# A SUFFICIENT CONDITION FOR THE REAL JACOBIAN CONJECTURE IN $\mathbb{R}^2$

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**Abstract.** Let $F = (f, g)\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial map such that $\det DF(x, y)$ is different from zero for all $(x, y) \in \mathbb{R}^2$. We provide some new sufficient conditions for the injectivity of $F$. The proofs are based on the qualitative theory of differential equations.

## 1. Introduction and statement of the main result

Let $F = (f, g)\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth map such that $\det DF(x, y)$ is different from zero for all $(x, y) \in \mathbb{R}^2$. By the Inverse Function Theorem, it is clear that $F$ is a local diffeomorphism, but it is not always injective. There are very general well known additional conditions to ensure that $F$ is a global diffeomorphism, see for instance (Cobo et al. 2002, Fernandes et al. 2004, Plastock 1974).

If $F$ is a polynomial map, the statement that $F$ is injective is known as the *real Jacobian conjecture*. This conjecture is false, because Pinchuk constructed, in (Pinchuk 1994), a non-injective polynomial map with nonvanishing Jacobian determinant. Thus it is natural to ask for additional conditions in order that this conjecture holds. In (Braun et al. 2006, Braun et al. 2010), for instance, it was shown that for the injectivity of $F$ it is enough to assume that the degree of $f$ is less than or equal to 4. If we assume that $\det DF(x)$ is a constant different from zero, then to know if $F$ is injective is an open problem largely known as the *Jacobian conjecture* (see (Bass et al. 1982) and (Jiang 2005) for

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details and for surveys on the Jacobian conjecture and related problems). In (Braun et al. 2016) the authors provide a sufficient condition for the validity of the real Jacobian conjecture. More precisely they proved the following theorem.

**Theorem 1.** *Let $F = (f, g)\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial map such that $\det DF$ is nowhere zero. If the higher homogeneous terms of the polynomials $ff_x + gg_x$ and $ff_y + gg_y$ have no linear factors in common, then $F$ is injective.*

The next example shows that the sufficient condition provided by Theorem 1 are not necessary.

**Example 1.** *Let $F = (f, g)\colon \mathbb{R}^2 \to \mathbb{R}^2$ with $f(x, y) = x^3 + y^3 + x$ and $g(x, y) = y$. Here $\det DF(x, y) = 3x^2 + 1$. Note that*

$$\begin{aligned} ff_x + gg_x &= 3x^2(x^3 + y^3) + 4x^3 + y^3 + x, \\ ff_y + gg_y &= 3y^2(x^3 + y^3) + 3xy^3 + y \end{aligned}$$

*whose higher homogeneous terms have the factor $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$ in common. Clearly the map $F$ is injective.*

The purpose of this paper is to provide new sufficient conditions for the validity of the real Jacobian conjecture in the case in which the higher homogeneous terms of either the polynomials $f$ and $g$ or of $ff_x+gg_x$ and $ff_y+gg_y$ have real linear factors in common of multiplicity one.

In all the paper we will denote by $G_k$ homogeneous polynomials of degree $k$.

**Theorem 2.** *Let $F = (f, g)\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial map such that $\det DF$ is nowhere zero, $F(0, 0) = (0, 0)$ and $m = \deg(f) \geq n = \deg(g)$. Assume that*

- (i) *the higher homogeneous terms of the polynomials $ff_x + gg_x$ and $ff_y+gg_y$ have real linear factors in common all with multiplicity one;*
- (ii) *if $ax + by$ is a real factor in common of the higher homogeneous part of $ff_x + gg_x$ and of $ff_y + gg_y$, then*
  - (ii.1) *if $n = m$, either $f_{m-1}(1, -a/b) = g_{m-1}(1, -a/b) = 0$, or $(f_{m-1}^2 + g_{m-1}^2)(1, -a/b) \neq 0$ and*

$$(g_{m-1}R_{m-1} - f_{m-1}S_{m-1})(1, -a/b) \neq 0,$$

*where* $R_{m-1} = R_{m-1}(x, y)$ *and* $S_{m-1} = S_{m-1}(x, y)$ *are the polynomials defined by*

$$R_{m-1} = \frac{f_m}{ax + by} \quad \text{and} \quad S_{m-1} = \frac{g_m}{ax + by};$$

(ii.2) *if* $n < m - 1$, *then* $f_{m-1}(1, -a/b) = 0$.

*Then* $F$ *is injective.*

Note that when $n = m - 1$ condition (ii) in Theorem 2 is empty. We recall that Example 1 satisfies the assumptions of Theorem 2 (here $f_2 = 0$, $m = 3$ and $n = 1$).

The proof of Theorem 2 is given in Section 3.

Unfortunately when the functions $f$ and $g$ are not polynomials, then in general we cannot do the compactification of Poincaré and use the Poincaré disc, so the approach for proving Theorem 2 that we do for a polynomial map does not work in general for the non-polynomial map.

We stress that our approach is based in the approach in (Braun et al. 2015, Braun et al. 2016) and is different from the approach followed in (Cima et al. 1995, Cima et al. 1996). Indeed, our proofs rely only on the qualitative theory of ordinary differential equations, following ideas started by (Gavrilov 1997) and Sabatini in (Sabatini 1998), see also (Braun et al. 2016), while the proofs in (Cima et al. 1995, Cima et al. 1996) are based mainly on the algebraic structure of polynomial maps.

Another example when the degrees of $f$ and $g$ are the same is the following.

**Example 2.** *Let* $F = (f, g)\colon \mathbb{R}^2 \to \mathbb{R}^2$ *with* $f(x, y) = x^3 + y^3 + x$ *and* $g(x, y) = x^3 + y^3 + y$. *Here* $\det DF(x, y) = 3(x^2 + y^2) + 1$. *Note that the higher homogeneous terms of* $ff_x + gg_x$ *and* $ff_y + gg_y$ *have the factor* $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$ *in common. But from (Braun et al. 2006) we know that the map* $F$ *is injective.*

The following example shows that the sufficient conditions in Theorem 2 are not necessary.

**Example 3.** *Let* $F = (f, g)\colon \mathbb{R}^2 \to \mathbb{R}^2$ *with* $f(x, y) = x + y^n$ *with* $n \geq 1$ *and* $g(x, y) = y$. *Here* $\det DF(x, y) = 1$. *Note that the higher homogeneous terms of* $ff_x + gg_x$ *and of* $ff_y + gg_y$ *are* $y^n$ *and* $ny^{2n-1}$, *respectively. Then they have the linear common factor* $y^n$ *with multiplicity* $n \geq 1$. *However, the map* $F$ *is clearly injective.*

## 2. Preliminary results

A singular point $q$ of a vector field defined in $\mathbb{R}^2$ is a *center* if it has a neighborhood filled of periodic orbits with the unique exception of $q$. The *period annulus* of the center $q$ is the maximal neighborhood $P$ of $q$ such that all the orbits contained in $P$ are periodic except, of course the point $q$. A center is *global* if its period annulus is the whole $\mathbb{R}^2$.

The next result due to (Gravilov 1997) or (Sabatini 1998) (see an extension of it in (Braun et al. 2016)), will play a main role in the proof of Theorem 2.

**Theorem 3.** *Let $F = (f, g) \colon \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial map with nowhere zero Jacobian determinant such that $F(0,0) = (0,0)$. Then the following statements are equivalent.*

(a) *The origin is a global center for the polynomial vector field $X = (-ff_y - gg_y, ff_x + gg_x)$.*
(b) *$F$ is a global diffeomorphism of the plane onto itself.*

Let $\mathcal{X}$ be a planar polynomial vector field of degree $n$ and

$$\mathbb{S}^2 = \{y = (y_1, y_2, y_3) \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\}$$

(the *Poincaré sphere*). The *Poincaré compactification* of $\mathcal{X}$, denoted by $p(\mathcal{X})$, is an induced vector field on $\mathbb{S}^2$ defined as follows. For more details see Chapter 5 of (Dumortier et al. 2006).

Denote by $T_y\mathbb{S}^2$ the tangent space to $\mathbb{S}^2$ at the point $y$. Assume that $\mathcal{X}$ is defined in the plane $T_{(0,0,1)}\mathbb{S}^2 \equiv \mathbb{R}^2$. Consider the *central projection* $f \colon T_{(0,0,1)}\mathbb{S}^2 \to \mathbb{S}^2$, i.e. we associate to each point $q \in T_{(0,0,1)}\mathbb{S}^2 = \mathbb{R}^2$ the two points on $\mathbb{S}^2$ which are in the intersection of the straight line through $q$ and $(0,0,0) \in \mathbb{R}^3$ with the sphere $\mathbb{S}^2$. This map defines two copies of $\mathcal{X}$, one in the open northern hemisphere of $\mathbb{S}^2$ and other in the open southern hemisphere. Denote by $\mathcal{X}_1$ the vector field $Df \circ \mathcal{X}$ defined on $\mathbb{S}^2$ except on its equator $\mathbb{S}^1 = \{y \in \mathbb{S}^2 : y_3 = 0\}$. Clearly $\mathbb{S}^1$ is identified to the *infinity* of $\mathbb{R}^2$. In order to extend $\mathcal{X}_1$ to a vector field on $\mathbb{S}^2$ (including $\mathbb{S}^1$) it is necessary that $\mathcal{X}$ satisfies suitable conditions. In the case that $\mathcal{X}$ is a planar polynomial vector field of degree $n$, then $p(\mathcal{X})$ is the only analytic extension of $y_3^{n-1}\mathcal{X}_1$ to $\mathbb{S}^2$. Knowing the behavior of $p(\mathcal{X})$ around $\mathbb{S}^1$, we know the behavior of $\mathcal{X}$ at infinity. The Poincaré compactification has the property that $\mathbb{S}^1$ is invariant under the flow of $p(\mathcal{X})$.

The singular points of $\mathcal{X}$ are called the *finite singular points* of $\mathcal{X}$ or of $p(\mathcal{X})$, while the singular points of $p(\mathcal{X})$ contained in $\mathbb{S}^1$, i.e. at infinity, are called the *infinite singular points* of $\mathcal{X}$ or of $p(\mathcal{X})$. It is known that the infinity singular points appear in pairs diametrically opposed. The orthogonal projection $\pi(y_1, y_2, y_3) = (y_1, y_2)$ sends the closed northern hemisphere to the *Poincaré disc* $\mathbb{D}^2 = \{(y_1, y_2, y_3) \in \mathbb{R}^3 : y_1^2 + y_2^2 \leq 1 \text{ and } y_3 \neq 0\}$.

For working with the vector field $p(\mathcal{X})$ on the Poincaré sphere $\mathbb{S}^2$ we shall use the six local charts $U_k = \{x \in \mathbb{S}^2 : y_k > 0\}$ and $V_k = \{x \in \mathbb{S}^2 : y_k < 0\}$ for $k = 1, 2, 3$, while the corresponding diffeomorphisms $\phi_k \colon U_k \to \mathbb{R}^2$ and $\psi_k \colon V_k \to \mathbb{R}^2$ given by $\phi_k(y) = -\psi_k(y) = (y_m/y_k, y_n/y_k) = (u, v)$ for $m < n$ and $m, n \neq k$. The points of $\mathbb{S}^1$ in any chart have its $v$-coordinate equal to zero.

The expression of $p(\mathcal{X})$ in the local chart $(U_1, \phi_1)$ is

$$\begin{aligned} \dot{u} &= v^d \Big[ -uP\Big(\frac{1}{v}, \frac{u}{v}\Big) + Q\Big(\frac{1}{v}, \frac{u}{v}\Big)\Big], \\ \dot{v} &= -v^{d+1} P\Big(\frac{1}{v}, \frac{u}{v}\Big), \end{aligned}$$

if $\mathcal{X} = (P, Q)$ and $d$ is the maximum degree of $P$ and $Q$. The expression for $(U_2, \phi_2)$ is

$$\begin{aligned} \dot{u} &= v^d \Big[ P\Big(\frac{u}{v}, \frac{1}{v}\Big) - uQ\Big(\frac{u}{v}, \frac{1}{v}\Big)\Big], \\ \dot{v} &= -v^{d+1} Q\Big(\frac{u}{v}, \frac{1}{v}\Big), \end{aligned}$$

and for $(U_3, \phi_3)$ is

$$\dot{u} = P(u, v), \quad \dot{v} = Q(u, v).$$

The expression of $p(\mathcal{X})$ in the local chart $(V_k, \psi_k)$ is the same as for the chart $(U_k, \phi_k)$ multiplied by $(-1)^{d+1}$ for $k = 1, 2, 3$. For more details on the Poincaré compactification see Chapter 5 of (Dumortier et al. 2006).

The next result is the Poincaré-Hopf Theorem for the Poincaré compactification of the polynomial vector field. For a proof see Theorem 6.30 of (Dumortier et al. 2006).

**Theorem 4.** *Let $\mathcal{X}$ be a polynomial vector field. If $p(\mathcal{X})$ defined on the Poincaré sphere $\mathbb{S}^2$ has finitely many singular points, then the sum of their topological indices is two.*

We end this section with the following lemma.

**Lemma 5.** *Let $F = (f, g) \colon \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial map such that $\det DF$ is nowhere zero. Assume that $\deg(f) = \deg(g) = m$ and that*

*$f_m$ and $g_m$ have a real linear factor in common of multiplicity one, then the higher homogeneous terms of $ff_x + gg_x$ and $ff_y + gg_y$ also have the same real linear factor in common of multiplicity one.*

*Proof.* Assume that $f_m$ and $g_m$ have a real linear factor $ax + by$ in common of multiplicity one. We first observe that since $m > 1$, $(f_m^2 + g_m^2)_x \not\equiv 0$ and $(f_m^2 + g_m^2)_y \not\equiv 0$, because if for instance $(f_m^2 + g_m^2)_x = 0$, then $f_m = a_{0m} y^m$ and $g_m = b_{0m} y^m$ for $a_{0m}$ and $b_{0m} \in \mathbb{R}$, a contradiction. Thus the homogeneous parts of higher degree of $ff_x + gg_x$ and $ff_y + gg_y$ are $(f_m^2 + g_m^2)_x/2$ and $(f_m^2 + g_m^2)_y/2$ respectively. Then clearly, $(f_m^2 + g_m^2)_x$ and $(f_m^2 + g_m^2)_y$ have $ax + by$ as a real linear factor in common. We will see that it has also multiplicity one. Assume on the contrary that $ax + by$ is a factor of multiplicity $\ell > 1$ dividing the polynomials $(f_m^2 + g_m^2)_x$ and $(f_m^2 + g_m^2)_y$. So,

$$\partial H_{2m}/\partial x = (f_m^2 + g_m^2)_x/2 = (ax + by)^{\ell} T(x, y),$$

where $T(x, y)$ is a homogeneous polynomial not divisible by $ax + by$. Then, since they are polynomials, integrating in $x$ we get

$$H_{2m} = \overline{H}_{2m}(y) + (ax + by)^{\ell+1} T_1(x, y)$$

where $T_1(x, y)$ is not divisible by $ax + by$. Now using that

$$\partial H_{2m}/\partial y = (f_m^2 + g_m^2)_y/2 = (ax + by)^{\ell} S(x, y),$$

where $S(x, y)$ is a homogeneous polynomial not divisible by $ax + by$ we get that $\overline{H}_{2m}(y) = 0$, and

$$H_{2m} = (ax + by)^{\ell+1} T_1(x, y).$$

Since $H_{2m} = (f_m^2 + g_m^2)/2$ we conclude that $ax + by$ is a common factor of $f_m$ and $g_m$ of multiplicity $\ell > 1$, which is not possible. This concludes the proof of the lemma. □

## 3. Proof of Theorem 2

The following results are the main tools for proving Theorem 2. We recall that we say that a hyperbolic sector $h$ of an infinite singular point $q$ is *degenerated* if its two separatrices are contained in the equator of $\mathbb{S}^2$ (i.e. in $\mathbb{S}^1$). The proof of the two first lemmas is inspired in the proof of Theorem 2.2 of (Cima et al. 1993).

**Lemma 6.** *Let $q$ be an infinite singular point of a Hamiltonian system $\mathcal{X} = (P, Q) = (-H_y, H_x)$ such that it has some non-degenerated hyperbolic sector $h$. Then $q$ is an endpoint of the straight line $ax + by = 0$ at* *infinity in the Poincaré disc, where $ax+by$ is a linear common factor of the higher homogeneous points of $H_y$ and $H_x$.*

*Proof.* We denote by $m$ the degree of $P=-H_y$ and by $n$ the degree of $Q=H_x$. Without loss of generality we can assume that $m \geq n$, and that $q$ is the origin of the local chart $U_1$. Then by Theorem 2.2 (ii) of (Cima et al. 1993) the two separatrices $s_1$ and $s_2$ of the non-degenerated hyperbolic sector $h$ are tangent to the same direction $y=\lambda$. Hence, since the Hamiltonian $H$ takes the same value on the two separatrices $s_1$ and $s_2$ we can write $H_{m+1}=(y-\lambda)^2\tilde{H}_{m-1}(x,y)$, with the degree of $\tilde{H}$ equal to $m-1$. Therefore, the maximum degree of $Q=H_x$ in the variable $x$ is $m-2$, and the maximum degree of $P=-H_y$ in the variable $x$ is $m-1$. Hence, if $Q_m=H_{m+1,x}=\sum_{i=0}^{m} a_i x^{m-i}y^i$, then $a_0=a_1=0$, and if $P_n=-H_{m+1,y}=\sum_{i=0}^{n} b_i x^{m-i}y^i$, then $b_0=0$. Consequently, $Q_m=y^2\overline{Q}_{m-2}$ and $P_n=y\overline{P}_{n-1}$, where $\overline{Q}_{m-2}$ and $\overline{P}_{n-1}$ homogeneous polynomials of degrees $m-2$ and $n-1$, respectively. This completes the proof of the lemma. □

**Lemma 7.** *Let $q$ be an infinite singular point of a Hamiltonian system $\mathcal{X}=(P,Q)=(-H_y,H_x)$ such that it is not the endpoint of any straight line $ax+by=0$ at infinity in the Poincaré disc where $ax+by$ is a common linear factor of the higher homogeneous terms of $H_y$ and $H_x$. Then the topological index of $q$ is greater or equal zero, and when it is zero it is formed by two degenerated hyperbolic sectors.*

*Proof.* In view of Lemma 6 if we denote by $h$ the number of hyperbolic sectors of $q$, then any hyperbolic sector of $q$ must be degenerated and so $h \leq 2$. Since the index of $q$ is equal to $1+(e-h)/2$ where $e$ and $h$ are the number of elliptic and hyperbolic sectors at the local phase portrait of $q$ (see Proposition 6.32 of (Dumortier 2006)), it follows that the index of $q$ is greater than or equal to zero. It is zero when $e=0$ and $h=2$ which clearly implies that $q$ is formed by two degenerated hyperbolic sectors and the lemma follows. □

The next result is the more important of this paper, because using it Theorem 2 will follow easily.

**Theorem 8.** *Under the assumptions of Theorem 2, let $q$ be an infinite singular point of the polynomial Hamiltonian vector field $\mathcal{X}=(-H_y,H_x)$ with $H=(f^2+g^2)/2$. The topological index of $q$ is greater than or equal to 0, and when it is 0 the singular point $q$ in the Poincaré sphere is formed by two degenerate hyperbolic sectors.*

*Proof.* Assume first that $q$ is an infinite singular point which is not an endpoint of any straight line $ax+by=0$, where $ax+by$ is a linear common factor of the higher homogeneous terms of $ff_x+gg_x$ and $ff_y+gg_y$. Then it follows from Lemma 7 that the topological index of $q$ is greater than or equal to 0, and when it is 0 the singular point $q$ in the Poincaré sphere is formed by two degenerate hyperbolic sectors.

Now assume that $q$ is an infinite singular point which is an endpoint of a straight line $ax+by=0$, where $ax+by$ is a linear common factor of the higher homogeneous terms of $ff_x+gg_x$ and $ff_y+gg_y$. Since $H=(f^2+g^2)/2$, the homogeneous terms of higher degree of $H$ are $H_{2m}$. Proceeding as in the proof of Lemma 5 we get that $H_{2m}=(ax+by)^2\overline{H}_{2m-2}(x,y)$ where $\overline{H}_{2m-2}(x,y)$ is a homogeneous polynomial of degree $2m-2$. If we write $H(x,y)=\sum_{k=0}^{2m}H_j(x,y)$ where each $H_j$ is a homogeneous polynomial of degree $j$, then it is clear that

$$\begin{aligned} H_{2m}(x,y) &= f_m^2/2+g_p^2/2,\\ H_{2m-1}(x,y) &= f_mf_{m-1}+g_pg_{p-1},\\ H_{2m-2}(x,y) &= f_{m-1}^2/2+f_mf_{m-2}+g_{p-1}^2/2+g_pg_{p-2}, \end{aligned}$$

where

$$g_p=\begin{cases} g_m & \text{if } n=m,\\ 0 & \text{if } n<m \end{cases},\qquad g_{p-1}=\begin{cases} g_{m-1} & \text{if } n=m,\\ g_n & \text{if } n=m-1,\\ 0 & \text{if } n<m-1 \end{cases}$$

and

$$g_{p-2}=\begin{cases} g_{m-2} & \text{if } n=m,\\ 0 & \text{if } n<m. \end{cases}$$

We can assume that $b\neq 0$ (otherwise we can interchange $x$ with $y$) and so we can write

$$\begin{aligned} H_{2m}(x,y) &= (ax+by)^2\Big[R_{m-1}^2(x,y)+S_{p-1}^2(x,y)\Big]/2,\\ \tilde{H}_{2m}(X,Y) &= H_{2m}(x,y)\Big|_{\substack{y=(Y-aX)/b\\ x=X}}\\ &= AX^{2m-2}Y^2+O(Y^3), \end{aligned}$$

with $AX^{2m-2}=A_1X^{2m-2}+A_2X^{2m-2}$, where

$$A_1X^{2m-2}=R_{m-1}^2(X,-aX/b)/2 \quad\text{and}\quad A_2X^{2m-2}=S_{p-1}^2(X,-aX/b)/2.$$

Since $ax + by$ does not divide $\overline{H}(x,y)$, then $A \neq 0$. Here

$$S_{p-1}(x,y) = \begin{cases} 0 & \text{if } n < m \\ S_{m-1}(x,y) & \text{if } n = m. \end{cases}$$

Moreover,

$$\begin{aligned} H_{2m-1}(x,y) &= (ax+by)\Big[R_{m-1}(x,y)f_{m-1}(x,y) + S_{p-1}(x,y)g_{p-1}(x,y)\Big], \\ \tilde{H}_{2m-1}(X,Y) &= H_{2m-1}(x,y)\Big|_{\substack{y=(Y-aX)/b \\ x=X}} \\ &= BX^{2m-2}Y + O(Y^2), \end{aligned}$$

where $BX^{2m-2} = (R_{m-1}f_{m-1} + S_{p-1}g_{p-1})(X, -aX/b)$. Finally,

$$\begin{aligned} H_{2m-2}(x,y) &= f_{m-1}^2(x,y)/2 + (ax+by)R_{m-1}(x,y)f_{m-2}(x,y) \\ &\quad + g_{p-1}^2(x,y)/2 + (ax+by)S_{p-1}(x,y)g_{p-2}(x,y), \\ \tilde{H}_{2m-2}(X,Y) &= H_{2m-2}(x,y)\Big|_{\substack{y=(Y-aX)/b \\ x=X}} \\ &= CX^{2m-2} + O(Y), \end{aligned}$$

where $CX^{2m-2} = (f_{m-1}^2 + g_{p-1}^2)(X, -aX/b)/2$.

The expression corresponding to the vector field on the local chart $U_1$ is

(1)
$$\begin{aligned} \dot{u} &= v^{2m-1}\Big[uH_y\Big(\frac{1}{v},\frac{u}{v}\Big) + H_x\Big(\frac{1}{v},\frac{u}{v}\Big)\Big] \\ &= 2mH_{2m}(1,u) + (2m-1)vH_{2m-1}(1,u) + (2m-2)v^2H_{2m-2}(1,u) \\ &\quad + O_3(v), \\ \dot{v} &= v^{2m+1}H_y\Big(\frac{1}{v},\frac{u}{v}\Big) = vH_{2m,y}(1,u) + v^2H_{2m-1,y}(1,u) + O_3(v), \end{aligned}$$

where $H_{j,y}$ denotes the derivative of the homogeneous polynomial $H_j$ with respect to the variable $y$.

Now we do the change of variables $u \to U$ given by $U = a + bu$ and system (1) becomes

(2)

$$\begin{aligned}\dot{U} &= 2bm\tilde{H}_{2m}(1,U) + b(2m-1)v\tilde{H}_{2m-1}(1,U) + 2b(m-1)v^2\tilde{H}_{2m-2}(1,U)\\ &\quad + O_3(v)\\ &= 2bmAU^2 + b(2m-1)BUv + 2b(m-1)Cv^2 + O_3(U,v),\\ \dot{v} &= bv\tilde{H}_{2m,U} + bv^2\tilde{H}_{2m-1,U} + O_3(v)\\ &= 2bAUv + bBv^2 + O_3(U,v).\end{aligned}$$

We consider different cases.

*Case 1: $n = m$ and $f_{m-1}(1,-a/b) = g_{m-1}(1,-a/b) = 0$.* In this case $B = C = 0$ and system (2) becomes, after a reparameterization of the time $t = \tau/(2bA)$,

$$\begin{aligned}U' &= mU^2 + O_3(U,v),\\ v' &= Uv + O_3(U,v),\end{aligned} \tag{3}$$

where the prime denotes derivative in $\tau$. Note that $U = v = 0$ is a singular point which is linearly zero. We need to make a blow up. Doing the blow up $v = Uz$, and then simplifying a common factor $U$ (by a reparameterization of the time) we get the system

$$\begin{aligned}U' &= U(m + O_1(U))),\\ z' &= Z((1-m) + O_1(U)).\end{aligned} \tag{4}$$

Note that on $U = 0$ the unique singular point of (4) is the origin $(U,z) = (0,0)$ that is a saddle because the Jacobian matrix at this point has eigenvalues $(m, 1-m)$ and $m > 1$. Moreover, the saddle has the separatrices at $U = 0$ and at $z = 0$. Going back through the rescaling and the blow up we have for system (3) that

(i) the $U$-axis is invariant. On the negative half $U$-axis an orbit enters the origin, and on the positive half $U$-axis an orbit exist from the origin; and

(ii) if some other orbit enters or exits the origin, it must be tangent to the $v$-axis.

The local phase portraits at the origin of an analytic differential system of the form

$$\begin{aligned}\dot{x} &= P_2(x,y) + P_3(x,y) + \cdots,\\ \dot{y} &= Q_2(x,y) + Q_3(x,y) + \cdots,\end{aligned} \tag{5}$$

with $P_2^2 + Q_2^2 \not\equiv 0$, and $P_k$, $Q_k$ homogeneous polynomials of degree $k$, have been classified in (Jiang et al. 2005), taking into account the directions that the orbits can enter or exit the origin. Looking at the 65 possible local phase portraits at the origin of system (5), there are only three possible local phase portraits satisfying (i) and (ii), which are the phase portraits 2, 7 and 31 of Figure 3 of (Jiang et al. 2005). But looking at the quadratic homogeneous parts of systems (5) realizing the local phase portraits 2 and 31, they are different from the quadratic homogenous part of system (3). The local phase portrait 7 is formed by two degenerated hyperbolic sectors. This completes the proof of the case 1.

*Case 2: $n = m$, $(f_{m-1}^2 + g_{m-1}^2)(1, -a/b) \neq 0$ and $(f_{m-1}S_{m-1})(1, -a/b) \neq (g_{m-1}R_{m-1})(1, -a/b)$.* In this case we get that $C \neq 0$. Moreover, system (2) becomes, after a reparameterization of the time $t = \tau/b$,

$$\begin{aligned} U' &= 2mAU^2 + (2m-1)BUv + 2(m-1)Cv^2 + O_3(U, v), \\ V' &= 2AUv + Bv^2 + O_3(U, v), \end{aligned}$$

where the prime denotes derivative in $\tau$. Note that $U = v = 0$ is a singular point which is linearly zero. We need to do a blow up. Doing the blow up $v = Uz$, and then simplifying by a common factor $U$ (by a reparameterization of the time) we get the system

$$\begin{aligned} U' &= U(2Am + B(2m-1)z + 2C(m-1)z^2 + O_1(U)), \\ z' &= 2z(1-m)(A + Bz + Cz^2 + O_1(U)) \end{aligned} \tag{6}$$

where

$$\begin{aligned} A &= (R_{m-1}^2 + S_{m-1}^2)(1, -a/b)/2, \\ B &= (R_{m-1}f_{m-1} + S_{m-1}g_{m-1})(1, -a/b), \\ C &= (f_{m-1}^2 + g_{m-1}^2)(1, -a/b)/2. \end{aligned}$$

Note that on $U = 0$ the singular points of (6) are on

$$z = 0, \quad z_{\pm} = -\frac{B \pm \sqrt{-\big((R_{m-1}g_{m-1} - S_{m-1}f_{m-1})(1, -a/b)\big)^2}}{2C}.$$

In view of the assumptions we have that $(R_{m-1}g_{m-1} - S_{m-1}f_{m-1})(1, -a/b) \neq 0$. So, the unique singular point on $U = 0$ is the origin $(0, 0)$ which is a saddle because the Jacobian matrix at this point has eigenvalues $Am$ and $2(1-m)A$ and $m > 1$ (we recall that $A > 0$). Moreover, the saddle has the separatrices at $U = 0$ and at $z = 0$. The rest of the proof in this case follows as in Case 1.

*Case 3: $n < m-1$ and $f_{m-1}(1,-a/b) = 0$.* Taking also into account that $g_p = g_{p-1} = 0$ we get that $B = C = 0$, and system (2) becomes, after a reparameterization of the time $t = \tau/(2bA)$, the system (3). Hence, this case follows as in Case 1.

*Case 4: $n = m-1$.* In this case $g_p = 0$ and $g_{p-1} = g_n \neq 0$. Again, $U = v = 0$ is a singular point which is linearly zero. Doing the blow up $u = Uz$ and simplifying by a factor $U$ we get the system

$$\begin{aligned} U' &= U(2Am + B(2m-1)z + 2C(m-1)z^2 + O_1(U)), \\ z' &= 2z(1-m)(A + Bz + Cz^2 + O_1(U)), \end{aligned}$$

where

$$\begin{aligned} A &= R_{m-1}^2(1,-a/b)/2, \\ B &= (R_{m-1}f_{m-1})(1,-a/b), \\ C &= (f_{m-1}^2 + g_n^2)(1,-a/b)/2. \end{aligned}$$

The singular points on $U = 0$ are $(0,0)$ and $(0, z_\pm)$ with

$$z_\pm = -\Big(\frac{R_{m-1}}{f_{m-1} \pm i g_n}\Big)(1,-a/b).$$

Since $R_{m-1}g_n \neq 0$ we get that both $z_\pm$ are complex and so the unique singular point is the origin which is a saddle because the Jacobian matrix at this point has eigenvalues $2mA$ and $2(1-m)A$ with $A > 0$ and $m > 1$. Proceeding as we did in Case 1, we conclude that after doing the blowing down the point is formed by two degenerated hyperbolic sectors. This concludes the proof of the theorem. □

*Proof of Theorem 2.* Without loss of generality we can assume that $F(0,0) = (0,0)$. Indeed, we denote $(a_1,a_2) = F(0,0)$ and consider the translation $A(x,y) = (x-a_1, y-a_2)$. Taking the map $G = A \circ F$, we observe that $G(0,0) = (0,0)$ and $\det DG$ is nowhere zero, the degrees of the components of $G$ are the same than the degrees of the components of $F$, and the assumption of Theorem 2 still holds for $G$, because the higher order terms of $F$ and $G$ coincide. Moreover, $F$ is injective if and only if $G$ is injective. In what follows we will assume $F = G$.

We consider the function $H\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$H(x,y) = \frac{1}{2}(f(x,y)^2 + g(x,y)^2)$$

and its associated Hamiltonian vector field $\mathcal{X} = (P,Q)$, that is, $P = -H_y = -ff_y - gg_y$ and $Q = H_x = ff_x + gg_x$. We claim that each finite singular point of $\mathcal{X}$ is a center, and consequently it has index 1. Indeed, $q \in \mathbb{R}^2$ is a singular point of $\mathcal{X}$ if and only if

$$\begin{pmatrix} f_x(q) & g_x(q) \\ f_y(q) & g_y(q) \end{pmatrix} \begin{pmatrix} f(q) \\ g(q) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which gives that $f(q) = g(q) = 0$, because $\det DF(q) \neq 0$. Let $V$ be a neighborhood of $q$ in which $F$ is injective. We have that $H$ is positive in all the points of $V$ different from $q$, while $H(q) = 0$, which proves that $q$ is an isolated minimum of $H$. Then all the orbits of $\mathcal{X}$ in a neighborhood of $q$ (maybe smaller than the neighborhood $V$) are closed, proving that $q$ is a center of $\mathcal{X}$. By Theorem 3, as $F(0,0) = (0,0)$, in order to prove Theorem 2 it is enough to prove that $(0,0)$ is a global center of the vector field $\mathcal{X}$.

From Theorem 8, the index of every infinite singular point of $\mathcal{X}$ is greater than or equal to 0. Moreover, the index of every finite singular of $\mathcal{X}$ is equal to 1. Since the points $(0,0,1)$ and $(0,0,-1)$ of the Poincaré sphere are finite singular points of $p(\mathcal{X})$ (corresponding to the singular point $(0,0)$ of $\mathcal{X}$), each of them with index 1, it follows from Theorems 4 that $p(\mathcal{X})$ does not have other finite singular points, and every infinite singular point of $p(\mathcal{X})$ has index 0. Therefore, it follows from Theorem 8 that all the infinite singular points are formed by two degenerate hyperbolic sectors.

Now we will prove that the boundary of the period annulus $\mathcal{P}$ of the center of $p(\mathcal{X})$ located at $(0,0,1)$ is the equator $\mathbb{S}^1$. This of course will show that the center $(0,0)$ of $\mathcal{X}$ is global and from Theorem 3 the map $F$ is injective. Since there are no finite singular points in the northern hemisphere of $\mathbb{S}^2$, except the center at $(0,0,1)$, and all the infinite singular points are formed by two degenerate hyperbolic sectors, it follows that the boundary of the period annulus $P$ is either a finite periodic orbit $\gamma$ or it is $\mathbb{S}^1$.

If it is $\mathbb{S}^1$ we are done. If not, we consider the Poincaré map $\pi$ defined in a transversal section $S$ through $\gamma$. Since the vector field $p(\mathcal{X})$ is analytic, it follows that $\pi$ is also analytic, this is due to the fact that the flow of an analytic system is analytic (see for instance [13]). Since $\pi$ is the identity map in $\mathbb{S}^1 \cap \mathcal{P}$, it is also the identity map in a neighborhood of $\gamma$. But then the orbits in this neighborhood are also periodic, and $\gamma$ is not the boundary of $\mathcal{P}$, a contradiction. This completes the proof of Theorem 2. $\square$

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