^3 + 38u_5^3v_5^4 + 20u_5^5v_5^3 - (7+3b+d)u_5^4v_5^4 - 16u_5^3v_5^5 - 6u_5^6v_5^3 \\ & + (d-39)u_5^5v_5^4 + (17+3b+2d)u_5^4v_5^5 + 30u_5^6v_5^4 - 4(d-9)u_5^5v_5^5 - \\ & (8+b+d)u_5^4v_5^6 - 4u_5^7v_5^4 - 60u_5^6v_5^5 + 5(-3+d)u_5^5v_5^6 + 20u_5^7v_5^5 + \\ & 60u_5^6v_5^6 - 2(d-1)u_5^5v_5^7 - u_5^8v_5^5 - 40u_5^7v_5^6 - 30u_5^6v_5^7 + 5u_5^8v_5^6 + 40u_5^7v_5^7 + \\ & 6u_5^6v_5^8 - 10u_5^8v_5^7 - 20u_5^7v_5^8 + 10u_5^8v_5^8 + 4u_5^7v_5^9 - 5u_5^8v_5^9 + u_5^8v_5^{10}), \\ \dot{v}_5 = & -(v_5 - 1)(-1 + 2v_5 - 6u_5v_5 - v_5^2 + 12u_5v_5^2 - 14u_5^2v_5^2 - 6u_5v_5^3 + \\ & 28u_5^2v_5^3 + u_5^4v_5^2 - 16u_5^3v_5^3 - 14u_5^2v_5^4 - 4u_5^4v_5^3 + 32u_5^3v_5^4 + 4u_5^5v_5^3 - \\ & (3+b+d)u_5^4v_5^4 - 16u_5^3v_5^5 - 16u_5^5v_5^4 + 2(7+b+d)u_5^4v_5^5 + 6u_5^6v_5^4 - \\ & 2(d-11)u_5^5v_5^5 - (8+b+d)u_5^4v_5^6 - 24u_5^6v_5^5 + 4(-3+d)u_5^5v_5^6 + 4u_5^7v_5^5 + \\ & + 36u_5^6v_5^6 - 2(-1+d)u_5^5v_5^7 - 16u_5^7v_5^6 - 24u_5^6v_5^7 + u_5^8v_5^6 + 24u_5^7v_5^7 + \\ & 6u_5^6v_5^8 - 4u_5^8v_5^7 - 16u_5^7v_5^8 + 6u_5^8v_5^8 + 4u_5^7v_5^9 - 4u_5^8v_5^9 + u_5^8v_5^{10}).\end{aligned}$$

If $a + c \neq -1$ then the singular points of system (18) on the straight line $u_5 = 0$ are

$$(19) \quad (0, 1), \quad q_- = \left(0, \frac{1 - \sqrt{-a-c}}{a+c+1}\right), \quad q_+ = \left(0, \frac{1 + \sqrt{-a-c}}{a+c+1}\right).$$

The eigenvalues of the linear part of system (18) evaluated at the singular point $(0, 1)$ are $-c$ and $c + a$. If $-c(c + a) > 0$ then the singular point $(0, 1)$ is a hyperbolic node, and consequently some orbits start or end at the origin of the local chart U_1 , and system (5) could not be a global center. Hence we must assume that

$$(20) \quad -c(c + a) \leq 0.$$

If $-c(c + a) < 0$ then the singular point $(0, 1)$ is a hyperbolic saddle.

If $c + a > 0$ the singular points q_- and q_+ do not exist. Then going back through the changes of variables we obtain that the origin of the local chart U_1 is formed by two hyperbolic sectors, see Figure 1.

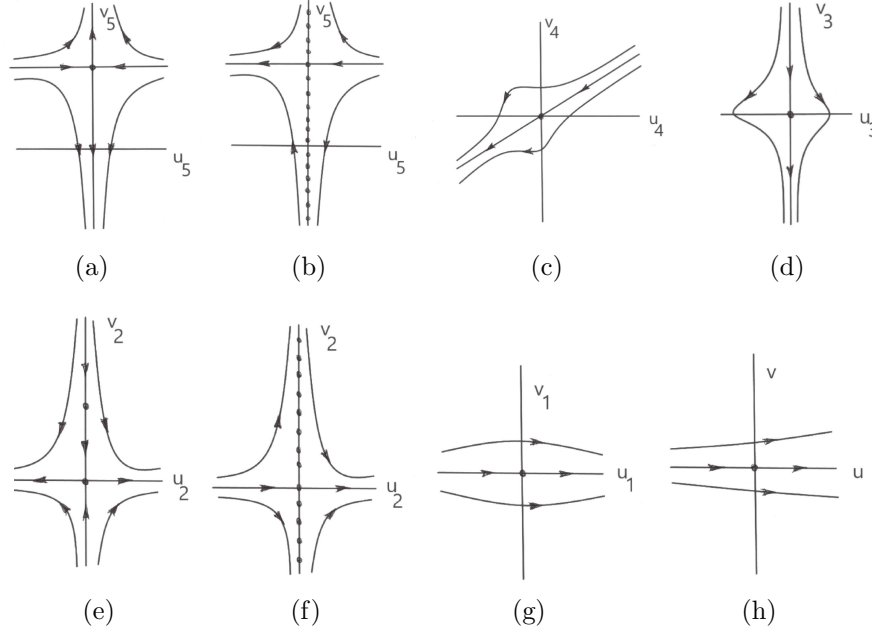


FIGURE 1. Figures of the blow up of the singular point located at the origin of the local chart U_1 of system (11) when $c \neq 0$.

If $c + a < 0$ and $c + a \neq -1$ then the determinant of the linear part of system (18) at the singular points q_{\pm} is

$$-\frac{2a(\sqrt{-a-c}-1)^2(a+c)}{(a+c+1)^2}.$$

If $c + a = -1$, i.e. $c = -1 - a$, then the singular points of system (18) on the straight line $u_5 = 0$ are $(0, 1)$ and $(0, 1/2)$. The eigenvalues

of the linear part of system (18) at the singular point $(0, 1/2)$ are 1 and $a/2$.

If $c + a = 0$, i.e. $a = -c$. Then the unique singular point on $u_5 = 0$ of system (18) is $(0, 1)$, whose eigenvalues are 0 and $-c$. If $c < 0$ then $(0, 1)$ is a semi-hyperbolic node, and consequently system (5) cannot have a global center. If $c > 0$ then it is a semi-hyperbolic saddle, and going back through the changes of variables the origin of U_1 is formed by two hyperbolic sectors.

Case 2: $-c(c - a) = 0$, i.e. $c = a \neq 0$. Then system (14) has the two singular points $(0, 0)$ and $(0, 1)$ on the straight line $u_2 = 0$. The point $(0, 0)$ is a semi-hyperbolic saddle if $a(b - d) < 0$ and a semi-hyperbolic node if $a(b - d) > 0$ but then the origin of U_1 is not formed by two hyperbolic sectors. If $d = b$ then the infinity is filled of singular points, and from Theorem 4 there are orbits ending or starting at infinity, so system (5) cannot have a global center. Therefore we assume that $a(b - d) < 0$.

3.3. The infinite singular points of the local chart U_2 when $b \neq 0$. Here we shall study the infinite singular points of the differential system (5) localized at the origin of the local chart U_2 . We recall that studying the infinite singular points in the local chart U_1 and at the origin of the local chart U_2 when the origin is a singular point, we are studying all the infinite singular points of a polynomial differential system.

System (5) in the chart U_2 writes

$$(21) \quad \dot{u} = (b - d)u^2 + (a - c)u^4 - v^4 - u^2v^4, \quad \dot{v} = -uv(d + cu^2 + v^4).$$

So the origin of U_2 is an infinite singular point. The linear part of system (21) at the origin is identically zero, so in order to determine its local phase portrait we must do blow ups. The characteristic directions at the origin of U_2 are given by the real linear factors of bu^2v .

Here we assume that $b \neq 0$. Then the vertical axis $u = 0$ is a characteristic direction at the origin of the local chart U_2 . Therefore before doing a vertical blow up we translate the direction $u = 0$ to $u = v$ doing the change of variables $(u, v) = (u_1 - v_1, v_1)$. Then system (21) becomes

$$(22) \quad \begin{aligned} \dot{u}_1 &= (b - d)u_1^2 - (2b - d)u_1v_1 + bv_1^2 + (a - c)u_1^4 - (4a - 3c)u_1^3v_1 + \\ &\quad 3(2a - c)u_1^2v_1^2 - (4a - c)u_1v_1^3 + (-1 + a)v_1^4 - u_1^2v_1^4 + u_1v_1^5, \\ \dot{v}_1 &= -du_1v_1 + dv_1^2 - cu_1^3v_1 + 3cu_1^2v_1^2 - 3cu_1v_1^3 + cv_1^4 - u_1v_1^5 + v_1^6. \end{aligned}$$

Now we do the vertical blow up $(u_1, v_1) \rightarrow (u_2, u_2 v_2)$ and system (22), after eliminating the common factor u_2 between \dot{u}_2 and \dot{v}_2 doing a rescaling of the time, writes in the new variables

(23)

$$\begin{aligned}\dot{u}_2 &= u_2(b - d - (2b - d)v_2 + (a - c)u_2^2 + bv_2^2 - (4a - 3c)u_2^2 v_2 + \\ &\quad 3(2a - c)u_2^2 v_2^2 - (4a - c)u_2^2 v_2^3 + (-1 + a)u_2^2 v_2^4 - u_2^4 v_2^4 + u_2^4 v_2^5), \\ \dot{v}_2 &= -v_2(b - 2bv_2 + au_2^2 + bv_2^2 - 4au_2^2 v_2 + 6au_2^2 v_2^2 - 4au_2^2 v_2^3 + \\ &\quad (a - 1)u_2^2 v_2^4).\end{aligned}$$

The singular points of system (23) on the straight line $u_2 = 0$ are $(0, 0)$ and $(0, 1)$. The linear part of system (23) evaluated at $(0, 0)$ has eigenvalues $-b$ and $b - d$. If $-b(b - d) > 0$ then the singular point $(0, 0)$ is a hyperbolic node, and consequently going back through the changes of variables there would be orbits ending or starting at the origin of the local chart U_2 , and consequently system (5) could not be a global center. Hence we must assume that

$$(24) \quad -b(b - d) \leq 0.$$

We consider the following two cases because $b \neq 0$.

Case 1: $-b(b - d) < 0$. Then the singular point $(0, 0)$ is a hyperbolic saddle. We translate the singular point $(0, 1)$ to the origin of coordinates in order to study its local phase portrait doing the change $(u_2, v_2) = (u_3, 1 + v_3)$. Therefore system (23) in the variables (u_3, v_3) becomes

(25)

$$\begin{aligned}\dot{u}_3 &= -u_3(dv_3 - u_3^2 + bv_3^2 - 4u_3^2 v_3 - 6u_3^2 v_3^2 + u_3^4 v_3 + (c - 4)u_3^2 v_3^3 + 4u_3^4 v_3^2 + \\ &\quad (a - 1)u_3^2 v_3^4 + 6u_3^4 v_3^3 + 4u_3^4 v_3^4 + u_3^4 v_3^5), \\ \dot{v}_3 &= -(1 + v_3)(-u_3^2 + bv_3^2 - 4u_3^2 v_3 - 6u_3^2 v_3^2 - 4u_3^2 v_3^3 + (a - 1)u_3^2 v_3^4).\end{aligned}$$

The characteristic directions at the origin are the real linear factors of $u_3(-u_3^2 + (b + d)v_3^2)$. Then the vertical axis $u_3 = 0$ is a characteristic direction at the origin of system (25). Therefore before doing the vertical blow up we translate the direction $u_3 = 0$ to $u_3 = v_3$ doing the

change of variables $(u_3, v_3) = (u_4 - v_4, v_4)$. Then system (25) becomes

$$\begin{aligned}\dot{u}_4 = & u_4^2 + (-2 + d)u_4v_4 - (b + d - 1)v_4^2 - u_4^3 + 8u_4^2v_4 + (b - 13)u_4v_4^2 - \\ & 2(b - 3)v_4^3 - 4u_4^3v_4 + 22u_4^2v_4^2 - 32u_4v_4^3 + 14v_4^4 - 6u_4^3v_4^2 + 28u_4^2v_4^3 - \\ & 38u_4v_4^4 + 16v_4^5 + u_4^5v_4 - 5u_4^4v_4^2 + (6 + c)u_4^3v_4^3 - (a + 3c - 7)u_4^2v_4^4 + \\ & (2a + 3c - 17)u_4v_4^5 - (a + c - 8)v_4^6 + 4u_4^5v_4^2 - 20u_4^4v_4^3 + (39 + a)u_4^3v_4^4 - \\ & 4(9 + a)u_4^2v_4^5 + 5(3 + a)u_4v_4^6 - 2(1 + a)v_4^7 + 6u_4^5v_4^3 - 30u_4^4v_4^4 + \\ & 60u_4^3v_4^5 - 60u_4^2v_4^6 + 30u_4v_4^7 - 6v_4^8 + 4u_4^5v_4^4 - 20u_4^4v_4^5 + 40u_4^3v_4^6 - \\ & 40u_4^2v_4^7 + 20u_4v_4^8 - 4v_4^9 + u_4^5v_4^5 - 5u_4^4v_4^6 + 10u_4^3v_4^7 - 10u_4^2v_4^8 + 5u_4v_4^9 - \\ & v_4^{10}, \\ \dot{v}_4 = & u_4^2 - 2u_4v_4 - (b - 1)v_4^2 + 5u_4^2v_4 - 10u_4v_4^2 - (b - 5)v_4^3 + 10u_4^2v_4^2 - \\ & 20u_4v_4^3 + 10v_4^4 + 10u_4^2v_4^3 - 20u_4v_4^4 + 10v_4^5 - (a - 5)u_4^2v_4^4 + \\ & 2(a - 5)u_4v_4^5 - (a - 5)v_4^6 - (a - 1)u_4^2v_4^5 + 2(a - 1)u_4v_4^6 - (a - 1)v_4^7.\end{aligned}$$

Now we do the vertical blow up $(u_4, v_4) \rightarrow (u_5, u_5v_5)$, and after eliminating the common factor u_5 of \dot{u}_5 and \dot{v}_5 doing a rescaling of the time, and system (3.2) writes

$$\begin{aligned}\dot{u}_5 = & -u_5(-1 + u_5 - (-2 + d)v_5 - 8u_5v_5 + (b + d - 1)v_5^2 + 4u_5^2v_5 - \\ & (b - 13)u_5v_5^2 - 22u_5^2v_5^2 + 2(b - 3)u_5v_5^3 - u_5^4v_5 + 6u_5^3v_5^2 + 32u_5^2v_5^3 + \\ & 5u_5^4v_5^2 - 28u_5^3v_5^3 - 14u_5^2v_5^4 - 4u_5^5v_5^2 - (6 + c)u_5^4v_5^3 + 38u_5^3v_5^4 + \\ & 20u_5^5v_5^3 + (a + 3c - 7)u_5^4v_5^4 - 16u_5^3v_5^5 - 6u_5^6v_5^3 - (39 + a)u_5^5v_5^4 - \\ & (2a + 3c - 17)u_5^4v_5^5 + 30u_5^6v_5^4 + 4(9 + a)u_5^5v_5^5 + (a + c - 8)u_5^4v_5^6 - \\ & 4u_5^7v_5^4 - 60u_5^6v_5^5 - 5(3 + a)u_5^5v_5^6 + 20u_5^7v_5^5 + 60u_5^6v_5^6 + 2(1 + a)u_5^5v_5^7 - \\ & u_5^8v_5^5 - 40u_5^7v_5^6 - 30u_5^6v_5^7 + 5u_5^8v_5^6 + 40u_5^7v_5^7 + 6u_5^6v_5^8 - 10u_5^8v_5^7 - \\ & 20u_5^7v_5^8 + 10u_5^8v_5^8 + 4u_5^7v_5^9 - 5u_5^8v_5^9 + u_5^8v_5^{10}), \\ \dot{v}_5 = & (v_5 - 1)(-1 + 2v_5 - 6u_5v_5 + (b + d - 1)v_5^2 + 12u_5v_5^2 - 14u_5^2v_5^2 + \\ & 2(b - 3)u_5v_5^3 + 28u_5^2v_5^3 + u_5^4v_5^2 - 16u_5^3v_5^3 - 14u_5^2v_5^4 - 4u_5^4v_5^3 + \\ & 32u_5^3v_5^4 + 4u_5^5v_5^3 + (a + c - 3)u_5^4v_5^4 - 16u_5^3v_5^5 - 16u_5^5v_5^4 - \\ & 2(a + c - 7)u_5^4v_5^5 + 6u_5^6v_5^4 + 2(11 + a)u_5^5v_5^5 + (a + c - 8)u_5^4v_5^6 - \\ & 24u_5^6v_5^5 - 4(3 + a)u_5^5v_5^6 + 4u_5^7v_5^5 + 36u_5^6v_5^6 + 2(1 + a)u_5^5v_5^7 - 16u_5^7v_5^6 - \\ & 24u_5^6v_5^7 + u_5^8v_5^6 + 24u_5^7v_5^7 + 6u_5^6v_5^8 - 4u_5^8v_5^7 - 16u_5^7v_5^8 + 6u_5^8v_5^8 + \\ & 4u_5^7v_5^9 - 4u_5^8v_5^9 + u_5^8v_5^{10}).\end{aligned}$$

If $b + d \neq 1$ then the singular points of system (26) on the straight line $u_5 = 0$ are

$$(27) \quad (0, 1), \quad r_- = \left(0, \frac{-\sqrt{b+d}-1}{b+d-1}\right), \quad r_+ = \left(0, \frac{\sqrt{b+d}-1}{b+d-1}\right).$$

The eigenvalues of the linear part of system (26) evaluated at the singular point $(0, 1)$ are $-b$ and $b + d$. If $-b(b + d) > 0$ then the singular point $(0, 1)$ is a hyperbolic node, and consequently some orbits start or

end at the origin of the local chart U_2 , and system (5) could not be a global center. Hence we must assume that

$$(28) \quad -b(b+d) \leq 0.$$

If $-b(b+d) < 0$ the singular point $(0, 1)$ is a hyperbolic saddle.

If $b+d < 0$ the singular points r_- and r_+ do not exist. Then going back through the changes of variables we obtain that the origin of the local chart U_2 is formed by two hyperbolic sectors, see again Figure 1.

If $b+d > 0$ and $b+d \neq 1$, then the determinant of the linear part of system (26) at the singular points r_{\pm} is

$$(29) \quad -\frac{2d(b+d)(\sqrt{b+d}+1)^2}{(b+d-1)^2}.$$

If $b+d = 1$, i.e. $d = 1-b$ the singular points of system (26) on the straight line $u_5 = 0$ are

$$(30) \quad (0, 1), \quad (0, 1/2).$$

The eigenvalues of the linear part of system (26) at the singular point $(0, 1/2)$ are -1 and $d/2$.

Case 2: $b-d = 0$. Then the singular point $(0, 0)$ of system (23) is a semi-hyperbolic node if $b > 0$, and consequently system (5) cannot have a global center. But if $b < 0$ then the singular point $(0, 0)$ of system (23) is a semi-hyperbolic saddle, and in order to know the local phase portrait of the origin of U_2 we need to study the local phase portrait of the singular point $(0, 1)$ of system (23). Since $d = b < 0$ from (27) system (26) on $u_5 = 0$ has a unique singular point the $(0, 1)$. The linear part of system (26) at $(0, 1)$ has eigenvalues $2b$ and $-b$, so this singular point is a hyperbolic saddle. Again going back through the changes of variables we get that the origin of the local chart U_2 is formed by two hyperbolic sectors.

4. THE PROOFS

In statement (i) we characterize all systems (5) with $c = 0$ having a global center.

Proof of statement (i) of Theorem 3. Assume that $c = 0$. Then, from (12), we have that $-a(b-d) \leq 0$, otherwise system (5) cannot have a global center, and the origin of the local chart U_1 is the unique infinite singular point in this chart. First we study its local phase portrait.

Since the linear part of the singular point localized at the origin of system (11) is identically zero, we do the blow up $(u, v) = (u_1, u_1 v_1)$ to system (11) and this system becomes

$$(31) \quad \dot{u}_1 = -u_1^2(a + (b-d)u_1^2 - u_1^2 v_1^4 - u_1^4 v_1^4), \quad \dot{v}_1 = -u_1^3 v_1(d + v_1^4).$$

Doing a rescaling of the time we eliminate the common factor u_1^2 between \dot{u}_1 and \dot{v}_1 obtaining the system

$$(32) \quad \dot{u}_1 = -(a + (b-d)u_1^2 - u_1^2 v_1^4 - u_1^4 v_1^4), \quad \dot{v}_1 = -u_1 v_1(d + v_1^4).$$

Going back through these changes of variables we obtain that the origin of U_1 is formed by two elliptic sectors separated by two parabolic ones if $a < 0$ (see Figure 2(a)), or by two hyperbolic sectors separated by two parabolic ones if $a > 0$ (see Figure 2(b)). So system (31) cannot have a global center if a is not zero, because there are orbits which end or start at the origin of the local chart U_1 due to the existence of the parabolic sectors. Hence in what follows we consider $a = 0$ and the differential system (5) reduces to

$$(33) \quad \dot{x} = y(-1 + bx^2y^2), \quad \dot{y} = x(1 + dy^4).$$

Clearly $d \geq 0$, otherwise the origin of coordinates would not be the unique finite singular point, and consequently system (33) could not have a global center.

Now system (32) is

$$(34) \quad \dot{u}_1 = (d-b)u_1^2 + u_1^2 v_1^4 + u_1^4 v_1^4, \quad \dot{v}_1 = -u_1 v_1(d + v_1^4).$$

We consider two cases.

Case 1: $d \neq b$. Then we eliminate the common factor u_1 between the two components of system (34) rescaling the time, and we obtain the system

$$(35) \quad \dot{u}_1 = u_1((d-b) + v_1^4 + u_1^2 v_1^4), \quad \dot{v}_1 = -v_1(d + v_1^4).$$

The unique singular point on $u_1 = 0$ is the origin.

Subcase 1.1: $d > 0$. The origin of system (35) is a hyperbolic node if $(d-b)d < 0$, and going back through the changes of variables the node provides orbits which end or start at the origin of the local chart U_1 , and again in this case system (5) cannot have a global center.

If $(d-b)d > 0$ then, since $d > 0$ we get that $d > b$, and the origin of system (35) is a hyperbolic saddle, and going back through the changes of variables we obtain that the origin of the local chart U_1 is formed by two hyperbolic sectors, see Figure 3.

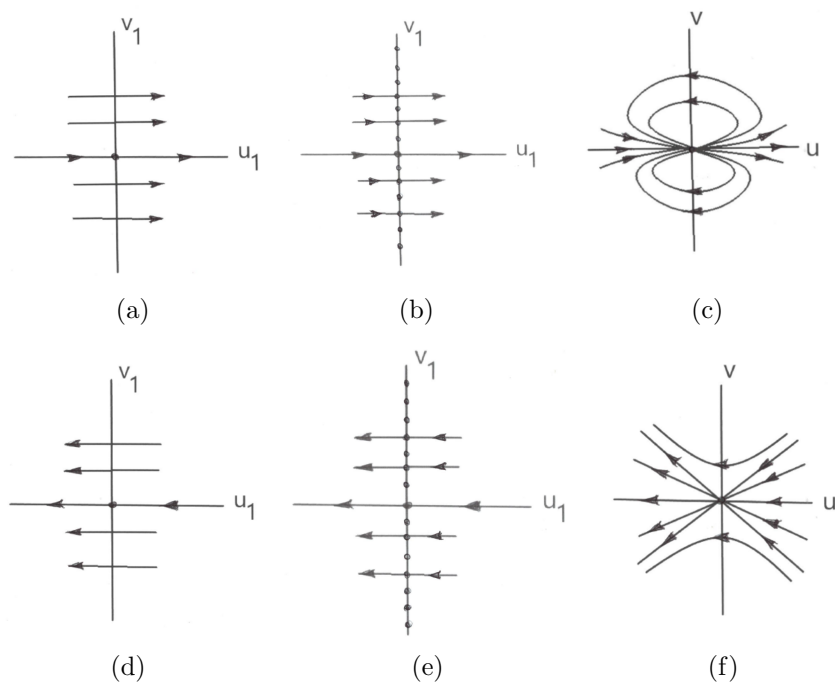


FIGURE 2. Figures of the blow up of the singular point located at the origin of the local chart U_1 of system (11) when $c = 0$: (a), (b) and (c) for $a < 0$, and (d), (e) and (f) for $a > 0$.

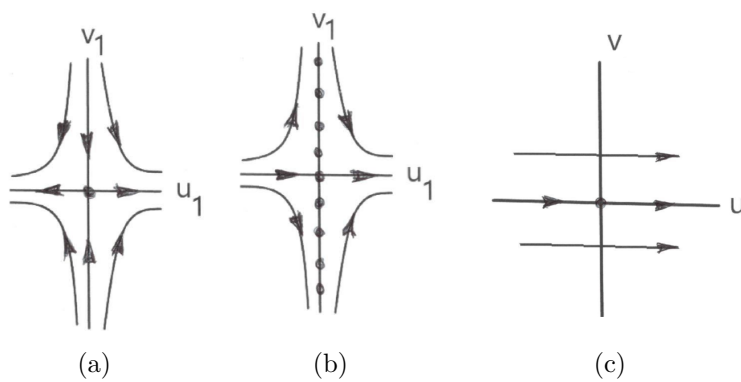


FIGURE 3. Figures of the blow up of the singular point located at the origin of the local chart U_1 of system (11) when $c = a = 0$.

Now we must study the local phase portrait at the origin of the local chart U_2 , and when the local phase portrait of this origin is formed by

two hyperbolic sectors system (5) will have a global center by Proposition 5.

Since $c = a = 0$ from (21) system (5) in the local chart U_2 writes

$$(36) \quad \dot{u} = (b - d)u^2 - v^4 - u^2v^4, \quad \dot{v} = -uv(d + v^4).$$

Clearly the origin of the chart U_2 is an infinite singular point whose linear part is identically zero. Its characteristic directions are the real linear factors of bu^2v . We consider two subcases.

Subcase 1.1.1: $b \neq 0$. Then, from (24) we have that $-b(b - d) < 0$, and from (15) we get that $-b(b + d) \leq 0$.

If $b + d < 0$, from the last part of subsection 3.3, the origin of the local chart U_2 is formed by two hyperbolic sectors. Hence from Proposition 5 system (5) has a global center. Note that under these assumptions $b < 0$.

If $b + d = 0$ then, from (29) the unique singular point on $u_5 = 0$ is $(0, 1)$, which is a semi-hyperbolic saddle. Going back through the changes of variables we obtain that the origin of the local chart U_2 is formed by two hyperbolic sectors, and system (5) has a global center.

If $b + d > 0$ and $b + d \neq 1$, then from (29) the singular points r_{\pm} are hyperbolic saddles. Going back through the changes of variables, that from now on we do not produce in a figure the distinct steps because they are similar to the ones already described, we get that the origin of the local chart U_2 is formed by two hyperbolic sectors separated by two parabolic ones. Therefore system (5) cannot have a global center.

If $b + d = 1$. Then, from (30) the unique singular points on $u_5 = 0$ of system (26) are $(0, 1)$ and $(0, 1/2)$. Since $(d - b)d = (2d - 1)d > 0$ we get that $d > 1/2$, and from $(d - b)d = (1 - 2b)d > 0$ we have $b < 1/2$. The eigenvalues of $(0, 1)$ are 1 and $-b = d - 1$, and the eigenvalues of $(0, 1/2)$ are -1 and $d/2$. So $(0, 1/2)$ always is a hyperbolic saddle. If $d > 1$ then $(0, 1)$ is a node, and system (5) cannot have a global center. Note that d cannot be 1 because then $b = 0$, a contradiction because we are in subcase 1.1.1. If $d \in (1/2, 1)$ then $(0, 1)$ is a hyperbolic saddle, and going back through the changes of variables we obtain that the origin of U_2 is formed by two hyperbolic sectors separated by two parabolic ones, hence system (5) cannot have a global center.

Subcase 1.1.2: $b = 0$. Doing the blow up $(u, v) = (u_1, u_1v_1)$ system (36), after eliminating the common factor u_1^2 between \dot{u}_1 and \dot{v}_1 , becomes

$$\dot{u}_1 = -d - u_1^2v_1^4 - u_1^4v_1^4, \quad \dot{v}_1 = u_1v_1^5.$$

Going back through the changes of variables we obtain that the origin of the local chart U_2 is formed by two hyperbolic sectors separated by two parabolic ones (see Figure 2(b)), so system (5) cannot have a global center.

Subcase 1.2: $d = 0$. Note that $b \neq 0$ since we are in Case 1 in which $d \neq b$. In this case the origin of system (35) is a semi-hyperbolic node if $b > 0$ and a semi-hyperbolic saddle if $b < 0$. Hence if $b > 0$ system (5) cannot have a global center, and if $b < 0$ going back through the changes of variables we obtain that the origin of the local chart U_1 is formed by two hyperbolic sectors, see Figure 3.

Now we must study the local phase portrait at the origin of the local chart U_2 . Since $b < 0$ from (27) we have that $(0, 1)$ is the unique singular point of system (26) on $u_5 = 0$, which is a hyperbolic saddle, going back through the changes of variables it follows that the origin of U_2 is formed by two hyperbolic sectors, so system (5) has a global center.

Case 2: $d = b$. If $b = 0$ the differential system (5) becomes the linear differential center $\dot{x} = -y$, $\dot{y} = x$ which clearly has a global center because their periodic orbits are the circles $x^2 + y^2 = \text{constant} > 0$. If $b \neq 0$ system (31) becomes

$$(37) \quad \dot{u} = v^4 + u^2v^4, \quad \dot{v} = -uv(bu^2 - v^4).$$

The line $v = 0$ is filled of singular points, the eigenvalues of the linear part of system (37) at the singular point $(0, u)$ are 0 and $-bu^3$. Since $d = b > 0$ by Theorem 4 there is an orbit ending at the infinite singular point $(0, u)$ of the local chart U_1 if $u \neq 0$, so in this case system (5) cannot have a global center.

In summary statement (i) is proved. \square

In all the remainder statements we study systems (5) with $c \neq 0$ having a global center. In view of subsection 3.2 we only need to study the cases for which $(c - a)(b - d) \leq 0$ (see (12)), $-c(c - a) < 0$ (see (15)) and $-c(a + c) \leq 0$ (see (20)). We separate the study in different subcases which will correspond to the statements in the theorem.

In particular in statement (ii) we classify all systems (5) with $c \neq 0$, $(c - a)(b - d) \leq 0$, $-c(c - a) < 0$, $-c(a + c) < 0$, $a + c > 0$ and $b = 0$.

Proof of statement (ii) of Theorem 3. Here we study systems (5) with $c \neq 0$, $(c - a)(b - d) \leq 0$, $-c(c - a) < 0$, $-c(a + c) < 0$, $a + c > 0$ and $b = 0$. So $c > 0$, $c - a > 0$ and $d \geq 0$.

Since $b = 0$ system (5) reduces to

$$(38) \quad \dot{x} = y(-1 + ax^4), \quad \dot{y} = x(1 + cx^2y^2 + dy^4).$$

Therefore $a \leq 0$, otherwise there are invariant vertical lines and the origin of system (38) cannot be a global center. Note that from Proposition 2 the unique finite singular point is the origin of coordinates.

From subsection 3.2 we know that the origin of the local chart U_1 is the unique singular point formed by two hyperbolic sectors. Hence we must study the local phase portrait at the origin of the local chart U_2 .

From (21) system (38) in the local chart U_2 becomes

$$(39) \quad \dot{u} = -du^2 + (a - c)u^4 - v^4 - u^2v^4, \quad \dot{v} = -uv(d + cu^2 + v^4).$$

Note that $(0, 0)$ is the unique infinite singular point in U_2 and its linear part is identically zero. So we do the blow up $(u, v) = (u_1, u_1v_1)$, and after eliminating the common factor u_1^2 between \dot{u}_1 and \dot{v}_1 doing a rescaling of the time we get the system

$$(40) \quad \dot{u}_1 = -d + (a - c)u_1^2 - u_1^2v_1^4 - u_1^4v_1^4, \quad \dot{v} = u_1v_1(v_1^4 - a).$$

If $d > 0$ going back through the blow up we get that the origin of U_2 is formed by two hyperbolic sectors separated by two parabolic ones and so system (5) cannot have a global center. If $d = 0$ doing another rescaling of the time we remove the common factor u_1 between the two components of system (40) and we get the system

$$(41) \quad \dot{u}_1 = (a - c)u_1 - u_1v_1^4 - u_1^3v_1^4, \quad \dot{v} = v_1(v_1^4 - a).$$

The unique singular point of system (41) on $u_1 = 0$ is the origin. If $a < 0$ the origin is a hyperbolic saddle, and if $a = 0$ the origin is a semi-hyperbolic saddle. In both cases going back through the changes of variables we obtain that the origin of the local chart U_2 is formed by two hyperbolic sectors, and by Proposition 5 system (5) has a global center. \square

In statement (iii) we classify all systems (5) with $c \neq 0$, $(c - a)(b - d) \leq 0$, $-c(c - a) < 0$, $-c(a + c) < 0$, $a + c > 0$ and $b \neq 0$.

Proof of statement (iii) of Theorem 3. We study system (5) with $c \neq 0$, $(c - a)(b - d) \leq 0$, $-c(c - a) < 0$, $-c(a + c) < 0$, $a + c > 0$ and $b \neq 0$. Then $c > 0$, $c - a > 0$ and $b - d \leq 0$.

Again from subsection 3.2 we know that the origin of the local chart U_1 is the unique singular point formed by two hyperbolic sectors. Hence we must study the local phase portrait at the origin of the local chart U_2 .

From condition (24) we get that $-b(b-d) \leq 0$, so $b < 0$ if $b-d < 0$. From (28) $-b(b+d) \leq 0$, then we distinguish the following cases.

Case 1: $b-d < 0$ and $-b(b+d) < 0$. Then $b+d < 0$ and from subsection 3.3, the origin of U_2 is formed by two hyperbolic sectors. By Proposition 2 the unique finite singular point is the origin of coordinates. Hence by Proposition 5 system (5) has a global center.

Case 2: $b-d < 0$ and $b+d = 0$. Then $d = -b$ and $2b < 0$. So $b < 0$. From (27) $(0, 1)$ is the unique singular point of system (26) on $u_5 = 0$, which is a semi-hyperbolic saddle. So going back through the changes of variables we get that the origin of the local chart U_2 is formed by two hyperbolic sectors. By Propositions 2 and 5 system (5) has a global center.

Case 3: $d = b$. Then $b+d = 0$ and from subsection 3.3, the origin of U_2 is formed by two hyperbolic sectors. By Propositions 2 and 5 system (5) has a global center.

In short we get that the conditions to have a global center are $c > 0$, $c-a > 0$, $c+a > 0$, $b-d < 0$ and $b+d \leq 0$ which can be written as $c > 0$, $b < 0$, $|c| > |a|$ and $|b| \geq |d|$ as in statement (iii) of the theorem. \square

Proposition 6. *Systems (5) with $c \neq 0$, $(c-a)(b-d) \leq 0$, $-c(c-a) < 0$, $-c(a+c) < 0$ and $a+c < 0$, have no global centers.*

Proof. Under the assumptions of the proposition we have that $c > 0$ and $a+c < 0$. We consider two cases.

Case 1: $a+c = -1$. In this case the singular points of system (18) on $u_5 = 0$ are $(0, 1)$ and $(0, 1/2)$, see the last part of subsection 3.2. The singular point $(0, 1)$ has eigenvalues -1 and $-c$. So it is a node, and system (5) cannot have a global center.

Case 2: $a+c \neq -1$. In this case $-c(c+a) > 0$ and the singular point $(0, 1)$ of system (18) is a node, implying that again system (5) cannot have a global center. \square

In statement (iv) we classify all systems (5) with $c \neq 0$, $(c-a)(b-d) \leq 0$, $-c(c-a) < 0$ and $a+c = 0$.

Proof of statement (iv) of Theorem 3. We study the systems (5) with $c \neq 0$, $(c-a)(b-d) \leq 0$, $-c(c-a) < 0$ and $a+c = 0$. So $a = -c$. Then from the last part of subsection 3.2 we have that $c > 0$ and the

origin of U_1 is formed by two hyperbolic sectors. We must study the local phase portrait at the origin of U_2 . We consider two cases.

Case 1: $b \neq 0$. Then from (24) we have that $-b(b-d) \leq 0$, and from (28) we obtain that $-b(b+d) \leq 0$. Now we consider different subcases.

Subcase 1.1: $b+d < 0$. Then $b < 0$ and from the last part of subsection 3.3 we get that the origin of the local chart U_2 is formed by two hyperbolic sectors, and from Propositions 2 and 5 system (5) has a global center.

Subcase 1.2: $b+d = 0$. Then the unique singular point of system (26) on $u_5 = 0$ is $(0, 1)$. Since $(b-d)(c-a) = (2b)(2c) \leq 0$ and $c > 0$ we obtain that $b < 0$. Therefore the singular point $(0, 1)$ is a semi-hyperbolic saddle, and going back through the changes of variables we obtain that the origin of the local chart U_2 is formed by two hyperbolic sectors, and from Propositions 2 and 5 system (5) has a global center.

Subcase 1.3: $b+d > 0$. Then, from $-b(b+d) < 0$ we get that $b > 0$. Since $(b-d)(c-a) = (b-d)2c \leq 0$, we have $b-d \leq 0$. Since $-b(b-d) \leq 0$ and $b > 0$, then $b-d \geq 0$. Therefore $d = b$.

If $2b \neq 1$ then the three singular points of system (26) given in equation (27) on $u_5 = 0$ are hyperbolic saddles (see (29)). Going back through the changes of variables the origin of the local chart U_2 is formed by two hyperbolic sectors separated by two parabolic ones, so system (5) cannot have a global center.

If $2b = 1$, from (30) we have that $(0, 1)$ and $(0, 1/2)$ are the unique singular points of system (26) on $u_5 = 0$ and both are hyperbolic saddles. Again going back through the changes of variables the origin of the local chart U_2 is formed by two hyperbolic sectors separated by two parabolic ones, so system (5) cannot have a global center.

Case 2: $b = 0$. Now system (5) in the local chart U_2 becomes

$$\dot{u} = -du^2 - 2cu^4 - v^4 - u^2v^4, \quad \dot{v} = -uv(d + cu^2 + v^4).$$

Doing the blow up $(u, v) = (u_1, u_1v_1)$ and eliminating the common factor u_1^2 between \dot{u}_1 and \dot{v}_1 and rescaling the time we obtain the system

$$\dot{u}_1 = -d - 2cu_1^2 - u_1^2v_1^4 - u_1^4v_1^4, \quad \dot{v}_1 = u_1v_1(c + v_1^4).$$

Going back through the changes of variables we get that the origin of the local chart U_2 is formed by two hyperbolic sectors separated by two parabolic ones, so system (5) cannot have a global center.

In short we get that the conditions to have a global center are $c > 0$, $b < 0$, $c + a = 0$, $b - d < 0$ and $b + d \leq 0$ which can be written as $c > 0$, $b < 0$, $c + a = 0$ and $|b| \geq |d|$ as in statement (iv) of the theorem. \square

Finally in statement (v) we classify all systems (5) with $c \neq 0$, $(c - a)(b - d) \leq 0$ and $c = a$.

Proof of statement (v) of Theorem 3. We consider the differential system (5) with $c \neq 0$, $(c - a)(b - d) \leq 0$ and $c = a$. Then from subsection 3.2 the unique infinite singular point in the local chart U_1 is $(0, 0)$.

From Case 2 of subsection 3.2 we have that $a(b - d) < 0$ and the singular point $(0, 0)$ of system (14) is a semi-hyperbolic saddle, and we must study the local phase portrait of the singular point $(0, 1)$ of system (14). Now we consider three cases.

Case 1: $a < 0$ and $a \neq -1/2$. Then, system (18) has on $u_5 = 0$ the three singular points given in (19), that is,

$$(0, 1), \quad q_- = \left(0, \frac{1 - \sqrt{-2a}}{2a + 1}\right), \quad q_+ = \left(0, \frac{1 + \sqrt{-2a}}{2a + 1}\right).$$

These three singular points are hyperbolic saddles. Going back through the changes of variables we obtain that the local phase portrait at the origin of U_1 is formed by six hyperbolic sectors, so system (5) cannot have a global center.

Case 2: $a = -1/2$. From the end of subsection 3.2 system (18) has on $u_5 = 0$ the two singular points $(0, 1)$ and $(0, 1/2)$, and both are hyperbolic saddles. Again going back through the changes of variables we obtain that the local phase portrait at the origin of U_1 is formed by six hyperbolic sectors, so system (5) cannot have a global center.

Case 3: $a > 0$. Then $b < d$, and system (18) has on $u_5 = 0$ a unique singular point given in (19) which is $(0, 1)$ and it is a hyperbolic saddle. Going back through the changes of variables the origin of U_1 is formed by two hyperbolic sectors. So we need to study the origin of U_2 .

Since $a > 0$ then $b < d$. We consider two subcases.

Subcase 3.1: $b \neq 0$. From (24) we have that $-b(b - d) < 0$, and consequently $b < 0$. From subsection 3.3 we consider the following two subcases.

Subcase 3.1.1: $b + d \neq 1$. Then from (28) we get that $-b(b + d) \leq 0$. So $b + d \leq 0$.

If $b + d < 0$ from (27) system (26) has a unique singular points on $u_5 = 0$ given in (27), which is $(0, 1)$ and it is a hyperbolic saddle. Going back through the changes of variables we get that the origin of U_2 is formed by two hyperbolic sectors. Then system (5) has a global center.

If $b + d = 0$ again from (27) system (26) has a unique singular points on $u_5 = 0$ given in (27), which is $(0, 1)$, but now it is a semi-hyperbolic saddle. Proceeding as in case $b + d < 0$ we obtain that system (5) has a global center.

Subcase 3.1.2: $b + d = 1$. From (27) system (26) has two singular points on $u_5 = 0$ given in (27), which are $(0, 1)$ and $(0, 1/2)$. The eigenvalues of $(0, 1)$ are 1 and $d - 1 > 0$ and so it is a hyperbolic node and consequently system (5) cannot have a global center.

Subcase 3.2 : $b = 0$. Then $d > 0$. In this case doing one vertical blow up we obtain that the origin of U_2 is formed by two hyperbolic sectors separated by two parabolic ones (see Figure 2(b)). So system (5) cannot have a global center.

This completes the proof of statement (v) of Theorem 3. \square

DATA AVAILABILITY

Our manuscript has no associated data.

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REFERENCES

- [1] M.J. ÁLVAREZ, A. FERRAGUD AND X. JARQUE, *A survey on the blow up technique*, Int. J. Bifur. Chaos **21** (2011), 3103–3118.
- [2] A.A. ANDRONOV, E.A. LEONTOVICH, I.I. GORDON AND A.G. MUIER, *Qualitative theory of second-order dynamic systems*, John Wiley & Sons, New York, Toronto, 1973.
- [3] N.N. BAUTIN, *On the number of limit cycles which appear with the variation of coefficients from an equilibrium position of focus or center type*, Mat. Sbornik **30** (1952), 181–196, Amer. Math. Soc. Transl. Vol. **100** (1954), 1–19.
- [4] R. CONTI, *Uniformly isochronous centers of polynomial systems in \mathbb{R}^2* , Lecture Notes in Pure and Appl. Math. **152** (1994), 21–31.

- [5] R. CONTI, *Centers of planar polynomial systems: A review*, Le Matematiche **LIII** (1998), 207–240.
- [6] H. DULAC, *Détermination et intégration d’une certaine classe d’équations différentielle ayant par point singulier un centre*, Bull. Sci. Math. Sér. (2) **32** (1908), 230–252.
- [7] F. DUMORTIER, J. LLIBRE AND J.C. ARTÉS, *Qualitative theory of planar differential systems*, UniversiText, Springer-Verlag, New York, 2006.
- [8] M. GALEOTTI AND M. VILLARINI, *Some properties of planar polynomial systems of even degree*, Ann. Mat. Pura Appl. **161**(4) (1992), 299–313.
- [9] J.D. GARCÍA-SALDAÑA, J. LLIBRE AND C. VALLS, *Linear type global centers of linear systems with cubic homogeneous nonlinearities*, Rendiconti del Circolo Matematico di Palermo Series 2, 69 (2020), 771–785.
- [10] M. GOPAL, *Control Systems: Principles and Design*, 2nd Ed. Tata McGraw-Hill Education, 2002.
- [11] M. HIRSCH, C. PUGH AND M. SHUB, *Invariant Manifolds*, Lecture Notes in Mathematics, Vol. 583. Springer-Verlag, 1977.
- [12] C. HUYGENS, *Horologium oscillatorium sive de motu pendularium*, (theory and design of the pendulum clock, dedicated to Louis XIV of France), 1673.
- [13] W. KAPTEYN, *New investigations on the midpoints of integrals of differential equations of the first degree*, Nederl. Akad. Wetensch. Verslag Afd. Natuurk. **19** (1911), 1446–1457; **20** (1912), 1354–1365; **21**, 27–33 (Dutch).
- [14] J. LLIBRE AND C. VALLS, *Polynomial differential systems with even degree have no global centers*, J. Math. Anal. and Appl. **503** (2021), 125281, 5 pp.
- [15] K.E. MALKIN, *Criteria for the center for a certain differential equation*, Volz. Mat. Sb. Vyp. **2** (1964), 87–91 (Russian)
- [16] W.I. MILHAM, *Time and timekeepers*, New York: MacMillan, 1945.
- [17] H. POINCARÉ, *Mémoire sur les courbes définies par les équations différentielles*, Journal de Mathématiques **37** (1881), 375–422; Oeuvres de Henri Poincaré, vol. I, Gauthier-Villars, Paris, 1951, pp 3–84.
- [18] N.I. VULPE AND K.S. SIBIRSKII, *Centro-affine invariant conditions for the existence of a center of a differential system with cubic nonlinearities*, Dokl. Akad. Nauk SSSR **301** (1988), 1297–1301 (Russian) ; translation in Soviet Math. Dokl. **38** (1989), 198–201.

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