TWIN POLYNOMIAL VECTOR FIELDS OF ARBITRARY DEGREE

JAUME LLIBRE[1] AND CLAUDIA VALLS[2]

Abstract. In this paper we study polynomial vector fields on $\mathbb{C}^2$ of degree larger than 2 with n^2 isolated singularities. More precisely, we show that if two polynomial vector fields share n^2-1 singularities with the same spectra (trace and determinant) and from these singularities n^2-2 have the same positions, then both vector fields have identical position and spectra at all the singularities. Moreover we also show that if two polynomial vector fields share n^2-1 singularities with the same positions and from these singularities n^2-2 have the same spectra, then both vector fields have identical position and spectra at all the singularities.

Moreover we also prove that generic vector fields of degree $n>2$ have no twins and that for any $n>2$ there exist two uniparametric families of twin vector fields, i.e. two different families of vector fields having exactly the same singular points and for each singular point both vector fields have the same spectrum.

1. Introduction and statement of the main results

Consider the polynomial vector fields on the affine plane $\mathbb{C}^2$ and denote by $\mathbb{P}_n$ the space of all polynomial vector fields

$$\chi = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y},$$

such that P and Q are polynomials and have degree at most n. By Bezout's theorem, a generic element of $\mathbb{P}_n$ has exactly n^2 isolated singularities. We denote by $\widehat{\mathbb{P}}_n$ the space of vector fields $\mathbb{P}_n$ that have n^2 isolated singularities. Since χ has the maximum number of singularities the determinant of the linear part of χ at each singular point is nonzero. So the eigenvalues at any singular point are nonzero, i.e. all singular points are non degenerate (see for more details [7]). The space $\widehat{\mathbb{P}}_n$ is endowed with a structure of a complex affine space identifying all the $(n+1)(n+2)$ coefficients of the polynomials P and Q with a point of $\mathbb{C}^{(n+1)(n+2)}$. This topology in the set of the polynomial vector fields of degree n is called the *topology of the coefficients* and $\widehat{\mathbb{P}}_n$ is an open subset of $\mathbb{P}_n$.

We will say that a property is a *generic property* for the class of vector fields $\widehat{\mathbb{P}}_n$ if the set of vector fields having this property contains a (non-empty) Zariski open and dense subset of $\widehat{\mathbb{P}}_n$.

We say that two vector fields χ_1 and χ_2 are *affine equivalent* if there exists an affine map T so that

$$\chi_2(x,y) = DT \cdot \chi_1(T^{-1}(x,y)).$$

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We denote by $\mathrm{Sing}\,(\chi)$ the set of singular points of the vector field χ. If $p \in \mathrm{Sing}\,(\chi)$, we define the *spectrum* of χ at p as the two eigenvalues of the linearization matrix

$$\left.\begin{pmatrix} P_x & Q_y \\ Q_x & Q_y \end{pmatrix}\right|_{(x,y)=p},$$

that is, the spectrum is the unordered set of eigenvalues of $D\chi(p)$.

Note that if $M \in GL_2(\mathbb{C})$ then the spectrum of M carries exactly the same information as the pair

$$\mathrm{Spec}\,(M) = (\mathrm{tr}M, \det M)$$

being $\mathrm{tr}M$ the trace of the matrix M and $\det M$ the determinant of the matrix M. In all the paper the spectrum of a vector field χ will be thought as the pair of the trace and the determinant.

If X is a topological space and $m \geq 1$ let S_m denote the symmetric group of m elements. Then X^m/S_m is the quotient of the usual action of S_m on X^m permuting its components. The set of spectra of singularities of a generic polynomial vector field χ of degree n belongs to the space $\mathcal{S}_n = (\mathbb{C}^2)^{n^2}/S_{n^2}$ which is an irreducible affine algebraic variety. We have a well defined map $\mathrm{Spec}_n \colon \widehat{\mathbb{P}}_n \to \mathcal{S}_n$.

We say that two vector fields from the class $\widehat{\mathbb{P}}_n$ have *the same spectra of singularities* if they have the same image under the map Spec_n. Note that the above definition takes into account the spectra of singularities and does not take into account the position of them. Our main aim is to understand the pair of vector fields that share both position and spectra of singularities, that is, our main aim is to provide results in the following spirit: consider two polynomial vector fields χ and $\widehat{\chi}$ having each of them n^2 isolated singularities and label these points as $p_1, \ldots, p_{n^2}$ and $\widehat{p}_1, \ldots, \widehat{p}_{n^2}$, respectively. Assume that

$$p_i = \widehat{p}_i \quad \text{for } i = 1, \ldots, M \quad \text{and} \quad D\chi(p_i) \sim D\widehat{\chi}(\widehat{p}_i) \quad \text{for } i = 1, \ldots, N \tag{1}$$

where $A \sim B$ denotes that the matrices A and B are *similar* that is, they have the same spectrum. Then for certain values of M and N we want to see when

$$p_i = \widehat{p}_i \quad \text{and} \quad D\chi(p_i) \sim D\widehat{\chi}(\widehat{p}_i) \quad \text{for all } i = 1, \ldots, n^2.$$

Once it has been established that two vector fields agree on position of their singularities and their corresponding spectra, it is natural to ask whether these vector fields are identical or not. This question gives rise to the concept of *twin vector fields*: two *different* vector fields are said to be twins if they agree on position and spectra at all their singular points. This corresponds to $M = N = n^2$ in (1).

The corresponding question for quadratic polynomial vector fields, that is, when $n = 2$ was studied in [8]. There the author proved that if two quadratic vector fields have the same spectra (namely $M = n^2 = 4$ in equation (1)), then after an affine transformation we can achieve that all points share the same position and spectra (that is $M = N = 4$ in (1)). Furthermore it is proved that a *generic* quadratic vector field indeed admits a unique twin vector field. The results of [8] are inspired in the papers [5, 6]. Similar results for polynomial vector fields of degree greater than $n = 2$ are not currently available in the literature and this is the main aim of this paper. The fact that $n > 2$ makes the analysis much more intricate in particular because even when $n = 3$ after an affine transformation we can achieve that three points share the same position but there are still six free points.

The following are our results. We always assume that we are under the assumptions of equation (1).

Theorem 1. *The following statements hold for $n \geq 2$:*

(i) *If two vector fields have the same spectra at $N = n^2 - 1$ singular points, then they have the same spectra at all n^2 singular points;*
(ii) *If in addition they have the same position at the original $M = n^2 - 1$ points, then they also have the same position for the final singular point.*

Note that it follows from Theorem 1 that if $n \geq 2$ and equation (1) is satisfied with $M = N = n^2 - 1$, then both vector fields χ and $\widehat{\chi}$ have the same singular points and the same spectra. This corresponds to $M = N = n^2$ in equation (1), with the additional constraint that $\chi \neq \widehat{\chi}$.

The proof of Theorem 1 is given in section 2. The particular case for $n = 2$ was first treated in [8] and the proof for any n follows the same lines but we include it in the paper for completeness.

Theorem 2. *Consider equation (1) with $M = n^2 - 2$ and $N = n^2 - 1$ with $n > 2$. Then both vector fields χ and $\widehat{\chi}$ have the same singular points and the same spectra and so $M = N = n^2$ in equation (1).*

The proof of Theorem 2 is given in section 3.

Theorem 3. *Consider equation (1) with $M = n^2 - 1$ and $N = n^2 - 2$ with $n > 2$. Then both vector fields χ and $\widehat{\chi}$ have the same singular points and the same spectra and so $M = N = n^2$ in equation (1).*

The proof of Theorem 3 is given in section 4.

We recall here that as mentioned above, the space $\widehat{\mathbb{P}}_n$ has dimension $(n+1)(n+2)$. Let χ be a given vector field from the class $\widehat{\mathbb{P}}_n$. Requiring that another vector field $\widehat{\chi} \in \widehat{\mathbb{P}}_N$ agrees on position on M points imposes $2M$ conditions on the parameters of $\widehat{\chi}$ (that is on the polynomials defining $\widehat{\chi}$). Similarly, requiring that $D\chi(p_i) \sim D\widehat{\chi}(\widehat{p}_i)$ at N singular points imposes $2N$ conditions on the parameters. Thus we impose $2M + 2N$ conditions on $(n+1)(n+2)$ parameters. For the values of M and N in Theorems 2 and 3, we have

$$2M + 2N = 4n^2 - 6 > (n+1)(n+2).$$

Even on the cubic case the gap is quite big. The problem of finding the smallest value of $2M + 2N$ that guarantees that if equation (1) is satisfied then all singular points agree on position and spectra, although is an interesting question, it is a very hard problem and is out of the scope of this paper.

In Theorems 2 and 3 we provide conditions and study the situation of having two vector fields agree on position and spectra at all their singularities. Now we want to establish whether the two vector fields are identical or not. We will see that when $n > 2$ vector fields that are twin vector fields are rather uncommon. The proof of this claim is the content of the following result. We set $R_x = \partial R/\partial x$ and $R_y = \partial R/\partial y$ for any polynomial R.

Theorem 4. *Let $\chi = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y} \in \widehat{\mathbb{P}}_n$. The following statements hold:*

(a) *If this vector field has a twin vector field then the four polynomials*

$$P_x, Q_x, P_y, Q_y \tag{2}$$

are linearly dependent over $\mathbb{C}$.

(b) *If the polynomials in (2) are linearly dependent over $\mathbb{C}$ and admit a non-trivial linear combination*

$$(a-1)P_x + bQ_x + cP_y + (d-1)Q_y \equiv 0 \tag{3}$$

satisfying $ad - bc = 1$, then χ admits a twin vector field.

The proof of Theorem 4 is given in section 5.

We have the following corollary from Theorem 4.

Corollary 5. *A generic vector field of degree $n > 2$ has no twins.*

Note that the polynomials P_x, Q_x, P_y and Q_y will generically be of degree $n-1$. The space of polynomials of two variables of degree $n-1$ has dimension $n(n+1)/2 > 4$, whenever $n > 2$. That is the reason why twin vector fields are uncommon in higher degrees while in the quadratic case P_x, Q_x, P_y and Q_y are always linearly dependent and it is not surprising that in [8] the author proves that generic quadratic vector fields admit always a twin vector field.

Taking $P_y = 0$ or $Q_x = 0$ in Theorem 4 we have the family of twin vector fields provided in the following theorem.

Theorem 6. *Let $P(x)$ and $Q(y)$ be polynomials each having n distinct roots. Let*

$$\chi = P(x)\frac{\partial}{\partial x} + Q(y)\frac{\partial}{\partial y}$$

and consider the following one-parameter family of vector fields

$$\left\{ (P(x) + bQ(y))\frac{\partial}{\partial x} + Q(y)\frac{\partial}{\partial y},\ b \in \mathbb{C} \right\},$$
$$\left\{ P(x)\frac{\partial}{\partial x} + (cP(x) + Q(y))d\tfrac{\partial}{\partial y},\ b \in \mathbb{C} \right\}.$$

Then each vector field in each family is a twin vector field of χ (except for $b = 0$ or $c = 0$) when the vector fields are identical.

Our final comment is the following: whenever $N = n^2 - 1$ in system (1) the Euler-Jacobi formula (see section 2 for its explicit statement) easily implies that the two vector fields have the same spectra (see the first part of the proof of Theorem 1). Moreover, when $M = n^2 - 1$ in system (1), the Cayley-Bacharach theorem (see section 4 for its explicit statement) easily implies that the two vector fields have the same singular set (see the first part in the proof of Theorem 3). Therefore, the first non trivial cases are $M = n^2 - 1$ with $N = n^2 - 2$ and $M = n^2 - 2$ with $N = n^2 - 1$ which are precisely the two cases treated in Theorems 2 and 3, respectively.

2. Proof of Theorem 1

The main result that we use for proving Theorems 2, 3 and 6 is the Euler-Jacobi formula which can be stated as follows.

Theorem 7 (Euler-Jacobi formula). *If P and Q are polynomials in $\mathbb{C}[x,y]$ of degree n whose divisors intersect transversally in n^2 different points $p_1,\ldots,p_{n^2}\in\mathbb{C}^2$ and $g(x,y)$ is a polynomial of degree at most $2n-3$, then*

$$\sum_{k=1}^{n^2}\frac{g(p_k)}{J(p_k)}=0,$$

where $J(x,y)$ is the Jacobian determinant of P and Q, that is,

$$J(x,y)=\det\frac{\partial(P,Q)}{\partial(x,y)}=\det\begin{pmatrix}P_x & Q_y\\ Q_x & Q_y\end{pmatrix}=\det\big(D\chi(x,y)\big).$$

For a proof of Theorem 7 see [1].

Proof of Theorem 1. Assume that two vector fields χ and $\widehat{\chi}$ have the same spectra at $N=n^2-1$ singular points. Denote by a_{n^2} and $\widehat{a}_{n^2}$ the determinant of $D\chi$ and $D\widehat{\chi}$ at p_{n^2}, and by b_{n^2} and $\widehat{b}_{n^2}$ the traces of $D\chi$ and $D\widehat{\chi}$ at p_{n^2}, respectively. Note that from the Euler-Jacobi formula with $g_1(x,y)=1$ and $g_2(x,y)=\mathrm{tr}(D\chi(x,y))$ we get

$$\frac{1}{a_{n^2}}=\frac{1}{\widehat{a}_{n^2}}\quad\text{which yields}\quad a_{n^2}=\widehat{a}_{n^2},$$

and

$$\frac{b_{n^2}}{a_{n^2}}=\frac{\widehat{b}_{n^2}}{\widehat{a}_{n^2}}\quad\text{which yields}\quad b_{n^2}=\widehat{b}_{n^2}.$$

Therefore, both vector fields χ and $\widehat{\chi}$ have the same spectra at all n^2 singular points. This completes the proof of statement (i).

Now assume that in addition we have $M=n^2-1$ and we denote by $p_{n^2}=(x_{n^2},y_{n^2})$ and $\widehat{p}_{n^2}=(\widehat{x}_{n^2},\widehat{y}_{n^2})$ the positions of p_{n^2} of χ and of $\widehat{p}_{n^2}$ of $\widehat{\chi}$, respectively. Applying the Euler-Jacobi formula with $g_1(x,y)=x$ and $g_2(x,y)=y$ we get

$$\frac{x_{n^2}}{a_{n^2}}=\frac{\widehat{x}_{n^2}}{\widehat{a}_{n^2}}\quad\text{and}\quad\frac{y_{n^2}}{a_{n^2}}=\frac{\widehat{y}_{n^2}}{\widehat{a}_{n^2}}$$

which yields $x_{n^2}=\widehat{x}_{n^2}$ and $y_{n^2}=\widehat{y}_{n^2}$. Therefore, both vector fields χ and $\widehat{\chi}$ have the same singular locus and the proof of statement (ii) is complete. □

3. Proof of Theorem 2

We continue to denote by a_{n^2} and $\widehat{a}_{n^2}$ the determinant of $D\chi(p_{n^2})$ and $D\widehat{\chi}(p_{n^2})$, respectively, and by b_{n^2} and $\widehat{b}_{n^2}$ the traces of $D\chi(p_{n^2})$ and $D\widehat{\chi}(p_{n^2})$, respectively. It follows from the proof of Theorem 1 (i) that $a_{n^2}=\widehat{a}_{n^2}$ and $b_{n^2}=\widehat{b}_{n^2}$. Therefore, the spectrum of p_{n^2} and $\widehat{p}_{n^2}$ is the same and so $M=n^2$.

Now we also continue to denote by $p_{n^2}=(x_{n^2},y_{n^2})$ and $p_{n^2-1}=(x_{n^2-1},y_{n^2-1})$ the positions of the points for the vector field χ, and by $\widehat{p}_{n^2}=(\widehat{x}_{n^2},\widehat{y}_{n^2})$ and $\widehat{p}_{n^2-1}=(\widehat{x}_{n^2-1},\widehat{y}_{n^2-1})$ the positions of the points for the vector field $\widehat{\chi}$. Applying the Euler

Jacobi formula with $g_1(x,y) = x$, $g_2(x,y) = y$, $g_3(x,y) = \mathrm{tr}(D\chi(p))x$, $g_4(x,y) = \mathrm{tr}(D\chi(p))y$ and taking into account that $a_i = \widehat{a}_i$ and $b_i = \widehat{b}_i$ for $i = 1, \dots, n^2$, we get

$$
\begin{aligned}
\frac{x_{n^2-1}}{a_{n^2-1}} + \frac{x_{n^2}}{a_{n^2}} &= \frac{\widehat{x}_{n^2-1}}{a_{n^2-1}} + \frac{\widehat{x}_{n^2}}{a_{n^2}},\\
\frac{y_{n^2-1}}{a_{n^2-1}} + \frac{y_{n^2}}{a_{n^2}} &= \frac{\widehat{y}_{n^2-1}}{a_{n^2-1}} + \frac{\widehat{y}_{n^2}}{a_{n^2}},\\
\frac{b_{n^2-1}x_{n^2-1}}{a_{n^2-1}} + \frac{b_{n^2}x_{n^2}}{a_{n^2}} &= \frac{b_{n^2-1}\widehat{x}_{n^2-1}}{a_{n^2-1}} + \frac{b_{n^2}\widehat{x}_{n^2}}{a_{n^2}},\\
\frac{b_{n^2-1}y_{n^2-1}}{a_{n^2-1}} + \frac{b_{n^2}y_{n^2}}{a_{n^2}} &= \frac{b_{n^2-1}\widehat{y}_{n^2-1}}{a_{n^2-1}} + \frac{b_{n^2}\widehat{y}_{n^2}}{a_{n^2}}.
\end{aligned}
\tag{4}
$$

Multiplying by b_{n^2-1} the first equality in (4) and subtracting the third from it, and doing the same with the second and the fourth equalities we get

$$
\frac{(b_{n^2-1} - b_{n^2})x_{n^2}}{a_n^2} = \frac{(b_{n^2-1} - b_{n^2})\widehat{x}_{n^2}}{a_n^2}, \quad \frac{(b_{n^2-1} - b_{n^2})y_{n^2}}{a_n^2} = \frac{(b_{n^2-1} - b_{n^2})\widehat{y}_{n^2}}{a_n^2}.
$$

We have two cases: either $b_{n^2-1} - b_{n^2} \neq 0$ or $b_{n^2-1} - b_{n^2} = 0$.

If $b_{n^2-1} - b_{n^2} \neq 0$, then $x_{n^2} = \widehat{x}_{n^2}$ and $y_{n^2} = \widehat{y}_{n^2}$. Then the third and second equalities in (4) imply that $x_{n^2-1} = \widehat{x}_{n^2-1}$ as well as $y_{n^2-1} = \widehat{y}_{n^2-1}$. In short $p_{n^2-1} = \widehat{p}_{n^2-1}$, $p_{n^2} = \widehat{p}_{n^2}$ and so $N = n^2$. This completes the proof of the theorem in this case.

If $b_{n^2-1} = b_{n^2}$ then we apply the Euler-Jacobi formula with $g_1(x,y) = x^2$, $g_2(x,y) = x^3$, $g_3(x,y) = y^2$ and $g_4(x,y) = y^3$ and we get

$$
\begin{aligned}
\frac{x^2_{n^2-1}}{a_{n^2-1}} + \frac{x^2_{n^2}}{a_{n^2}} &= \frac{\widehat{x}^2_{n^2-1}}{a_{n^2-1}} + \frac{\widehat{x}^2_{n^2}}{a_{n^2}}, \quad \text{i.e} \quad \frac{x^2_{n^2-1} - \widehat{x}^2_{n^2-1}}{a_{n^2-1}} = \frac{\widehat{x}^2_{n^2} - x^2_{n^2}}{a_{n^2}},\\
\frac{x^3_{n^2-1}}{a_{n^2-1}} + \frac{x^3_{n^2}}{a_{n^2}} &= \frac{\widehat{x}^3_{n^2-1}}{a_{n^2-1}} + \frac{\widehat{x}^3_{n^2}}{a_{n^2}}, \quad \text{i.e} \quad \frac{x^3_{n^2-1} - \widehat{x}^3_{n^2-1}}{a_{n^2-1}} = \frac{\widehat{x}^3_{n^2} - x^3_{n^2}}{a_{n^2}},\\
\frac{y^2_{n^2-1}}{a_{n^2-1}} + \frac{y^2_{n^2}}{a_{n^2}} &= \frac{\widehat{y}^2_{n^2-1}}{a_{n^2-1}} + \frac{\widehat{y}^2_{n^2}}{a_{n^2}}, \quad \text{i.e} \quad \frac{y^2_{n^2-1} - \widehat{y}^2_{n^2-1}}{a_{n^2-1}} = \frac{\widehat{y}^2_{n^2} - y^2_{n^2}}{a_{n^2}},\\
\frac{y^3_{n^2-1}}{a_{n^2-1}} + \frac{y^3_{n^2}}{a_{n^2}} &= \frac{\widehat{y}^3_{n^2-1}}{a_{n^2-1}} + \frac{\widehat{y}^3_{n^2}}{a_{n^2}}, \quad \text{i.e} \quad \frac{y^3_{n^2-1} - \widehat{y}^3_{n^2-1}}{a_{n^2-1}} = \frac{\widehat{y}^3_{n^2} - x^3_{n^2}}{a_{n^2}}.
\end{aligned}
\tag{5}
$$

Note that the first and second relations in (4) can be written as

$$
\frac{x_{n^2-1} - \widehat{x}_{n^2-1}}{a_{n^2-1}} = \frac{\widehat{x}_{n^2} - x_{n^2}}{a_{n^2}}, \quad \frac{y_{n^2-1} - \widehat{y}_{n^2-1}}{a_{n^2-1}} = \frac{\widehat{y}_{n^2} - y_{n^2}}{a_{n^2}}.
\tag{6}
$$

It follows from the first identity in (6) that either $\widehat{x}_{n^2-1} = x_{n^2-1}$ and so $\widehat{x}_{n^2} = x_{n^2}$, or $\widehat{x}_{n^2-1} \neq x_{n^2-1}$ and $\widehat{x}_{n^2} \neq x_{n^2}$. In this last case from the first and second relations in (5) we get

$$
\begin{aligned}
\widehat{x}_{n^2-1} + x_{n^2-1} &= \widehat{x}_{n^2} + x_{n^2},\\
\widehat{x}^2_{n^2-1} + \widehat{x}_{n^2-1}x_{n^2-1} + x^2_{n^2-1} &= \widehat{x}^2_{n^2} + \widehat{x}_{n^2}x_{n^2} + x^2_{n^2}
\end{aligned}
\tag{7}
$$

So taking the square of the first relation in (7) and substracting it in the second one we obtain

$$
\widehat{x}_{n^2}x_{n^2} = \widehat{x}_{n^2-1}x_{n^2-1}.
$$

Therefore, using (6) and (7) we have

$$\begin{aligned}\frac{1}{a_{n^2}^2}(\widehat{x}_{n^2}-x_{n^2})^2 &= \frac{1}{a_{n^2-1}^2}(\widehat{x}_{n^2-1}-x_{n^2-1})^2\\ &= \frac{1}{a_{n^2-1}^2}(\widehat{x}_{n^2-1}^2-2\widehat{x}_{n^2-1}x_{n^2-1}+x_{n^2-1}^2)\\ &= \frac{1}{a_{n^2-1}^2}(\widehat{x}_{n^2}^2-2\widehat{x}_{n^2}x_{n^2}+x_{n^2}^2) = \frac{1}{a_{n^2-1}^2}(\widehat{x}_{n^2}-x_{n^2})^2\end{aligned}$$

and so $a_{n^2-1} = \pm a_{n^2}$.

Now proceeding exactly as we did with x but now with y using (6) and (7) we get that either $\widehat{y}_{n^2-1} = y_{n^2-1}$ and so $\widehat{y}_{n^2} = y_{n^2}$, or $\widehat{y}_{n^2-1} \neq y_{n^2-1}$ and $\widehat{y}_{n^2} \neq y_{n^2}$ and again $a_{n^2-1} = \pm a_{n^2}$.

In short we have four possibilities that we will study separately. As we will see, only Case 1 may occur.

Case 1*:* $\widehat{x}_{n^2-1} = x_{n^2-1}$, $\widehat{x}_{n^2} = x_{n^2}$, $\widehat{y}_{n^2-1} = y_{n^2-1}$ and $\widehat{y}_{n^2} = y_{n^2}$. In this case, $\widehat{p}_{n^2-1} = p_{n^2-1}$, $\widehat{p}_{n^2} = p_{n^2}$, and so $N = n^2$, as claimed in the statement of the theorem.

Case 2*:* $\widehat{x}_{n^2-1} = x_{n^2-1}$, $\widehat{x}_{n^2} = x_{n^2}$, $\widehat{y}_{n^2-1} \neq y_{n^2-1}$, $\widehat{y}_{n^2} \neq y_{n^2}$ and $a_{n^2-1} = \pm a_{n^2}$. Note that from (7) we have $\widehat{x}_{n^2-1} = x_{n^2-1} = x_{n^2} = \widehat{x}_{n^2}$. Moreover, if $a_{n^2-1} = a_{n^2}$ then from (6) and (7) we obtain

$$\widehat{y}_{n^2-1} - y_{n^2-1} = \widehat{y}_{n^2} - y_{n^2} \quad \text{and} \quad \widehat{y}_{n^2-1} + y_{n^2-1} = \widehat{y}_{n^2} + y_{n^2}$$

and so $\widehat{y}_{n^2-1} = \widehat{y}_{n^2}$ and $y_{n^2-1} = y_{n^2}$, but then $\widehat{p}_{n^2-1} = \widehat{p}_{n^2}$ and $p_{n^2-1} = p_{n^2}$ which is not possible.

On the other hand, if $a_{n^2-1} = -a_{n^2}$ then

$$\widehat{y}_{n^2-1} - y_{n^2-1} = -\widehat{y}_{n^2} + y_{n^2} \quad \text{and} \quad \widehat{y}_{n^2-1} + y_{n^2-1} = \widehat{y}_{n^2} + y_{n^2}$$

which yields $\widehat{y}_{n^2-1} = y_{n^2}$ and $y_{n^2-1} = \widehat{y}_{n^2}$, but then $\widehat{p}_{n^2-1} = p_{n^2}$ and $p_{n^2-1} = \widehat{p}_{n^2}$ and in this case both vector fields χ and $\widehat{\chi}$ have the same singular points and the determinants of p_{n^2-1} and p_{n^2} have different signs. Now it follows from the second and fourth relations in (5) with $a_{n^2-1} = -a_{n^2}$ and $\widehat{x}_{n^2-1} = x_{n^2}$, and $x_{n^2-1} = \widehat{x}_{n^2}$, $\widehat{y}_{n^2-1} = y_{n^2}$ and $y_{n^2-1} = \widehat{y}_{n^2}$ that

$$-\frac{\widehat{x}_{n^2}^3 - \widehat{x}_{n^2-1}^3}{a_n^2} = \frac{\widehat{x}_{n^2}^3 - \widehat{x}_{n^2-1}^3}{a_n^2}$$

and

$$-\frac{\widehat{y}_{n^2}^3 - \widehat{y}_{n^2-1}^3}{a_n^2} = \frac{\widehat{y}_{n^2}^3 - \widehat{y}_{n^2-1}^3}{a_n^2}$$

which yields $\widehat{x}_{n^2}^3 = \widehat{x}_{n^2-1}^3$ and $\widehat{y}_{n^2}^3 = \widehat{y}_{n^2-1}^3$ and so $\widehat{x}_{n^2} = \widehat{x}_{n^2-1}$ and $\widehat{y}_{n^2} = \widehat{y}_{n^2-1}$, but then $\widehat{p}_{n^2-1} = \widehat{p}_{n^2}$ and $p_{n^2-1} = p_{n^2}$ which is not possible.

Case 3*:* $\widehat{x}_{n^2-1} \neq x_{n^2-1}$, $\widehat{x}_{n^2} \neq x_{n^2}$, $\widehat{y}_{n^2-1} = y_{n^2-1}$, $\widehat{y}_{n^2} = y_{n^2}$ and $a_{n^2-1} = \pm a_{n^2}$. In this case interchanging the roles of x and y in Case 2 we arrive to the same conclusions as in Case 2, that is, this case is not possible.

Case 4: $\widehat{x}_{n^2-1} \neq x_{n^2-1}$, $\widehat{x}_{n^2} \neq x_{n^2}$, $\widehat{y}_{n^2-1} \neq y_{n^2-1}$, $\widehat{y}_{n^2} \neq y_{n^2}$ and $a_{n^2-1} = \pm a_{n^2}$. If $a_{n^2-1} = a_{n^2}$ then from (6) and (7) we get

$$\widehat{x}_{n^2-1} - x_{n^2-1} = \widehat{x}_{n^2} - x_{n^2} \quad \text{and} \quad \widehat{x}_{n^2-1} + x_{n^2-1} = \widehat{x}_{n^2} + x_{n^2},$$

which yields $\widehat{x}_{n^2-1} = \widehat{x}_{n^2}$ and $x_{n^2-1} = x_{n^2}$ and proceeding analogously as in Case 2 we get that $\widehat{y}_{n^2-1} = \widehat{y}_{n^2}$ and $y_{n^2-1} = y_{n^2}$. But then $\widehat{p}_{n^2-1} = \widehat{p}_{n^2}$ and $p_{n^2-1} = p_{n^2}$ which is not possible.

If $a_{n^2-1} = -a_{n^2}$ then

$$\widehat{x}_{n^2-1} - x_{n^2-1} = -\widehat{x}_{n^2} + x_{n^2} \quad \text{and} \quad \widehat{x}_{n^2-1} + x_{n^2-1} = \widehat{x}_{n^2} + x_{n^2}$$

which yields $\widehat{x}_{n^2-1} = x_{n^2}$ and $x_{n^2-1} = \widehat{x}_{n^2}$, and proceeding analogously as in Case 2 we get that $\widehat{y}_{n^2-1} = y_{n^2}$ and $y_{n^2-1} = \widehat{y}_{n^2}$. But then $\widehat{p}_{n^2-1} = p_{n^2}$ and $p_{n^2-1} = \widehat{p}_{n^2}$ and in this case both vector fields χ and $\widehat{\chi}$ have the same singular points and the determinants of p_{n^2-1} and p_{n^2} have different signs. Proceeding as in Case 2 we reach a contradiction. This concludes the proof of the theorem.

4. Proof of Theorem 3

To prove Theorem 3 we state the well-known Cayley-Bacharach theorem for polynomials. See [3, Theorem CB5] for a proof.

Theorem 8 (Cayley-Bacharach Theorem). *Let $X_1, X_2 \in \mathbb{P}^2$ be plane curves of degrees d_1 and d_2 respectively, intersecting in $d_1 d_2$ points $\Gamma = X_1 \cap X_2 = \{p_1, \ldots, p_{d_1 d_2}\}$, and assume that Γ is the disjoint union of subsets Γ' and Γ''. Set $s = d_1 + d_2 - 3$. If $k \leq s$ is a non-negative integer, then the dimension of the vector space of polynomials of degree k vanishing on Γ' modulo those containing Γ is equal to the failure of Γ'' to impose independent conditions on the polynomials of degree $s - k$.*

We recall that the failure of Γ'' to impose independent conditions on polynomials of degree $s-k$ is the difference between the cardinality of Γ'' and the rank of the linear conditions on the polynomials of degree $s - k$ imposed by Γ''.

Proof of Theorem 3. Take $\chi = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y} \in \widehat{\mathbb{P}}_n$. Taking into account that since $n > 2$ the curves of degree n passing through the n^2 points of $P \cap Q$ provide an overdetermined linear system, it follows from Theorem 8 (with $X_1 = P$, $X_2 = Q$, $d_1 = d_2 = k = n$, Γ' being $n^2 - 1$ points and Γ'' only one point) that any polynomial R of degree n which vanishes on $n^2 - 1$ points in $P \cap Q$ also vanishes on the remaining one. Therefore, since by assumption $M = n^2 - 1$, we immediately conclude that the two vector fields χ and $\widehat{\chi}$ have the same singular locus and so $M = n^2$.

We take the same notation as in the proof of Theorem 2. More precisely, we denote by $a_{n^2-1}, a_{n^2}, \widehat{a}_{n^2-1}, \widehat{a}_{n^2}$ the determinants of $D\chi(p_{n^2-1})$, $D\chi(p_{n^2})$, $D\widehat{\chi}(p_{n^2-1})$ and $D\widehat{\chi}(\widehat{p}_{n^2})$, respectively, and by $b_{n^2-1}, b_{n^2}, \widehat{b}_{n^2-1}, \widehat{b}_{n^2}$ the traces of $D\chi(p_{n^2-1})$, $D\chi(p_{n^2})$,$D\widehat{\chi}(p_{n^2-1})$ and $D\widehat{\chi}(\widehat{p}_{n^2})$, respectively. Moreover, the points will be denoted as $p_k = (x_k, y_k)$ for $k = 1, \ldots, n^2$.

Note that from the Euler-Jacobi fromula with $g_1(x,y) = 1$, $g_2(x,y) = \text{tr}(D\chi(x,y))$, $g_3(x,y) = \text{tr}(D\chi(p)x$, $g_4(x,y) = x$, $g_5(x,y) = \text{tr}(D\chi(p)y)$ and $g_6(x,y) = y$ we also get

$$
\begin{aligned}
\frac{1}{a_{n^2-1}}+\frac{1}{a_{n^2}} &= \frac{1}{\widehat{a}_{n^2-1}}+\frac{1}{\widehat{a}_{n^2}} \\
\frac{b_{n^2-1}}{a_{n^2-1}}+\frac{b_{n^2}}{a_{n^2}} &= \frac{\widehat{b}_{n^2-1}}{\widehat{a}_{n^2-1}}+\frac{\widehat{b}_{n^2}}{\widehat{a}_{n^2}}, \\
\frac{b_{n^2-1}x_{n^2-1}}{a_{n^2-1}}+\frac{b_{n^2}x_{n^2}}{a_{n^2}} &= \frac{\widehat{b}_{n^2-1}x_{n^2-1}}{\widehat{a}_{n^2-1}}+\frac{\widehat{b}_{n^2}\widehat{x}_{n^2}}{\widehat{a}_{n^2}}, \\
\frac{x_{n^2-1}}{a_{n^2-1}}+\frac{x_{n^2}}{a_{n^2}} &= \frac{x_{n^2-1}}{\widehat{a}_{n^2-1}}+\frac{\widehat{x}_{n^2}}{\widehat{a}_{n^2}}, \\
\frac{b_{n^2-1}y_{n^2-1}}{a_{n^2-1}}+\frac{b_{n^2}y_{n^2}}{a_{n^2}} &= \frac{\widehat{b}_{n^2-1}y_{n^2-1}}{\widehat{a}_{n^2-1}}+\frac{\widehat{b}_{n^2}\widehat{y}_{n^2}}{\widehat{a}_{n^2}}, \\
\frac{y_{n^2-1}}{a_{n^2-1}}+\frac{y_{n^2}}{a_{n^2}} &= \frac{y_{n^2-1}}{\widehat{a}_{n^2-1}}+\frac{\widehat{y}_{n^2}}{\widehat{a}_{n^2}}
\end{aligned} \tag{8}
$$

Multiplying the first equation in (8) by x_{n^2-1} and subtracting it to the fourth identity, and multiplying the first equation in (8) by y_{n^2-1} and subtracting it to the sixth identity we get

$$
\frac{x_{n^2-1}-x_{n^2}}{a_{n^2}}=\frac{x_{n^2-1}-\widehat{x}_{n^2}}{\widehat{a}_{n^2}} \quad \text{and} \quad \frac{y_{n^2-1}-y_{n^2}}{a_{n^2}}=\frac{y_{n^2-1}-\widehat{y}_{n^2}}{\widehat{a}_{n^2}}. \tag{9}
$$

Since either $x_{n^2-1}\neq x_{n^2}$ or $y_{n^2-1}\neq y_{n^2}$, we get from (9) that $a_{n^2}=\widehat{a}_{n^2}$ and then the first identity in (8) implies that also $a_{n^2-1}=\widehat{a}_{n^2-1}$.

Now multiplying the second equation in (8) by x_{n^2-1} and subtracting it to the third identity, and multiplying the second equation in (8) by y_{n^2-1} and subtracting it to the fifth identity we get

$$
\begin{aligned}
\frac{b_{n^2-1}(x_{n^2-1}-x_{n^2})}{a_{n^2}} &= \frac{b_{n^2-1}(x_{n^2-1}-\widehat{x}_{n^2})}{a_{n^2}}, \\
\frac{b_{n^2-1}(y_{n^2-1}-y_{n^2})}{a_{n^2}} &= \frac{b_{n^2-1}(y_{n^2-1}-\widehat{y}_{n^2})}{a_{n^2}}.
\end{aligned} \tag{10}
$$

Since either $x_{n^2-1}\neq x_{n^2}$ or $y_{n^2-1}\neq y_{n^2}$, we get from (10) that $b_{n^2}=\widehat{b}_{n^2}$ and then the second identity in (8) implies that also $b_{n^2-1}=\widehat{b}_{n^2-1}$. In short $N=n^2$ and the proof of the theorem is complete. □

5. Proof of Theorem 4

To prove Theorem 6 we state the well-known Max Noether's fundamental theorem and a closely related proposition. For a proof of both of them see [4, Section 5 of Chapter 5]. Let $\mathbb{P}^2$ be the projective plane. For $F\in\mathbb{C}[X,Y,Z]$ we write $F_*=F(X,Y,1)$. Moreover, when $p=[x:y:1]$ then $\mathcal{O}_p(\mathbb{P}^2)$ is canonically isomorphic to $\mathcal{O}_{(x,y)}(\mathbb{C}^2)$ can be regarded as an element of the local ring $\mathcal{O}_p(\mathbb{P}^2)$. We recall that $\mathcal{O}_p(\mathbb{P}^2)$ is the ring of rational functions on $\mathbb{P}^2$ that are defined at p.

Let $p\in\mathbb{P}^2$, F,G curves with no common component through p and H be another curve. We say that *Noether's conditions* are satisfied at p (with respect to F,G

and H) if $H_* \in (F_*, G_*) \subset \mathcal{O}_p(\mathbb{P}^2)$ that is, there are $a, b \in \mathcal{O}_p(\mathbb{P}^2)$ such that $H_* = aF_* + bG_*$.

Theorem 9 (Max Noether's fundamental theorem)**.** *Let F, G, H be projective plane curves. Assume that F and G have no common components. Then there is an equation $H = AF + BG$ (with A and B forms of degrees deg(H) − deg(F) and deg(H) − deg(G), respectively) if and only if Noether's conditions are satsified at every $p \in F \cap G$.*

Proposition 10. *Let $P, Q, R \in \mathbb{C}[X, Y, Z]$ and p be a point in $P \cap Q$. If P and Q intersect transversally at p and $R(p) = 0$, then the Noether conditions holds at p.*

Proof of Theorem 4. Assume that χ has a twin vector field $\widehat{\chi}$. Using Max Noether's fundamental theorem since $\widehat{\chi}$ and χ have the same singular locus, there exist complex numbers a, b, c, d such that

$$\widehat{\chi} = (aP + bQ)\frac{\partial}{\partial x} + (cP + dQ)\frac{\partial}{\partial y}. \tag{11}$$

Consider the polynomial

$$R(x, y) = \mathrm{tr}D\widehat{\chi} - \mathrm{tr}D\chi = (a-1)P_x + bQ_x + cP_x + (d-1)Q_y.$$

If χ and $\widehat{\chi}$ have the same spectra, then

$$R(p_i) = 0 \quad \text{for } i = 1, \dots, n^2.$$

Since P and Q intersect transversally at each p_i because the maximum number of their intersection points is n^2, it follows from Proposition 10 and Theorem 9 that if $R \not\equiv 0$ taking into account that the degree of R is $n-1$, there exist two polynomials A and B with $\deg(A) = -1$ and $\deg(B) = 0$ such that $R = AP + BQ$, which is not possible. So, $R \equiv 0$ and condition in (3) means that the polynomials P_x, Q_x, P_y and Q_y are $\mathbb{C}$-linearly dependent. This concludes the proof of statement a) of the theorem.

Assume now that equation (3) is a non-trivial linear combination and satisfies $ad - bc = 1$. We claim that the vector field $\widehat{\chi}$ given in (11) is a twin vector field of χ. Note that if equation (3) is a trivial linear combination of P_x, P_y, Q_x, Q_y then $a = d = 1$ and $b = c = 0$ which thus gives $\chi = \widehat{\chi}$ implying that $\widehat{\chi}$ is not a twin of χ.

Now we prove the claim. It is clear that if $P = Q = 0$, then any linear combination of them will also be zero. For the converse, we note that we already obtained $aP + bQ$ and $cP + dQ$ by multiplying the vector $(P, Q)^T$ by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ which is invertible since $ad - bc = 1$. Hence, clearly $aP + bQ = cP + dQ = 0$ imply $P = Q = 0$. Therefore both χ and $\widehat{\chi}$ have the same singular locus $\{p_1, \dots, p_{n^2}\}$. Moreover, since equation (3) is a non-trivial linear combination, for $i = 1, \dots, n^2$ we have

$$\begin{aligned}\mathrm{tr}D\chi(p_i) - \mathrm{tr}\widehat{\chi}(p_i) &= (a-1)P_x(p_i) + bQ_x(p_i) + cP_y(q_i) + (d-1)Q_y(p_i)\\ &= R(p_i) = 0.\end{aligned}$$

Finally, note that for $i = 1, \dots, n^2$

$$\det \widehat{\chi}(p_i) = (ad - bc) \det \chi(p_i) = \det \chi(p_i),$$

and so the vector field $\widehat{\chi}$ given in (11) is a twin vector field of χ. This completes the proof of statement (b) and concludes the proof of the theorem. □

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[1] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

Email address: `jllibre@mat.uab.cat`

[2] Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais 1049–001, Lisboa, Portugal

Email address: `cvalls@math.ist.utl.pt`