Chaos, Solitons and Fractals 148 (2021) 111050

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

Phase portraits of the complex Abel polynomial differential systems

Jaume Llibre [a], Claudia Valls [b,*]

[a] Departament de Matemàtiques, Universitat Autònoma de Barcelona, Barcelona, 08193 Bellaterra, Catalonia, Spain
[b] Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais Lisboa, 1049–001, Portugal

ARTICLE INFO

Article history:
Received 30 January 2021
Revised 13 April 2021
Accepted 7 May 2021

MSC:
Primary 34A05
Secondary 34C05
37C10

Keywords:
Complex Abel system
Poincaré compactification
Dynamics at infinity

ABSTRACT

In this paper we characterize the phase portraits of the complex Abel polynomial differential equations

$\dot{z} = (z-a)(z-b)(z-c),$

with $z \in \mathbb{C}$, $a, b, c \in \mathbb{C}$. We give the complete description of their topological phase portraits in the Poincaré disc, i.e. in the compactification of $\mathbb{R}^2$ adding the circle $\mathbb{S}^1$ of the infinity.

1. Introduction and statement of the main results

Numerous problems of applied mathematics, or in physics, chemistry, economics,...are modeled by polynomial differential systems. Excluding linear systems the quadratic polynomial differential systems are the ones with the lowest degree of complexity, and the large bibliography on them proves their relevance, see the books [1,15,16] and the surveys [3,4]. After the quadratic polynomial differential systems come the cubic ones, which also have many applications. Among the cubic polynomial differential systems we emphasize the Abel systems, see for instance the papers [2,6,8,9,11] where the Abel systems are applied to modelize problems from Ecology, control theory for electrical circuits and cosmology, respectively.

In this paper we characterize the phase portraits of the complex Abel differential equations

$$\dot{z} = (z-a)(z-b)(z-c), \tag{1}$$

with $z \in \mathbb{C}$, $a, b, c \in \mathbb{C}$ and the dot means derivative with respect to the time $t \in \mathbb{R}$. We write $z = x + iy$, $a = a_1 + ia_2$, $b = b_1 + ib_2$, $c = c_1 + ic_2$, with $x, y \in \mathbb{R}$ and $a_i, b_i, c_i \in \mathbb{R}$ for $i = 1, 2$. The complex differential Eq. (1) becomes the real differential system

$$\begin{aligned}\dot{x} = & -a_1b_1c_1 + a_2b_2c_1 + a_2b_1c_2 + a_1b_2c_2 + (a_1b_1 - a_2b_2 + a_1c_1 + b_1c_1 \\ & -a_2c_2 - b_2c_2)x - (a_2b_1 + a_1b_2 + a_2c_1 + b_2c_1 + a_1c_2 + b_1c_2)y \\ & -(a_1+b_1+c_1)x^2 + 2(a_2+b_2+c_2)xy + (a_1+b_1+c_1)y^2 + x^3 - 3xy^2, \\ \dot{y} = & -a_2b_1c_1 - a_1b_2c_1 - a_1b_1c_2 + a_2b_2c_2 + (a_2b_1 + a_1b_2 + a_2c_1 + b_2c_1 \\ & +a_1c_2 + b_1c_2)x + (a_1b_1 - a_2b_2 + a_1c_1 + b_1c_1 - a_2c_2 - b_2c_2)y \\ & -(a_2+b_2+c_2)x^2 - 2(a_1+b_1+c_1)xy + (a_2+b_2+c_2)y^2 + 3x^2y - y^3.\end{aligned} \tag{2}$$

The objective of this work is to classify the phase portraits of the Abel polynomial differential systems (2) in the Poincaré disc modulo topological equivalence. As any polynomial differential system, system (2) can be extended to an analytic system on a closed disc $\mathbb{D}$ of radius one, whose interior is diffeomorphic to $\mathbb{R}^2$ and its boundary, the circle $\mathbb{S}^1$, plays the role of the infinity. This closed disc is denoted by $\mathbb{D}^2$ and called the *Poincaré disc*, because the technique for doing such an extension is the *Poincaré compactification* for a polynomial differential system in $\mathbb{R}^2$, which is described in details in Chapter 5 of [5], see also Section 2.1. In this paper we shall use the notation of that chapter. By using this compactification technique the dynamics of system (2) in a neighborhood of the infinity can be studied.

See Sectioin 2.2 for the definition of equivalent topological phase portraits, and for seeing that it is sufficient to draw the sep-

* Corresponding author.
E-mail addresses: jllibre@mat.uab.cat (J. Llibre), cvalls@math.tecnico.ulisboa.pt (C. Valls).

https://doi.org/10.1016/j.chaos.2021.111050
0960-0779/