PHASE PORTRAITS OF THE COMPLEX ABEL POLYNOMIAL DIFFERENTIAL SYSTEMS

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Abstract. In this paper we characterize the phase portraits of the complex Abel polynomial differential equations

$$\dot{z} = (z-a)(z-b)(z-c),$$

with $z \in \mathbb{C}$, $a, b, c \in \mathbb{C}$. We give the complete description of their phase portraits in the Poincaré disc (i.e. in the compactification of $\mathbb{R}^2$ adding the circle $\mathbb{S}^1$ of the infinity) modulo topological equivalence.

1. Introduction and statement of the main results

Numerous problems of applied mathematics, or in physics, chemistry, economics, ... are modeled by polynomial differential systems. Excluding linear systems the quadratic polynomial differential systems are the ones with the lowest degree of complexity, and the large bibliography on them proves their relevance, see the books [1, 12, 13] and the surveys [3, 4]. After the quadratic polynomial differential systems come the cubic ones, which also have many applications. Among the cubic polynomial differential systems we emphasize the Abel systems, see for instance the papers [2, 6, 8] where the Abel systems are applied to modelize problems from Ecology, control theory for electrical circuits and cosmology, respectively.

In this paper we characterize the phase portraits of the complex Abel differential equations

$$\dot{z} = (z-a)(z-b)(z-c), \tag{1}$$

with $z \in \mathbb{C}$, $a, b, c \in \mathbb{C}$ and the dot means derivative with respect to the time $t \in \mathbb{R}$. We write $z = x + iy$, $a = a_1 + ia_2$, $b = b_1 + ib_2$, $c = c_1 + ic_2$, with $x, y \in \mathbb{R}$ and $a_i, b_i, c_i \in \mathbb{R}$ for $i = 1, 2$. The complex differential equation (1) becomes the real differential system

$$\begin{aligned} \dot{x} &= -a_1b_1c_1 + a_2b_2c_1 + a_2b_1c_2 + a_1b_2c_2 + (a_1b_1 - a_2b_2 + a_1c_1 + b_1c_1 \\ &\quad - a_2c_2 - b_2c_2)x - (a_2b_1 + a_1b_2 + a_2c_1 + b_2c_1 + a_1c_2 + b_1c_2)y \\ &\quad - (a_1 + b_1 + c_1)x^2 + 2(a_2 + b_2 + c_2)xy + (a_1 + b_1 + c_1)y^2 + x^3 - 3xy^2, \\ \dot{y} &= -a_2b_1c_1 - a_1b_2c_1 - a_1b_1c_2 + a_2b_2c_2 + (a_2b_1 + a_1b_2 + a_2c_1 + b_2c_1 \\ &\quad + a_1c_2 + b_1c_2)x + (a_1b_1 - a_2b_2 + a_1c_1 + b_1c_1 - a_2c_2 - b_2c_2)y \\ &\quad - (a_2 + b_2 + c_2)x^2 - 2(a_1 + b_1 + c_1)xy + (a_2 + b_2 + c_2)y^2 + 3x^2y - y^3. \end{aligned} \tag{2}$$

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The objective of this work is to classify the phase portraits of the Abel polynomial differential systems (2) in the Poincaré disc modulo topological equivalence. As any polynomial differential system, system (2) can be extended to an analytic system on a closed disc $\mathbb{D}$ of radius one, whose interior is diffeomorphic to $\mathbb{R}^2$ and its boundary, the circle $\mathbb{S}^1$, plays the role of the infinity. This closed disc is denoted by $\mathbb{D}^2$ and called the *Poincaré disc*, because the technique for doing such an extension is the *Poincaré compactification* for a polynomial differential system in $\mathbb{R}^2$, which is described in details in Chapter 5 of [5], see also subsection 2.1. In this paper we shall use the notation of that chapter. By using this compactification technique the dynamics of system (2) in a neighborhood of the infinity can be studied.

See subsection 2.2 for the definition of equivalent topological phase portraits, and for seeing that it is sufficient to draw the separatrices in the Poincaré disc and an orbit in every canonical region for characterizing a phase portrait in the Poincaré disc.

Theorem 1. *There are only six different topologically phase portraits of the Abel differential system* (2) *in the Poincaré disc, see Figure 1.*

In Figure 1 the phase portrait
(a) is realized when $a_1 = a_2 = -1$, $b_1 = b_2 = 0$, and $c_1 = c_2 = -1/2$;
(b) is realized when $a_1 = a_2 = 0$, $b_1 = -3/2$, $b_2 = -3$, $c_1 = -2$, $c_2 = -1$;
(c) is realized when $a_1 = a_2 = b_1 = 0$, $b_2 = 1$, $c_1 = 0$, $c_2 = 2$;
(d) is realized when $a_1 = a_2 = 0$, $b_1 = 2$, $b_2 = -1$, $c_1 = -2$, $c_2 = -1$;
(e) is realized when $a_1 = a_2 = 0$, $b_1 = -8$, $b_2 = -6$, $c_1 = -3$, $c_2 = -1$;
(f) is realized when $a_1 = -1$, $a_2 = 0$, $b_1 = -1$, $b_2 = 0$, $c_1 = 0$, $c_2 = 1$;
(g) is realized when $a_1 = a_2 = b_1 = b_2 = 0$, $c_1 = -1$, $c_2 = -1/2$;
(h) is realized when $a_1 = a_2 = b_1 = b_2 = c_1 = 0$, $c_2 = -1$; and
(i) is realized when $a_1 = a_2 = b_1 = b_2 = c_1 = c_2 = 0$.

2. Preliminaries

2.1. **Poincaré compactification.** For a complete description of the Poincaré compactification method we refer to chapter 5 of [5]. In what follows we remember some formulas.

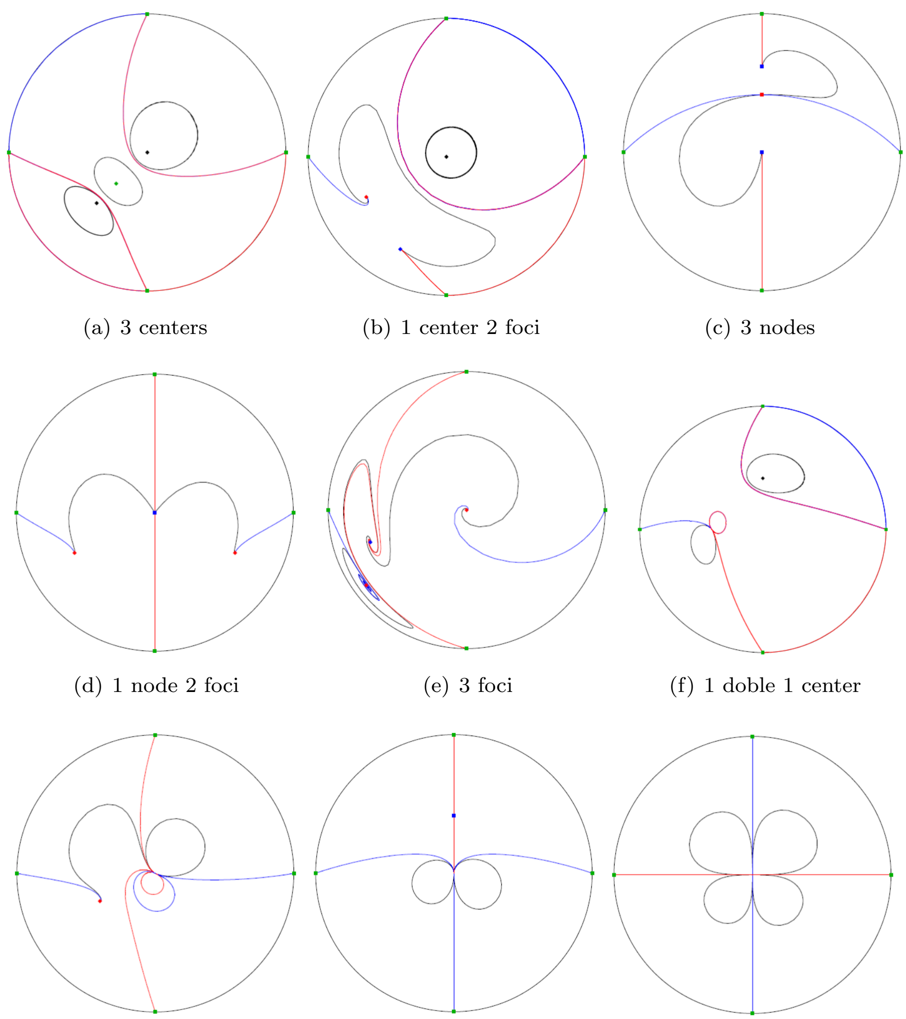
Consider a polynomial differential system in $\mathbb{R}^2$ with degree 3.

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y)$$

or equivalently its associated polynomial vector field $X = (P, Q)$. As we said before, any polynomial differential system can be extended to an analytic differential system on a closed disc of radius one centered at their origin of coordinates, whose interior is diffeomorphic to $\mathbb{R}^2$ and its boundary, the circle $\mathbb{S}^1$, plays the role of the infinity.

In order to study the compactified vector field we consider four open charts covering the disc $\mathbb{D}$:

$$\phi_1 : \mathbb{R}^2 \longrightarrow U_1, \quad \phi_1(x, y) = (1/v, u/v),$$
$$\phi_2 : \mathbb{R}^2 \longrightarrow U_2, \quad \phi_1(x, y) = (u/v, 1/v)$$



(a) 3 centers (b) 1 center 2 foci (c) 3 nodes

(d) 1 node 2 foci (e) 3 foci (f) 1 doble 1 center

(g) 1 doble 1 focus (h) 1 doble 1 node (i) 1 triple

Figure 1. The nine topological phase portraits in the Poincaré disc of system (2) if we distinguish between nodes and foci, without this distintion the phase portraits (c), (d) and (e) are topologically equivalent, and also the phase portraits (g) and (h) are topologically equivalent. So there are only six different topological phase portraits.

and

$$\psi_k : \mathbb{R}^2 \longrightarrow V_k, \quad \psi_k(x,y) = -\phi_k(x,y), \quad k = 1,2$$

with

$$\begin{aligned} U_1 &= \{(u,v) \in \mathbb{D} : u^2 + v^2 \leq 1 \quad \text{and} \quad u > 0\}, \\ U_2 &= \{(u,v) \in \mathbb{D} : u^2 + v^2 \leq 1 \quad \text{and} \quad v > 0\}, \\ V_1 &= \{(u,v) \in \mathbb{D} : u^2 + v^2 \leq 1 \quad \text{and} \quad u < 0\}, \\ V_2 &= \{(u,v) \in \mathbb{D} : u^2 + v^2 \leq 1 \quad \text{and} \quad v < 0\}. \end{aligned}$$

The Poincaré compactification is denoted by $p(X)$. The expression of $p(X)$ in the chart U_1 is

$$\dot{u} = v^3(-uP + Q), \quad \dot{v} = -v^4 P,$$

where P and Q are evaluated at $(1/v, u/v)$.

The expression of $p(X)$ in the chart U_2 is

$$\dot{u} = v^3(P - uQ), \quad \dot{v} = -v^4 Q,$$

where P and Q are evaluated at $(u/v, 1/v)$. Moreover in all these local charts the points (u, v) of the infinity have its coordinate $v = 0$.

The expression for the extend differential system in the local chart V_i, $i = 1, 2$ is the same as in U_i.

2.2. **Topological equivalence of two polynomial vector fields.** Let X_1 and X_2 be two polynomial vector fields on $\mathbb{R}^2$. We say that they are *topologically equivalent* if there exists a homeomorphism on the Poincaré disc $\mathbb{D}$ which preserves the infinity $\mathbb{S}^1$ and sends the orbits of $\pi(p(X_1))$ to orbits of $\pi(p(X_2))$, preserving or reversing the orientation of all the orbits.

A *separatrix* of the Poincaré compactification $\pi(p(X))$ is one of following orbits: all the orbits at the infinity $\mathbb{S}^1$, the finite singular points, the limit cycles, and the two orbits at the boundary of a hyperbolic sector at a finite or an infinite singular point, see for more details on the separatrices [9, 10].

The set of all separatrices of $\pi(p(X))$, which we denote by Σ_X, is a closed set (see [10]).

A *canonical region* of $\pi(p(X))$ is an open connected component of $\mathbb{D} \setminus \Sigma_X$. The union of the set Σ_X with an orbit of each canonical region form the *separatrix configuration* of $\pi(p(X))$ and is denoted by Σ'_X. We denote the number of separatrices of a phase portrait in the Poincaré disc by S, and its number of canonical regions by R.

Two separatrix configurations Σ'_{X_1} and Σ'_{X_2} are *topologically equivalent* if there is a homeomorphism $h : \mathbb{D} \longrightarrow \mathbb{D}$ such that $h(\Sigma'_{X_1}) = \Sigma'_{X_2}$.

According to the following theorem which was proved by Markus [9], Neumann [10] and Peixoto [11], it is sufficient to investigate the separatrix configuration of a polynomial differential system, for determining its global phase portrait.

Theorem 2. *Two Poincaré compactified polynomial vector fields $\pi(p(X_1))$ and $\pi(p(X_2))$ with finitely many separatrices are topologically equivalent if and only if their separatrix configurations Σ'_{X_1} and Σ'_{X_2} are topologically equivalent.*

3. Infinite singular points

The following lemma summarizes the information at the infinite singular points.

Lemma 3. *There are two pairs of infinite singular points which are saddles located at the origins of the local chats U_1, U_2, V_1 and V_2.*

Proof. First we determine the local phase portrait of the infinite singular points in the local chart U_1. The expression of system (2) in this chart is

$$\begin{aligned}\dot u &= 2u - (a_2+b_2+c_2)v - (a_1+b_1+c_1)uv + (a_2b_1+a_1b_2+a_2c_1+b_2c_1\\ &+ a_1c_2+b_1c_2)v^2 + 2u^3 - (a_2+b_2+c_2)u^2v - (a_2b_1c_1+a_1b_2c_1+a_1b_1c_2\\ &- a_2b_2c_2)v^3 - (a_1+b_1+c_1)u^3v + (a_2b_1+a_1b_2+a_2c_1+b_2c_1+a_1c_2\\ &+ b_1c_2)u^2v^2 + (a_1b_1c_1 - a_2b_2c_1 - a_2b_1c_2 - a_1b_2c_2)uv^3,\\ \dot v &= -v + (a_1+b_1+c_1)v^2 + 3u^2v - 2(a_2+b_2+c_2)uv^2 - (a_1b_1 - a_2b_2\\ &+ a_1c_1+b_1c_1 - a_2c_2 - b_2c_2)v^3 - (a_1+b_1+c_1)u^2v^2 + (a_2b_1+a_1b_2\\ &+ a_2c_1+b_2c_1+a_1c_2+b_1c_2)uv^3 + (a_1b_1c_1 - a_2b_2c_1 - a_2b_1c_2 - a_1b_2c_2)v^4.\end{aligned}$$

The singular points $(u,0)$ in the local chart U_1 satisfy $2u(1+u^2)=0$. So the unique singular point in the local chart U_1 is the origin. Computing the eigenvalues of the Jacobian matrix at the origin we obtain that they are 2 and -1. So the origin is a saddle.

Now we analyze the phase portrait in the local chart U_2, we need to the study the origin of U_2, the other infinite singular points have been studied in the local chart U_1. The expression of the system in this chart is

$$\begin{aligned}\dot u &= -2u + (a_1+b_1+c_1)v + (a_2+b_2+c_2)uv - (a_2b_1+a_1b_2+a_2c_1+b_2c_1\\ &+ a_1c_2+b_1c_2)v^2 - 2u^3 + (a_1+b_1+c_1)u^2v - (a_1b_1c_1 - a_2b_2c_1 - a_2b_1c_2\\ &- a_1b_2c_2)v^3 + (a_2+b_2+c_2)u^3v - (a_2b_1+a_1b_2+a_2c_1+b_2c_1+a_1c_2\\ &+ b_1c_2)u^2v^2 + (a_2b_1c_1+a_1b_2c_1+a_1b_1c_2 - a_2b_2c_2)uv^3\\ \dot v &= v - (a_2+b_2+c_2)v^2 - 3u^2v + 2(a_1+b_1+c_1)uv^2 - (a_1b_1 - a_2b_2 + a_1c_1\\ &+ b_1c_1 - a_2c_2 - b_2c_2)v^3 + (a_2+b_2+c_2)u^2v^2 - (a_2b_1+a_1b_2+a_2c_1+b_2c_1\\ &+ a_1c_2+b_1c_2)uv^3 + (a_2b_1c_1+a_1b_2c_1+a_1b_1c_2 - a_2b_2c_2)v^4.\end{aligned}$$

Note that the origin of the local chart U_2 is a singular point whose eigenvalues of the Jacobian matrix at this point are -2 and 1. So the origin is a saddle. This completes the proof of the lemma. ☐

4. Finite singular points

The finite singular points of system (2) are the real solutions of $\dot x = \dot y = 0$. Computing such solutions we obtain

$$(x,y) = (a_1,a_2),\ (x,y) = (b_1,b_2),\ (x,y) = (c_1,c_2).$$

Now we study the local phase portrait of each of the finite singular points.

We will use the following three propositions.

Proposition 4. *Let p be a singular point of system* (2) *such that their eigenvalues are purely imaginary. Then p is an isochronous center.*

The proof of Proposition 4 is given in [7, Corollary 2.7 (a)].

Proposition 5. *Let p be a singular point of system* (2) *of multiplicity $k \in \{2,3\}$. Then the local phase portrait of p is formed by $2(k-1)$ elliptic sectors separated by parabolic sectors.*

The proof of Proposition 5 is given in [7, Corollary 2.7 (b)].

Proposition 6. *System* (2) *has no limit cycles.*

The proof of Proposition 6 is given in [7, Theorem 1.3 (a)].

The finite singular point (a_1, a_2)

The eigenvalues associated to the finite singular point (a_1, a_2) are $\lambda_1 \pm i\lambda_2$ where

$$\lambda_1 = a_1^2 - a_2^2 - a_1b_1 + a_2b_2 - a_1c_1 + b_1c_1 + a_2c_2 - b_2c_2$$
$$\lambda_2 = b_2c_1 - a_2(b_1 + c_1) + b_1c_2 - a_1(b_2 + c_2 - 2a_2).$$

We have different possibilities. If $\lambda_2 \neq 0$ then we have a focus, hyperbolic if $\lambda_1 \neq 0$ and by Proposition 4 is a center if $\lambda_1 = 0$. If $\lambda_2 = 0$ and $\lambda_1 \neq 0$ then we have a node. If $\lambda_1 = \lambda_2 = 0$ then the singular point is linearly zero. In this case we have that either $b = a$ and $c \neq a$, or $b = c = a$. In view of Proposition 5 in the first case the singular point (a_1, a_2) is formed by two elliptic sectors separated by parabolic sectors and in the second case the singular point (a_1, a_2) is formed by four elliptic sectors separated by parabolic sectors.

The finite singular point (b_1, b_2)

The eigenvalues associated to the finite singular point (b_1, b_2) are $\lambda_3 \pm i\lambda_4$ where

$$\lambda_3 = -a_1b_1 + b_1^2 + a_2b_2 - b_2^2 + a_1c_1 - b_1c_1 - a_2c_2 + b_2c_2$$
$$\lambda_4 = a_2b_1 + a_1b_2 - 2b_1b_2 - a_2c_1 + b_2c_1 - a_1c_2 + b_1c_2.$$

We have different possibilities. If $\lambda_4 \neq 0$ then we have a focus, hyperbolic if $\lambda_3 \neq 0$ and by Proposition 4 is a center if $\lambda_3 = 0$. If $\lambda_4 = 0$ and $\lambda_3 \neq 0$ then we have a node. Finally, if $\lambda_4 = 0$ and $\lambda_3 = 0$, then the singular point is linearly zero. We can consider that $b \neq a$ because this case has been already studied. Then the condition $\lambda_4 = \lambda_3 = 0$ implies $b = c$. Since $c \neq a$ then in view of Proposition 5 the singular point (b_1, b_2) is formed by two elliptic sectors separated by parabolic sectors.

The finite singular point (c_1, c_2)

The eigenvalues associated to the finite singular point (c_1, c_2) are $\lambda_5 \pm i\lambda_6$ where

$$\lambda_5 = a_1b_1 - a_2b_2 - a_1c_1 - b_1c_1 + c_1^2 + a_2c_2 + b_2c_2 - c_2^2,$$
$$\lambda_6 = b_2c_1 + a_2(-b_1 + c_1) + b_1c_2 - 2c_1c_2 + a_1(-b_2 + c_2).$$

We have different possibilities. If $\lambda_6 \neq 0$ then we have a focus, hyperbolic if $\lambda_5 \neq 0$ and by Proposition 5 is a center if $\lambda_5 = 0$. If $\lambda_6 = 0$ and $\lambda_5 \neq 0$ then we have a node. Finally, if $\lambda_6 = \lambda_5 = 0$ then either $c = a$, or $c = b$, and these two cases have been studied before.

Combining the different possibilities for the finite singular points we have in principle the following cases:
three centers;
two centers and a node;

two centers and a focus;
one center and two nodes;
one center, one node and one focus;
one center and two foci;
three nodes;
two nodes and a focus;
one node and two foci;
three foci;
one point formed by two elliptic sectors separated by parabolic sectors and a center;
one point formed by two elliptic sectors separated by parabolic sectors and a node;
one point formed by two elliptic sectors separated by parabolic sectors and a focus; and
one point formed by four parabolic sectors separated by parabolic sectors.
However the cases:
two centers and a node;
two centers and a focus;
one center and two nodes;
one center, one node and one focus; and
two nodes and a focus
are not possible. We only show that the case two centers and a node is not possible, the other non possible cases are proved in a similar way. Indeed, the singular points a and b are centers if and only if $\lambda_1 = \lambda_3 = 0$ and c is a node if and only if $\lambda_6 = 0$. Then it is easy to check that the system $\lambda_1 = \lambda_3 = \lambda_6 = 0$ has no solutions in a_i, b_i and c_i for $i = 1, 2$.

In short, only nine possibilities for the finite singular points are indeed realizable, the ones of Figure 1.

5. Global phase portraits

Combining the study of the local phase portraits of the finite and infinite singular points, together with Proposition 6, which states that the Abel system (2) has no limit cycles, and combining the different possibilities for the finite singular points we conclude that the possible global phase portraits are the nine described in Figure 1.

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