THE MARKUS-YAMABE CONJECTURE FOR DISCONTINUOUS PIECEWISE LINEAR DIFFERENTIAL SYSTEMS IN $\mathbb{R}^n$ SEPARATED BY A CONIC$\times\mathbb{R}^{n-2}$

JAUME LLIBRE[1] AND CLÀUDIA VALLS[2]

Abstract. In 1960 Markus and Yamabe made the conjecture that if a C^1 differential system $\dot{x} = F(x)$ in $\mathbb{R}^n$ has a unique equilibrium point and $DF(x)$ is Hurwitz for all $x \in \mathbb{R}^n$, then the equilibrium point is a global attractor. This conjecture was completely solved in 1997 and it turned out to be true in $\mathbb{R}^2$ and false in $\mathbb{R}^n$ for all $n \geq 3$.

In [17] the authors extended the Markus–Yamabe conjecture to continuous and discontinuous piecewise linear differential systems in $\mathbb{R}^n$ separated by a hyperplane, they proved for the continuous systems that the extended conjecture is true in $\mathbb{R}^2$ and false in $\mathbb{R}^n$ for all $n \geq 3$, but for discontinuous systems they proved that the conjecture is false in $\mathbb{R}^n$ for all $n \geq 2$.

In this paper first we show that there are no continuous piecewise linear differential systems separated by a conic$\times\mathbb{R}^{n-2}$ except the linear differential systems in $\mathbb{R}^n$. And after we prove that the extended Markus–Yamabe conjecture to discontinuous piecewise linear differential systems in $\mathbb{R}^n$ separated by a conic$\times\mathbb{R}^{n-2}$ is false in $\mathbb{R}^n$ for all $n \geq 2$.

1. Introduction and statement of the results

Consider a C^1 differential system $\dot{x} = F(x)$ defined in $\mathbb{R}^n$ and having an equilibrium point at the origin of coordinates. If $DF(0)$ is Hurwitz (i.e. the eigenvalues of $DF(0)$ have negative real part), then by the Hartman-Grobman Theorem [11, 14] the origin is locally asymptotically stable. A natural question arises: which are the additional hypotheses that one may add to the function F in order that the origin is a global attractor.

Markus and Yamabe in 1960 (see [18]) made the following conjecture: If we have a C^1 differential system $\dot{x} = F(x)$ defined in $\mathbb{R}^n$ such that $DF(x)$ is Hurwitz for all $x \in \mathbb{R}^n$, and having a unique equilibrium point at the origin of coordinates, then the origin is a global attractor.

This conjecture follows easily when $n = 1$. This conjecture when $n = 2$ was proved independently by Gutierrez [12, 13] in 1993 and by Fessler [6, 7] in 1995. A simpler proof was then given by Glutsyuk in [9, 10]. The counterexample to Markus-Yamabe conjecture for $n > 3$ was given by Bernat and Llibre in [3] and the counterexample for $n \geq 3$ was given by Cima, van den Essen, Gasull, Hubbers and

2010 *Mathematics Subject Classification.* 34C05, 34C07, 34C08.
Key words and phrases. Markus-Yamabe conjecture, Hurwitz matrix, discontinuous piecewise linear differential systems.

Mañosas in [4]. In short, the Markus–Yamabe conjecture is true in $\mathbb{R}^2$ and false in $\mathbb{R}^n$ for $n \geq 3$.

We recall that an equilibrium point p is a *global attractor* if it is globally asymptotically stable that is, every solution tends to p as the time goes to infinity.

The natural step is to ask whether this conjecture is true for continuous systems and even more for discontinuous ones. Since the study of such systems is much more complicated, one natural thing to do is to start with the simpler ones, that is the continuous or discontinuous piecewise linear differential systems. The study of this class of continuous and discontinuous differential systems started with Andronov, Vitt and Khaikin in [1]. Due to the fact that these systems model many real phenomena and different modern devices, they have became a topic of great interest these last twenty years. For more details see for instance [2, 19] and the references therein.

In [**?**, 17] the authors extended the Markus–Yamabe conjecture to continuous and discontinuous piecewise linear differential systems formed by two pieces of $\mathbb{R}^n$ separated by a hyperplane. A Markus–Yannabe piecewise linear differential system is a piecewise linear differential system of the form

$$\dot{x} = \begin{cases} A^+x + b^+ & \text{if } x_1 \geq 0, \\ A^-x + b^- & \text{if } x_1 \leq 0, \end{cases} \tag{1}$$

such that $x = (x_1, \ldots, x_n)$, the matrices A^+ and A^- are Hurwitz, and either only one of the systems $\dot{x} = A^+x + b^+$ and $\dot{x} = A^-x + b^-$ has a real equilibrium point, or both systems have the same equilibrium point in $\{x_1 = 0\}$.

We recall that if $A^+x + b^+ = A^-x + b^-$ in all points $x = (0, x_2, \ldots, x_n)$, then we say that system (1) is a *continuous piecewise linear differential system*, and otherwise we say that it is a *discontinuous linear differential system.*

We recall that a linear differential system $\dot{x} = A^+x + b^+$ has a *real equilibrium point* if the equilibrium point $-(A^+)^{-1}b^+$ exits, and it is in the closed half-space $\{x_1 \geq 0\}$, otherwise we say that the equilibrium point is *virtual.* Similarly for the linear differential system $\dot{x} = A^-x + b^-$. More concretely in [17] it was proved the following result.

Theorem 1. *The following statements hold.*

(a) *The equilibrium point of all continuous Markus–Yamabe piecewise linear differential systems in $\mathbb{R}^2$ is a global attractor. Moreover, for all $n \geq 3$ there are continuous Markus–Yamabe piecewise linear differential systems in $\mathbb{R}^n$ for which their equilibrium point is not a global attractor.*

(b) *For all $n \geq 2$ there are discontinuous Markus–Yamabe piecewise linear differential systems in $\mathbb{R}^n$ for which their equilibrium point is not a global attractor.*

Using an affine change of coordinates, any conic can be written in one of the following nine canonical forms:

(p) $x_1^2 + x_2^2 = 0$ two complex straight lines intersecting at a real point;
(CL) $x_1^2 + 1 = 0$ two complex parallel straight lines;

(CE) $x_1^2 + x_2^2 + 1 = 0$ complex ellipse;
(DL) $x_1^2 = 0$ one double straight line;
(PL) $x_1^2 - 1 = 0$ two real parallel straight lines;
(LV) $x_1 x_2 = 0$ two real straight lines intersecting a real point;
(E) $x_1^2 + x_2^2 - 1 = 0$ ellipse;
(H) $x_1^2 - x_2^2 - 1 = 0$ hyperbola;
(P) $x_2 - x_1^2 = 0$ parabola.

We do not consider conics of type (p), (CL) or (CE) because they do not separate the plane in connected regions. Moreover the case (DL) is completely analogous to the one proved in [17] and so we do not consider it here. In short we are left with cases (PL), (LV), (E), (H) and (P).

These conics (PL), (LV), (E), (H) and (P) are extended to *hyperconics* as follows. The hyperconic (PL) is

$$\{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_1^2 - 1 = 0\}.$$

In a similar way are defined the hyperconics (LV), (E), (H) and (P).

In this paper we will focus on continuous and discontinuous piecewise linear differential systems formed by the pieces of $\mathbb{R}^n$ separated by a hyperconic $C(x) = 0$. Following the definition in [17] *a Markus–Yamabe piecewise linear differential system separated by a hyperconic* $C(x) = 0$ is a piecewise linear differential system in $\mathbb{R}^n$ of the form

$$\dot{x} = \begin{cases} A^+x + b^+ & \text{if } C(x) \geq 0, \\ A^-x + b^- & \text{if } C(x) \leq 0, \end{cases} \tag{2}$$

such that the matrices A^+ and A^- are Hurwitz and either only one of the systems $\dot{x} = A^+x + b^+$ and $\dot{x} = A^-x + b^-$ has a real equilibrium point, or both systems have the same equilibrium point in $\{C(x) = 0\}$.

Again if $A^+x + b^+ = A^-x + b^-$ in all points x such that $C(x) = 0$, then we say that system (2) is a *continuous piecewise linear differential* system. Otherwise we say that it is a *discontinuous linear differential system* separated by the hyperconic $C(x) = 0$. The dynamics of the discontinuous piecewise differential systems on the hyperconic of discontinuity is defined according with the definitions of the book of Filippov [8]. Moreover, a linear differential system $\dot{x} = A^+x + b^+$ has a *real equilibrium point* if the equilibrium point $-(A^+)^{-1}b^+$ exists, and it is contained in $\{C(x) \geq 0\}$, otherwise we say that the equilibrium point is *virtual*. Similarly for the linear differential system $\dot{x} = A^-x + b^-$.

Proposition 2. *There are no continuous Markus–Yamabe piecewise linear differential systems in $\mathbb{R}^n$ separated by a hyperconic of the form (PL), (LV), (H), (E) or (P), other than the linear differential systems.*

The proof of Proposition 2 is given in section 2.

Our main result is the following.

Theorem 3. *For all $n \geq 2$ there are discontinuous Markov-Yamabe piecewise linear differential systems separated by a hyperconic (PL), (LV), (E), (H) and (P) in $\mathbb{R}^n$ for which their equilibrium point is not a global attractor.*

The proof of Theorem 3 is given in section 2.

2. Proof of the results

Proof of Proposition 2. We will do it only for (P) since for the others the proof is completely analogous.

Note that setting $A^+x + b^+ = A^-x + b^-$ in all points x such that $C(x) = 0$, that is, in the points $(x_1, \ldots, x_n)$ with $x_2 = x_1^2$, and if $A^+ = (a_{ij}^+)_{1\le i,j\le n}$ and $A^- = (a_{ij}^-)_{1\le i,j\le n}$ we get from $\dot{x}_1$ the equality

$$a_{11}^+x_1 + a_{12}^+x_1^2 + a_{13}^+x_3 + \ldots + a_{1n}^+x_n + b_1^+ = a_{11}^-x_1 + a_{12}^-x_1^2 + a_{13}^-x_3 + \ldots + a_{1n}^-x_n + b_1^-$$

must be satisfied for all $x_1, x_3, \ldots, x_n$. Therefore $a_{1i}^+ = a_{1i}^-$ for $i = 1, \ldots, n$ and $b_1^+ = b_1^-$.

Doing a similar process with $\dot{x}_k$ for $k = 2, \ldots, n$ we get that $A^+ = A^-$ and $b^+ = b^-$. This proves the proposition for the hyperconic (P). The other cases are analogous. □

We recall that a *crossing limit cycle* is a periodic solution isolated in the set of all periodic solutions of the discontinuous piecewise linear differential system, which only have two points of intersection with the discontinuity set $C(x) = 0$.

Proof of Theorem 3(*PL*). It is sufficient to prove the theorem for $n = 2$, because then we can extend a discontinuous Markus–Yamabe piecewise linear differential system in $\mathbb{R}^2$ separated by a conic (PL) for which the unique equilibrium point of the system is not a global attractor, to a discontinuous Markus–Yamabe piecewise linear differential system in $\mathbb{R}^n$ with $n \ge 3$ separated by a hyperconic (PL) for which its unique equilibrium point will not be a global attractor by adding to the 2-dimensional system the equations

$$\dot{x}_k = -x_k \quad \text{for } k = 3, \ldots, n.$$

We consider the discontinuous piecewise linear differential system in $\mathbb{R}^2$ with coordinates $(x_1, x_2) = (x, y)$ separated by two real parallel straight lines, a conic (PL), defined by

$$\begin{aligned} \dot{x} &= 2 - x, \quad \dot{y} = -y, \qquad \text{in the region } |x| \ge 1,\\ \dot{x} &= -2 - x, \quad \dot{y} = -y, \quad \text{in the region } |x| \le 1. \end{aligned}$$

Note that this system is formed by two stable star nodes, i.e. there solutions leave on invariant straight lines. Clearly this is a discontinuous Markus–Yamabe piecewise linear differential system.

The star node at $(2, 0)$ of the system in the region $|x| \ge 1$ is real, and the start node at $(-2, 0)$ of the system in the region $|x| \le 1$ is virtual. Since all the orbits of the system in the region $|x| \le 1$ runs from the right to the left, these orbits cannot go the stable start node at $(2, 0)$, so this node is not a global attractor. This completes the proof of Theorem 3 for the discontinuous piecewise linear differential systems separated by a two real parallel straight lines. □

Proof of Theorem 3(LV). As in the proof for the hyperconic (LP) it is sufficient to prove the theorem for $n = 2$. Consider the discontinuous piecewise linear differential system in $\mathbb{R}^2$ separated by two real straight lines intersecting in a point, a conic (LV), defined by

$$\begin{aligned} \dot{x} &= 1 - x, \quad \dot{y} = 1 - y, && \text{in the region } xy \geq 0,\\ \dot{x} &= -1 - x, \quad \dot{y} = -1 - y, && \text{in the region } xy \leq 0. \end{aligned}$$

Note that this system is formed by two stable star nodes. Hence it is a discontinuous Markus–Yamabe piecewise linear differential system.

The star node at $(1, 1)$ of the system in the region $xy \geq 0$ is real, and the start node at $(-1, -1)$ of the system in the region $xy \leq 0$ is virtual. Since all the orbits in the quadrant $\{(x, y) : x < 0, y > 0\}$ of the system in the region $xy \leq 0$ runs from top to bottom, these orbits cannot go the stable start node at $(1, 1)$, so this node is not a global attractor. This completes the proof of Theorem 3 for the discontinuous piecewise linear differential systems separated by two real straight lines intersecting in a point. □

Proof of Theorem 3(E). As in the proof for the hyperconic (LP) it is sufficient to prove the theorem for $n = 2$. Consider a discontinuous piecewise linear differential system in $\mathbb{R}^2$ separated by an ellipse (E) defined by

$$\begin{aligned} \dot{x} &= x + y + \frac{1}{\sqrt{2}} + \frac{1}{2},\ \dot{y} = -2x - y - \frac{1}{\sqrt{2}}, && \text{in the region } x^2 + y^2 \geq 1,\\ \dot{x} &= x - \frac{5}{4}y + \frac{1}{\sqrt{2}} + \frac{1}{8},\ \dot{y} = x - y - \frac{1}{\sqrt{2}}, && \text{in the region } x^2 + y^2 \leq 1. \end{aligned} \tag{3}$$

Note that this discontinuous piecewise linear differential system is formed by two linear centers, one of them being virtual.

Moreover, the orbits in the region $x^2 + y^2 \geq 1$ are the level curves of the first integral

$$H_1 = 2x^2 + x(2y + \sqrt{2}) + y(y + \sqrt{2} + 1),$$

while in the region $x^2 + y^2 \leq 1$ the orbits are in the level curves of the first integral

$$H_2 = 4x^2 - 4x(2y + \sqrt{2}) + y(5y - 4\sqrt{2} - 1).$$

It was proved in [16] that this piecewise linear differential system has two crossing limit cycles. One of the crossing limit cycles has the points $(1, 0)$ and $(0, 1)$ of intersection with the ellipse (see Figure 1).

Now we shall see that this crossing limit cycle is unstable. The orbit of the system in the region $x^2 + y^2 \leq 1$ through the point

$$(\cos(-1/100), \sin(-1/100)) = (9999500004166.., -0.00999983333416..),$$

intersects the circle $x^2+y^2 = 1$ at the point $(-0.0129688565617.., 0.999915900843..)$. Then the orbit through this last point of the system in the region $x^2 + y^2 \geq 1$ intersects the circle $x^2 + y^2 = 1$ in the point $(0.999927123997.., -0.01207255955763..)$. So the crossing limit cycle in its inner part is unstable. Consider now the orbit of the system in the region $x^2 + y^2 \leq 1$ through the point

$$(\cos(1/100), \sin(1/100)) = (9999500004166.., 0.00999983333416..),$$

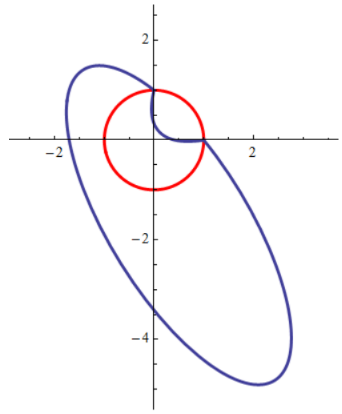

FIGURE 1. The crossing limit cycle of the discontinuous piecewise linear differential system (3)

intersects the circle $x^2+y^2=1$ at the point $(0.01288913205622.., 0.999916931687..)$. Then the orbit through this last point of the system in the region $x^2+y^2 \geq 1$ intersects the circle $x^2+y^2=1$ in the point $(0.999927745390.., 0.0120209815765..)$. So the crossing limit cycle in its outer part is also unstable.

We perturb the discontinuous piecewise linear differential system (3) as follows

$$\begin{aligned} &\dot{x} = (1-\varepsilon)x + y + \frac{1}{\sqrt{2}} + \frac{1}{2},\ \dot{y} = -2x - y - \frac{1}{\sqrt{2}}, \text{ in the region } x^2+y^2 \geq 1, \\ &\dot{x} = (1-\varepsilon)x - \frac{5}{4}y + \frac{1}{\sqrt{2}} + \frac{1}{8},\ \dot{y} = x - y - \frac{1}{\sqrt{2}}, \quad \text{in the region } x^2+y^2 \leq 1. \end{aligned} \tag{4}$$

with $\varepsilon > 0$ sufficiently small.

Note that the two matrices A^+ and A^- of system (4) are Hurwitz. So system (4) is a discontinuous Markus–Yamabe piecewise linear differential system having the unique real equilibrium point

$$P = \Big(\frac{1}{2(1-\varepsilon)}, \frac{2+\sqrt{2}(1-\varepsilon)}{2(1-\varepsilon)}\Big),$$

which is a stable focus of the region $\{x^2+y^2 \geq 1\}$. Since the crossing limit cycle of system (3) is unstable, it persists for system (4) for ε sufficiently small and so the equilibrium point P is not a global attractor. This completes the proof of Theorem 3 for the discontinuous piecewise linear differential systems separated by the ellipse (E). □

First proof of Theorem 3(H). As in the proof for the hyperconic (LP) it is sufficient to prove the theorem for $n = 2$. Consider a discontinuous piecewise linear differential system in $\mathbb{R}^2$ separated by a hyperbola (H) defined by

$$\begin{aligned} &\dot{x} = 2 - x, \quad \dot{y} = -y, \quad \text{in the region } x^2 - y^2 \geq 1, \\ &\dot{x} = -2 - x, \quad \dot{y} = -y, \quad \text{in the region } x^2 - y^2 \leq 1. \end{aligned} \tag{5}$$

Note that this system is formed by two stable star nodes. Therefore it is a discontinuous Markus–Yamabe piecewise linear differential system.

The star node at $(2,0)$ of the system iin the region $x^2-y^2\geq 1$ is real, and the start node at $(-2,0)$ of the system in the region $x^2-y^2\leq 1$ is virtual. Since all the orbits of the system in the region $x^2-y^2\leq 1$ runs from the right to the left, these orbits cannot go the stable start node at $(2,0)$, hence this node is not a global attractor. This completes the proof of Theorem 3 for the discontinuous piecewise linear differential systems separated by a hyperbola (H). □

Second proof of Theorem 3(*H*). As in the proof for the hyperconic (LP) it is sufficient to prove the theorem for $n=2$. Consider a discontinuous piecewise linear differential system in $\mathbb{R}^2$ separated by a hyperbola (H) defined by

$$
(6)\qquad
\begin{aligned}
&\dot{x}=1-\frac{1}{4}y, \quad \dot{y}=x, && \text{in the region } x^2-y^2\geq 1,\\
&\dot{x}=\frac{1}{40}(89-5\sqrt{89})-y, \quad \dot{y}=-1+x, && \text{in the region } x^2-y^2\leq 1.
\end{aligned}
$$

Note that this system is formed by two linear centers. The linear differential system in $\{x^2-y^2\leq 1\}$ has the real center at the point $(0,4)$, so it is a real equilibrium point. The linear differential system in $\{x^2-y^2\geq 1\}$ has a virtual equilibrium point. The first integrals of these systems are

$$
H_1(x,y)=(x-1)^2+\left(\frac{1}{40}(-89+5\sqrt{89})+y\right)^2
$$

and

$$
H_2(x,y)=\frac{1}{4}x^2+\left(\frac{1}{4}y-1\right)^2,
$$

respectively.

Proceeding as in the proof of the ellipse (E) we can prove that this piecewise linear differential system has one crossing limit cycle Γ with the points $(1,0)$ and $(\sqrt{89}/5, 8/5)$ of intersection with the hyperbola (see Figure 2). Moreover, proceeding as in the proofs of the conics (LP) and (LV) one can show that Γ is a stable limit cycle.

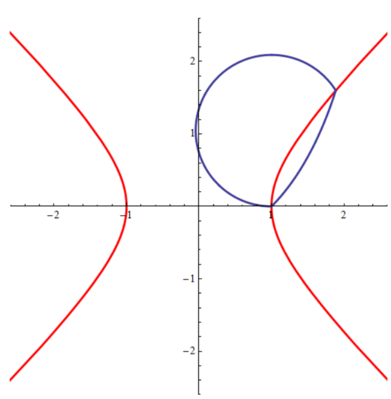

FIGURE 2. The crossing limit cycle of the discontinuous piecewise linear differential system (6)

Now we perturb the discontinuous piecewise linear differential system (6) as follows

$$\dot{x} = 1 - \varepsilon x - \frac{1}{4}y,\ \dot{y} = x - \varepsilon y, \qquad \text{in the region } x^2 - y^2 \geq 1,$$

$$\dot{x} = \frac{1}{40}(89 - 5\sqrt{89}) - \varepsilon x - y, \quad \dot{y} = -1 + x - \varepsilon y, \quad \text{in the region } x^2 - y^2 \leq 1.$$

Note that for $\varepsilon > 0$ sufficiently small the two matrices of this piecewise system A^+ and A^- are Hurwitz, and that this piecewise system has a unique real equilibrium

$$p = \left(\frac{4\varepsilon}{4\varepsilon^2 + 1}, \frac{4}{4\varepsilon^2 + 1}\right),$$

which is a stable focus, and it has a stable limit cycle near Γ_1 near the stable limit cycle Γ. Hence this local stable focus is not a global attractor. This completes the proof of Theorem 3 for the discontinuous piecewise linear differential systems separated by the hyperbola (H).

Note that since the focus p and the limit cycle Γ_1 are stable, implies by the Poincaré–Bendixson Theorem that an unstable limit cycle must exist between them. For more details on the Poincaré–Bendixson Theorem see Corollary 1.30 of [5]. □

Proof of Theorem 3(P). As in the proof for the hyperconic (LP) it is sufficient to prove the theorem for $n = 2$. Consider a discontinuous piecewise linear differential system in $\mathbb{R}^2$ separated by the parabola (P) defined by

$$\dot{x} = \frac{18}{5} - y, \quad \dot{y} = x - \frac{99}{50}, \tag{7}$$

in the region $y \geq x^2$ and

$$\dot{x} = \frac{720042289}{205000000} + \frac{189}{100}x - \frac{38221}{20500}y, \quad \dot{y} = \frac{8241}{2500} + \frac{41}{20}x - \frac{189}{100}y, \tag{8}$$

in the region $y \leq x^2$.

Note that this system is formed by two linear centers. The linear differential system in the region $y \geq x^2$ has a virtual center and the linear differential system in the region $y \leq x^2$ has the real center at the point

$$P_1 = \left(\frac{10096288221}{5125000000}, \frac{97022689}{25000000}\right).$$

The first integrals of these systems are

$$H_1(x, y) = \left(x - \frac{99}{50}\right)^2 + \left(y - \frac{18}{5}\right)^2$$

and

$$H_2(x, y) = 675762000x - 720042289y + 210125000x^2 - 387450000xy + 191105000y^2,$$

respectively.

Proceeding as in the proof of the ellipse (E) we can prove that this piecewise linear differential system has one crossing limit cycle Γ with the points (x_1, x_1^2) and (x_2, x_2^2), where $x_1 = 2.494245365886619..$ and $x_2 = 1.047324356752986..$, of intersection with the parabola (see Figure 3). Moreover, also as in the proofs of the ellipse (E) it follows that Γ is stable.

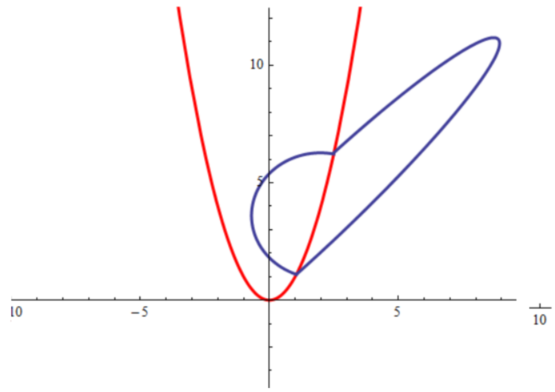

FIGURE 3. The crossing limit cycle of the discontinuous piecewise linear differential system (7)–(8)

Now we perturb the discontinuous piecewise linear differential system (6) as follows

$$\dot{x} = \frac{18}{5} - \varepsilon x - y, \quad \dot{y} = x - \frac{99}{50} - \varepsilon y,$$

in the region $y \geq x^2$ and

$$\dot{x} = \frac{720042289}{205000000} + \left(\frac{189}{100} - \varepsilon\right) x + \frac{38221}{20500} y, \quad \dot{y} = \frac{8241}{2500} + \frac{41}{20} x - \frac{189}{100} y,$$

in the region $y \leq x^2$.

Note that for $\varepsilon > 0$ sufficiently small the two matrices of this piecewise system A^+ and A^- are Hurwitz, and that this piecewise system has a unique real equilibrium near P_1 (still in the region $y \leq x^2$) which is a stable focus, and it has a stable limit cycle near Γ_2. Hence this local stable focus is not a global attractor. This completes the proof of Theorem 3 for the discontinuous piecewise linear differential systems separated by the parabola (P).

Again since the focus p and the limit cycle Γ_1 are stable, implies by the Poincaré–Bendixson Theorem that an unstable limit cycle must exist between them. □

ACKNOWLEDGEMENTS

This work is supported by the Ministerio de Ciencia, Innovación y Universidades, Agencia Estatal de Investigación grants MTM2016-77278-P (FEDER), the Agència de Gestió d'Ajuts Universitaris i de Recerca grant 2017SGR1617, and the H2020 European Research Council grant MSCA-RISE-2017-777911. The second author is partially supported by FCT/Portugal through UID/MAT/04459/2013.

References

[1] A. Andronov, A. Vitt and S. Khaikin, *Theory of Oscillations*, Pergamon Press, Oxford, 1966.

[2] M. Di Bernardo, C.J. Budd, A.R. Champneys and P. Kowalczyk, *Piecewise–Smooth Dynamical Systems: Theory and applications*, Appl. Math. Sci. Series 163, Springer–Verlag, London, 2008.

[3] J. Bernat and J. Llibre, *Counterexample to Kalman and Markus–Yamabe conjectures in dimension larger than* 3, Dynam. Contin. Discrete Impuls. Systems **2** (1996), 337–379.

[4] A. Cima, A. van den Essen, E.Hubbers and F. Ma nosas, *A polynomial counterexample to the Markus–Yamabe conjecture*, Adv. Math. **131** (1997), 453–457.

[5] F. Dumortier, J. Llibre and J. C. Artés, *Qualitative Theory of Planar Differential Systems*, UniversiText, Springer-verlag, New York, 2006.

[6] R. Fessler, *On the Markus–Yamabe conjecture, in Automorphisms of affine spaces* (Cura cao, 1994), Kluwer Acad. Publ., Dordrecht, 1995, pp 127–135.

[7] R. Fessler, *A proof of the two–dimensional Markus–Yamabe stability conjecture and a generalization*, Ann. Polon. Math. **62** (1995), 45–74.

[8] A.F. Filippov, *Differential Equations with Discontinuous Right–Hand Sides*, Translated from the Russian, Mathematics and its Applications (Soviet Series) **18**, Kluwer Academic Publishers Group, Dordrecht, 1988.

[9] A.A. Glutsyuk, *A complete solution of the Jacobian problem for vector fields on the plane*, Russian Math. Survey **49** no. 3 (1994), 185–186.

[10] A.A. Glutsyuk, *The asymptotic stability of the linearization of a vector field on the plane with a singular point implies global stability*, (Russian) Funktsional. Anal. i Prilozhen. **29** (1995), no. 4, 17–30; translation in Funct. Anal. Appl. **29** (1996), no. 4, 238–247.

[11] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*, Springer–Verlag, New York, 1983.

[12] C. Gutierrez, *A solution to the bidimensional global asymptotic stability conjecture*, conference given at the Workshop: Recent results on the global asymptotic stability Jacobian conjecture, 1993.

[13] C. Gutierrez, *A solution to the bidimensional global asymptotic stability conjecture*, Ann. Inst. H. Poincaré Anal. Non Linaire **12** (1995), no. 6, 627–671.

[14] P. Hartman, *Ordinary Differential Equations*, Reprint of the 2nd Edition, Birkhäuser, Boston, 1982.

[15] J. Llibre and L.A.S. Menezes, *The Markus–Yamabe conjecture does not hold for piecewise discontinuous linear differential systems separated by one straight line*, J. of Dynamics and Differential Equations (2020), https://doi.org/10.1007/s10884-020-09825-8.

[16] J. Llibre and M.A. Teixeira, *Piecewise linear differential systems with only centers can create limit cycles?*, Nonlinear Dynam. **91** (2018), 249–255.

[17] J. Llibre and X. Zhang, *The Markus–Yamabe conjecture for continuous and discontinuous piecewise linear differential systems*, preprint, 2019.

[18] L. Markus and H. Yamabe, *Global stability criteria for differential systems*, Osaka Math. J. **12** (1960), 305–317.

[19] D.J.W. Simpson, *Bifurcations in Piecewise–Smooth Continuous Systems*, World Scientific Series on Nonlinear Science A, vol 69, World scientific, Singapure, 2010.

[1] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

Email address: jllibre@mat.uab.cat

[2] Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais 1049–001, Lisboa, Portugal

Email address: cvalls@math.tecnico.ulisboa.pt