RATIONAL LIMIT CYCLES ON ABEL EQUATIONS

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Abstract. In this paper we deal with Abel equations $dy/dx = A(x)y^2 + B(x)y^3$, where $A(x)$ and $B(x)$ are real polynomials. We prove that these Abel equations can have at most three rational limit cycles and we characterize when this happens. Moreover, we provide examples of these Abel equations with three nontrivial rational limit cycles. We also prove that in this case the limit cycles cannot be hyperbolic.

1. Introduction and statement of the results

We study the Abel equations

$$\frac{dy}{dx} = A(x)y^2 + B(x)y^3, \tag{1}$$

where x, y are real variables and $A(x)$ and $B(x)$ are polynomials. The limit cycles of these equations have been intensively investigated mainly when the functions $A(x)$ and $B(x)$ are periodic (see for instance [1, 2, 3, 4, 5, 6, 7, 9, 12, 13, 15, 16, 17, 18, 19, 21, 22, 23, 24]), and also when $A(x)$ and $B(x)$ are polynomial (see for instance [8, 10, 11, 14, 20]). Here we are interested in the rational limit cycles of equation (1) when the functions $A(x)$ and $B(x)$ are polynomial.

A *periodic solution* of equation (1) is a solution $y(x)$ defined in the closed interval $[0, 1]$ such that $y(0) = y(1)$.

We say that a *limit cycle* is a periodic solution isolated in the set of periodic solutions of a differential equation (1). Without loss of generality we will assume that the period is 1.

The limit cycle is called a *polynomial limit cycle* if the periodic solution $y(x)$ is a polynomial in the variable x. In particular the authors of [14] proved that any polynomial limit cycle of system (1) is of the form $y = c$ with $c \in \mathbb{R}$, and that if a polynomial limit cycle exists with $c \neq 0$, then no other polynomial limit cycles can exist.

In this paper we want to consider the existence of *rational limit cycles* for system (1), i.e. we want to consider limit cycles of the form $y(x) = q(x)/p(x)$

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where $p, q \in \mathbb{R}[x]$ and $(p(x), q(x)) = 1$. As usual $\mathbb{R}[x]$ denotes the set of all real polynomials in the variable x. We will study only the rational limit cycles that are not polynomial limit cycles. In this scenario we will distinguish between trivial limit cycles (the polynomial ones) and non-trivial limit cycles (the rational limit cycles that are not polynomials).

In [17] the authors provide examples of differential equations (1) having one or two rational limit cycles, and that these limit cycles are hyperbolic.

Our main theorem is the following one.

Theorem 1. *System* (1) *has at most three rational limit cycles, and when it has three rational limit cycles* $1/y_i(x)$ *for* $i = 1, 2, 3$, *then there exist a polynomial* $S(x)$ *and two different constants* $c_1, c_2 \in \mathbb{R} \setminus \{0\}$ *such that* $S(0) = S(1)$ *and* $S(x) \neq 0$, $S(x) + c_1 \neq 0$ *and* $S(x) + c_2 \neq 0$ *for* $x \in [0, 1]$. *Moreover*

$$B(x) = S(x)S'(x)(S(x) + c_1)(S(x) + c_2), \tag{2}$$

and the three limit cycles and satisfy

$$\begin{aligned} y_1(x) &= S(x)(S(x) + c_1), \\ y_2(x) &= S(x)(S(x) + c_2), \\ y_3(x) &= (S(x) + c_1)(S(x) + c_2). \end{aligned} \tag{3}$$

We provide an example of system (1) with three rational limit cycles. The proof of Theorem 1 and the example are given in section 2.

Denote by $y(x, x_0)$ the solution of equation (1) such that $y(0, x_0) = x_0$. Clearly a zero of the function $\phi(x_0) = y(1, x_0) - x_0$ implies that $y(x, x_0)$ is a periodic solution of (1), and an isolated zero of $\phi(x_0)$ implies that $y(x, x_0)$ is a limit cycle of system (1). When we have a simple isolated zero of $\phi(x_0)$, i.e. $\phi(x_0) = 0$ and $\phi'(x_0) \neq 0$, then we say that $y(x, x_0)$ is a *hyperbolic limit cycle.*

Theorem 2. *When system* (1) *has three limit cycles, they are not hyperbolic.*

Theorem 2 is proved in section 3.

2. Proof of Theorem 1

It was proved in Lemma 2 of [17] the following result, but for completeness we provide it here.

Lemma 3. *The rational function* $y = q(x)/p(x)$ *with* $p(x)$ *non-constant is a periodic solution of system* (1) *if and only if* $q(x) = c \in \mathbb{R} \setminus \{0\}$, $p(0) = p(1)$ *and* $p(x)$ *has no zero in* $[0, 1]$ *and*

$$cB(x) + \frac{p(x)p'(x)}{c} + p(x)A(x) = 0. \tag{4}$$

Proof. For the reverse implication, we note that if $q(x) = c \in \mathbb{R} \setminus \{0\}$, $p(0) = p(1)$, $p(x)$ has no zero in $[0, 1]$ and equality (4) holds then it is clear that the rational function $y = c/p(x)$ is a periodic solution of system (1).

For the direct implication, we note that if $y(x) = q(x)/p(x)$ is a periodic solution of system (1) then $p(x) \neq 0$ for $x \in [0, 1]$. Let $g(x, y) = p(x)y - q(x)$. Then

$$\begin{aligned} 0 = \frac{dg(x,y)}{dx}|_{g(x,y)=0} &= p'(x)y + p(x)\frac{dy}{dx} - q'(x) \\ &= p'(x)y + p(x)(A(x)y^2 + B(x)y^3) - q'(x). \end{aligned}$$

Note that $g(x, y)$ is irreducible, so there exists a polynomial $k(x, y)$ so that

$$p'(x)y + p(x)(A(x)y^2 + B(x)y^3) - q'(x) = k(x, y)g(x, y). \tag{5}$$

Since the highest degree in y in the left-hand side is 3 and the highest degree in y in $g(x, y)$ is 1 we get that the highest degree in y in $k(x, y)$ is 2 and so it can be written as $k(x, y) = k_0(x) + k_1(x)y + k_2(x)y^2$, where $k_0, k_1, k_2 \in \mathbb{R}[x]$. Comparing the coefficients of y^0, y^1, y^2 and y^3 in (5) we get

$$\begin{aligned} q'(x) &= k_0(x)q(x), \\ p'(x) &= k_0(x)p(x) - k_1(x)q(x), \\ p(x)A(x) &= k_1(x)p(x) - k_2(x)q(x), \\ p(x)B(x) &= k_2(x)p_(x). \end{aligned} \tag{6}$$

From the first relation we get that $q(x)|q'(x)$. This implies that $q(x)$ is a constant that we denote by c, that is, $q(x) = c \in \mathbb{R}$. If $c = 0$ then $y = q(x)/p(x) = 0$. This is not possible and so $c \neq 0$. Moreover, $y = q(x)/p(x) = c/p(x)$ is a periodic solution, then $p(0) = p(1)$. From the second relation we get that $k_1(x) = -p'(x)/c$ and from the fourth relation we obtain $k_2(x) = B(x)$. Substituting them in the third relation we get (4) and the direct inclusion is proved. □

In view of Lemma 3 it is not restrictive to take $c = 1$ and consider all rational limit cycles of the form $y = 1/p(x)$ with $p(x)$ satisfying $p(0) = p(1)$ with $p(x)$ having no zero in $[0, 1]$ and satisfying (4).

From (4) we must have that $B(x)$ is multiple of $p(x)$ and so $B(x) = p(x)r(x)$ for some polynomial $r(x)$. Therefore, (4) becomes

$$r(x)+p'(x)+A(x) = 0 \quad \text{and so} \quad p(x) = \kappa - \int (A(s)+r(s))\, ds, \quad \kappa \in \mathbb{R}. \tag{7}$$

Assume that equation (1) has two rational limit cycles, $y(x) = 1/p_1(x)$ and $y(x) = 1/p_2(x)$ with $p_1(x), p_2(x) \in \mathbb{R}[x] \setminus \mathbb{R}$. Denote by $q(x) = (p_1(x), p_2(x))$, i.e. the maximum common divisor of the polynomials $p_1(x)$ and $p_2(x)$, and consequently

$$p_1(x) = q(x)s_1(x), \quad p_2(x) = q(x)s_2(x) \tag{8}$$

with $q(x), s_i(x) \in \mathbb{R}[x]$ and $(s_1(x), s_2(x)) = 1$. Note that in view of the above observation we must have that

$$B(x) = q(x)s_1(x)s_2(x)s_3(x) \tag{9}$$

for some $s_3(x) \in \mathbb{R}[x]$.

Lemma 4. *The following equalities hold*

$$s_3(x) = q'(x) \quad \textit{and} \quad s_1(x) - s_2(x) = c \in \mathbb{R}. \tag{10}$$

Proof. Note that in view of (7) we have

$$\begin{aligned} q(x)s_1(x) &= \kappa_0 - \int (A(s) + s_2(s)s_3(s))\, ds, \\ q(x)s_2(x) &= \kappa_1 - \int (A(s) + s_1(s)s_3(s))\, ds, \end{aligned} \tag{11}$$

with $\kappa_0, \kappa_1 \in \mathbb{R}$. Hence,

$$\begin{aligned} q'(x)s_1(x) + q(x)s_1'(x) &= -A(x) - s_2(x)s_3(x), \\ q'(x)s_2(x) + q(x)s_2'(x) &= -A(x) - s_1(x)s_3(x), \end{aligned}$$

and so

$$\begin{aligned} &q'(x)s_1(x) + q(x)s_1'(x) - q'(x)s_2(x) + q(x)s_2'(x)) \\ &= -s_2(x)s_3(x) + s_1(x)s_3(x) = (s_1(x) - s_2(x))s_3(x), \end{aligned}$$

which gives

$$q'(x)(s_1(x) - s_2(x)) + q(x)(s_1(x) - s_2(x))' = (s_1(x) - s_2(x))s_3(x),$$

that is

$$q(x)(s_1(x) - s_2(x))' = (s_1(x) - s_2(x))(s_3(x) - q'(x)).$$

Hence

$$\frac{(s_1(x) - s_2(x))'}{(s_1(x) - s_2(x))} = \frac{s_3(x)}{q(x)} - \frac{q'(x)}{q(x)}.$$

Therefore

$$s_1(x) - s_2(x) = \kappa_2 \frac{1}{q(x)} \exp\left(\int \frac{s_2(s)}{q(s)}\, ds \right), \tag{12}$$

for some $\kappa_2 \in \mathbb{R}$. Since $s_1(x) - s_2(x)$ must be a polynomial we must have $s_3(x) = \kappa_3 q'(x)$ for some $\kappa_3 \in \mathbb{R}$. Indeed note that we must have

$$\int \frac{s_3(s)}{q(s)}\, ds = \kappa_3 \log h(x), \quad \kappa_3 \in \mathbb{R}, \quad h(x) \in \mathbb{R}[x] \setminus \{0\}. \tag{13}$$

Let $H(x) = (q(x), s_3(x))$. Then

$$\frac{s_3(x)}{q(x)} = \frac{H(x)\bar{s}_3(x)}{H(x)\bar{q}(x)} = \frac{\bar{s}_3(x)}{\bar{q}(x)} = \kappa_3 \frac{h'(x)}{h(x)}.$$

Therefore $\bar{q}(x) = h(x)$ and $\bar{s}_3(x) = \kappa_3 \bar{q}'(x)$. So $q(x) = H(x)h(x)$. From (12) and (13) we have

$$s_1(x) - s_2(x) = \kappa_2 \frac{1}{q(x)} \left(\frac{q(x)}{H(x)} \right)^{\kappa_3}.$$

Since $s_1(x) - s_2(x)$ must be a polynomial and $\kappa_3 \neq 0$, it follows that $H(x)$ can be one. Hence,

$$s_3(x) = \kappa_3 q'(x) \quad \text{and} \quad s_1(x) - s_2(x) = \kappa_2 q(x)^{\kappa_3 - 1}. \tag{14}$$

On the other hand, doing a change of variables of the form $Y = \beta y$ where $\beta^2 = \text{sign}(\kappa_3)\kappa_3$, the Abel equation (1) becomes

$$\frac{dY}{dx} = \frac{A(x)}{\beta} Y^2 + \frac{B(x)}{\beta^2} Y^3 = \overline{A}(x) Y^2 + \overline{B}(x) Y^3. \tag{15}$$

Since $B(x) = q(x)s_1(x)s_2(x)\kappa_3 q'(x)$, then $\overline{B}(x) = \pm q(x)s_1(x)s_2(x)q'(x)$. In what follows we shall work with the Abel equation (15).

Repeating the previous computations starting with the Abel equation (15) we will arrive to equation (14) which now writes

$$s_3(x) = q'(x) \quad \text{and} \quad s_1(x) - s_2(x) = \kappa_2,$$

because $\kappa_3 = \pm 1$ and only can be one. This concludes the proof of the lemma. □

Note that from (8), (9) and Lemma 4 we have that

$$B(x) = q(x)q'(x)s_1(x)s_2(x). \tag{16}$$

Proof of Theorem 1. Assume that equation (1) has three rational limit cycles, $y = 1/p_1(x)$ and $y = 1/p_2(x)$ and $y_3 = 1/p_3(x)$ with $p_1, p_2, p_3 \in \mathbb{R}[x] \setminus \mathbb{R}$. Denote by $q_1(x) = (p_1(x), p_2(x))$, $q_2(x) = (p_1(x), p_3(x))$ and $q_3(x) = (p_2(x), p_3(x))$. In view of Lemma 4 we have

$$\begin{aligned} p_1(x) &= q_1(x)s_1(x) = q_2(x)s_2(x), \\ p_2(x) &= q_1(x)(s_1(x) + c_1) = q_3(x)s_3(x), \\ p_3(x) &= q_2(x)(s_2(x) + c_2) = q_3(x)(s_3(x) + c_3), \end{aligned} \tag{17}$$

for some polynomials $s_1(x), s_2(x), s_3(x)$ and constants $c_1, c_2, c_3 \in \mathbb{R} \setminus \{0\}$. Hence, we get

$$\begin{aligned} p_2(x) - p_1(x) &= q_1(x)c_1, \ c_1 \in \mathbb{R}, \\ p_3(x) - p_1(x) &= q_2(x)c_2, \ c_2 \in \mathbb{R}, \\ p_3(x) - p_2(x) &= q_3(x)c_3, \ c_3 \in \mathbb{R}, \end{aligned}$$

and so

$$q_2(x)c_2 = q_1(x)c_1 + q_3(x)c_3. \tag{18}$$

We consider two situations.

Case 1*: $q_1(x)$ and $s_2(x)$ are coprime.* Note that from (17) we have that $q_1(x)s_1(x) = q_2(x)s_2(x)$, and then from (18) we get

$$\frac{q_1(x)s_1(x)c_2}{s_2(x)} = q_2(x)c_2 = q_1(x)c_1 + q_3(x)c_3.$$

In particular there exists $T(x) \in \mathbb{R}[x]$ so that

$$q_3(x) = q_1(x)T(x),$$

and consequently

$$\frac{s_1(x)c_2}{s_2(x)} = c_1 + T(x)c_3,$$

which yields

$$s_1(x) = \frac{s_2(x)}{c_2}(c_1 + T(x)c_3).$$

Therefore from (17) we get

$$q_2(x)s_2(x) = q_1(x)s_1(x) = q_1(x)\frac{s_2(x)}{c_2}(c_1 + T(x)c_3),$$

and so

$$q_2(x) = q_1(x)\frac{c_1 + T(x)c_3}{c_2}.$$

Hence we have

$$\begin{aligned} p_1(x) &= q_1(x)s_1(x) = q_1(x)\frac{s_2(x)}{c_2}(c_1 + T(x)c_3), \\ p_2(x) &= q_1(x)(s_1(x) + c_1) = q_1(x)\left(\frac{s_2(x)}{c_2}(c_1 + T(x)c_3) + c_1\right), \\ p_3(x) &= q_2(x)(s_2(x) + c_2) = q_1(x)\frac{c_1 + T(x)c_3}{c_2}(s_2(x) + c_2). \end{aligned} \tag{19}$$

We consider two subcases.

Subcase 1.1*: Assume that $T(x)$ and $s_2(x) + c_2$ are coprime.* Then the maximum common divisor between $p_2(x)$ and $p_3(x)$ is $q_1(x)$. Indeed, we will show that

$$r_1(x) = \frac{s_2(x)}{c_2}(c_1 + T(x)c_3) + c_1$$

and

$$r_2(x) = (c_1 + T(x)c_3)(s_2(x) + c_2)$$

are coprime. Note that if x^* is a zero of $c_1 + T(x)c_3$ then we have that $r_2(x^*) = 0$ but $r_1(x^*) = c_1 \neq 0$. Moreover, if $\hat{x}$ is a solution of $s_2(x) + c_2 = 0$ then $r_2(\hat{x}) = 0$ but $r_1(\hat{x}) = -(c_1 + T(\hat{x})c_3) + c_1 = T(\hat{x})c_3 \neq 0$. Therefore, using $p_1(x)$ and $p_2(x)$ from (16) and (19) we can write

$$B(x) = q_1(x)q_1'(x)\frac{s_2(x)}{c_2}(c_1 + T(x)c_3)\left(\frac{s_2(x)}{c_2}(c_1 + T(x)c_3) + c_1\right),$$

and from $p_1(x)$ and $p_3(X)$ we can write

$$B(x) = q_1(x)q_1'(x)\left(\frac{s_2(x)}{c_2}(c_1 + T(x)c_3) + c_1\right)\frac{c_1 + T(x)c_3}{c_2}(s_2(x) + c_2),$$

and so

$$s_2(x) = s_2(x) + c_2$$

which is not possible because $c_2 \neq 0$.

Subcase 1.2: *Assume that $T(x)$ and $s_2(x) + c_2$ are not coprime.* Write

$$T(x) = \alpha_1(x)\alpha_2(x), \quad s_2(x) + c_2 = \alpha_1(x)\alpha_3(x),$$

where $\alpha_2, \alpha_3 \in \mathbb{R}[x]$ and $\alpha_1(x) \in \mathbb{R}[x] \setminus \mathbb{R}$. Then

$$p_3(x) = q_1(x)\alpha_1(x)\alpha_3(x)\frac{c_1 + T(x)c_3}{c_2},$$

$$p_2(x) = q_1(x)\frac{\alpha_1(x)}{c_2}(c_1\alpha_3(x) + s_2(x)\alpha_2(x)c_3).$$

We first note that the maximum common divisor between $p_2(x)$ and $p_3(x)$ is $q_1(x)\alpha_1(x)$. To do so, we will show that

$$r_3(x) = \alpha_3(x)(c_1 + T(x)c_3) \quad \text{and} \quad r_4(x) = c_1\alpha_3(x) + s_2(x)\alpha_2(x)c_3$$

are coprime. If x^* is a zero of $\alpha_3(x)$ then $r_3(x^*) = 0$ but $r_4(x^*) = s_2(x^*)\alpha_2(x^*)c_3 = -c_2\alpha_2(x^*)c_3$. Since $\alpha_2(x)$ and $\alpha_3(x)$ are coprime, we get that $\alpha_2(x^*) \neq 0$, and then $r_4(x^*) \neq 0$. Moreover, if $c_1 + T(\hat{x})c_3 = 0$ then $r_4(\hat{x}) = c_1 \neq 0$. So $r_3(x)$ and $r_4(x)$ are coprime.

From $p_1(x)$, $p_2(x)$, (16) and (19) we get

$$B(x) = q_1(x)q_1'(x)\frac{s_2(x)}{c_2}(c_1 + T(x)c_3)\frac{\alpha_1(x)}{c_2}(c_1\alpha_3(x) + s_2(x)\alpha_2(x)c_3). \tag{20}$$

Note that from $p_2(x)$, $p_3(x)$, (16) and (19) we have

$$B(x) = \frac{q_1(x)}{c_2^2}\alpha_1(x)(q_1(x)\alpha_1(x))'\alpha_3(x)(c_1 + T(x)c_3)(c_1\alpha_3(x) + s_2(x)\alpha_2(x)c_3). \tag{21}$$

Comparing (20) with (21) we obtain

$$\alpha_3(x)(q_1(x)\alpha_1(x))' = q_1'(x)(\alpha_1(x)\alpha_3(x) - c_2),$$

i.e.

$$-c_2q_1'(x) = -\alpha_3(x)q_1(x)\alpha_1'(x),$$

which is not possible unless either $\alpha_3(x) = 0$ or $\alpha_1'(x) = 0$, but then $q_1(x)$ would be constant, a contradiction. In short, Case 1 is not possible.

Case 2: *$q_1(x)$ and $s_2(x)$ are not coprime* We write

$$q_1(x) = R_1(x)R_2(x), \quad s_2(x) = R_1(x)R_3(x)$$

with $R_1(x), R_2(x), R_3(x) \in \mathbb{R}[x]$ and $R_1(x) \notin \mathbb{R}$.

We consider two different subcases.

Subcase 2.1*:* $R_3(x) = R \in \mathbb{R}$. So $s_2(x) = R_1(x)R$ and $q_1(x) = R_2(x)s_2(x)/R$.

We also consider two cases

2.1.1*:* $R_2(x) = R_2 \in \mathbb{R}$. From (8) we have $q_1(x)s_1(x) = q_2(x)s_2(x)$ and so $q_2(x) = R_2 s_1(x)/R$.Then

$$\begin{aligned} p_1(x) &= \frac{R_2}{R} s_1(x)s_2(x), \\ p_2(x) &= \frac{R_2}{R} s_2(x)(s_1(x) + c_1), \\ p_3(x) &= \frac{R_2}{R} s_1(x)(s_2(x) + c_2). \end{aligned}$$

From $p_1(x)$, $p_2(x)$, (16) and (19) we get

$$B(x) = \Big(\frac{R_2}{R}\Big)^2 s_2(x)s_1(x)(s_1(x) + c_1)s_2'(x),$$

and from $p_1(x)$, $p_3(x)$, (16) and (19) we obtain

$$B(x) = \Big(\frac{R_2}{R}\Big)^2 s_2(x)s_1(x)s_1'(x)(s_2(x) + c_2),$$

and so

$$s_2'(x)(s_1(x) + c_1) = s_1'(x)(s_2(x) + c_2),$$

which yields

$$\frac{s_2'(x)}{s_2(x) + c_2} = \frac{s_1'(x)}{s_1(x) + c_1}$$

and integrating

$$\log(s_2(x) + c_2) = \kappa + \log(s_1(x) + c_1), \quad \kappa \in \mathbb{R}$$

and so

$$s_2(x) + c_2 = \kappa_1(s_1(x) + c_1), \quad \kappa_1 = e^\kappa \in \mathbb{R}^+.$$

Hence

$$\begin{aligned} p_1(x) &= \frac{R_2}{R} s_1(x)(\kappa_1(s_1(x) + c_1) - c_2), \\ p_2(x) &= \frac{R_2}{R}(\kappa_1(s_1(x) + c_1) - c_2)(s_1(x) + c_1), \\ p_3(x) &= \frac{R_2}{R} s_1(x)\kappa_1(s_1(x) + c_1), \end{aligned} \tag{22}$$

and from (16) we obtain

$$B(x) = \Big(\frac{R_2}{R}\Big)^2 \kappa_1 s_1(x)s_1'(x)(s_1(x) + c_1)(\kappa_1(s_1(x) + c_1) - c_2).$$

Doing the rescaling $Y = \beta y$, we can assume that the constant $(R_2/R)^2\kappa_1 = 1$ and the constants R_2/R and κ_1 in the expression of $p_i(x)$ is one (see the proof of Lemma 4). Note that $B(x)$ is as in the statement of the theorem as well as $p_i(x)$ for $i = 1, 2, 3$.

2.1.2*: $R_2(x) \in \mathbb{R}[x] \setminus \mathbb{R}$.* Since $R_3(x) = R$ we have $s_2(x) = R_1(x)R$ and $q_1(x) = R_2(x)s_2(x)/R$. From (18) we get

$$\frac{R_2(x)s_1(x)c_2}{R} = R_2(x)s_2(x)c_1 + q_3(x)c_3(x),$$

and so

$$q_3(x) = \frac{R_2(x)}{c_3}\left(s_1(x)c_2 - \frac{c_1 s_2(x)}{R}\right).$$

Since $q_1(x)s_1(x) = q_2(x)s_2(x)$ we get $q_2(x) = R_2(x)s_1(x)/R$. In short

$$\begin{aligned} p_1(x) &= \frac{R_2(x)}{R} s_1(x)s_2(x), \\ p_2(x) &= \frac{R_2(x)}{R} s_2(x)(s_1(x) + c_1), \\ p_3(x) &= \frac{R_2(x)}{R} s_1(x)(s_2(x) + c_2). \end{aligned}$$

We consider two cases:

2.1.2.1*: $s_1(x)$ and $s_2(x)$ are coprime.* In this case the maximum common divisor between $p_2(x)$ and $p_3(x)$ is $R_2(x)$ and so from (16) we get

$$\begin{aligned} B(x) &= \frac{R_2(x)}{R^2} s_2(x)(R_2(x)s_2(x))' s_1(x)(s_1(x) + c_1) \\ &= \frac{R_2(x)}{R^2} s_2(x)s_1(x)R_2'(x)(s_1(x) + c_1)(s_2(x) + c_2), \end{aligned}$$

and so

$$(R_2(x)s_2(x))' = R_2'(x)(s_2(x) + c_2),$$

that is

$$R_2'(x)s_2(x) + R_2(x)s_2'(x) = R_2'(x)s_2(x) + R_2'(x)c_2,$$

which yields

$$\frac{R_2'(x)}{R_2(x)} = \frac{s_2'(x)}{c_2}.$$

Hence $R_2(x) = e^{c_2 s_2(x)}$ which is not possible.

2.1.2.2*: $s_1(x)$ and $s_2(x)$ are not coprime.* In this case we write

$$s_1(x) = \kappa(x)\hat{s}_1(x), \quad s_2(x) = \kappa(x)\hat{s}_2(x),$$

with $\kappa(x), \hat{s}_1(x), \hat{s}_2(x) \in \mathbb{R}[x]$ with $\kappa(x) \notin \mathbb{R}$. Then

$$\begin{aligned} p_1(x) &= \frac{R_2(x)}{R} \kappa^2(x)\hat{s}_1(x)\hat{s}_2(x), \\ p_2(x) &= \frac{R_2(x)}{R} \kappa(x)\hat{s}_2(x)(\kappa(x)\hat{s}_1(x) + c_1), \\ p_3(x) &= \frac{R_2(x)}{R} \kappa(x)\hat{s}_1(x)(\kappa(x)\hat{s}_2(x) + c_2). \end{aligned}$$

Then

$$\begin{aligned} B(x) &= \frac{R_2(x)}{R^2}\kappa(x)(R_2(x)\kappa(x))'\hat{s}_2(x)(\kappa(x)\hat{s}_1(x)+c_1)\hat{s}_1(x)(\kappa(x)\hat{s}_2(x)+c_2) \\ &= \frac{R_2(x)}{R^2}\kappa(x)\hat{s}_2(x)(R_2(x)\kappa(x)\hat{s}_2(x))'\hat{s}_1(x)(\kappa(x)\hat{s}_1(x)+c_1)\kappa(x), \end{aligned}$$

and so

$$(R_2(x)\kappa(x))'(\kappa(x)\hat{s}_2(x)+c_2) = \kappa(x)(R_2(x)\kappa(x)\hat{s}_2(x))',$$

which yields

$$(R_2(x)\kappa(x))'c_2 = (R_2(x)\kappa(x))\kappa(x)\hat{s}_2(x).$$

This is not possible because the left hand side has less dimension than the right hand side. In summary, Subcase 2.1.2 is not possible.

Subcase 2.2*:* $R_2(x) \in \mathbb{R}[x] \setminus \mathbb{R}$. We have $q_1(x) = R_1(x)R_2(x)$ and $s_2(x) = R_1(x)R_3(x)$. Then

$$\frac{R_2(x)s_1(x)c_2}{R_3(x)} = R_1(x)R_2(x)c_1 + q_3(x)c_3.$$

In particular there exists $T(x) \in \mathbb{R}[x]$ so that

$$q_3(x) = R_2(x)T(x),$$

and so

$$\frac{s_1(x)c_2}{R_3(x)} = R_1(x)c_1 + T(x)c_3,$$

which yields $s_1(x) = R_4(x)R_3(x)$. Therefore, from $p_1(x)$ in (8) we get

$$q_2(x)s_2(x) = q_1(x)s_1(x) = R_1(x)R_2(x)R_3(x)R_4(x) = q_2(x)R_1(x)R_3(x)$$

and so

$$q_2(x) = R_2(x)R_4(x).$$

Hence we have

$$\begin{aligned} p_1(x) &= q_1(x)s_1(x) = R_1(x)R_2(x)R_3(x)R_4(x), \\ p_2(x) &= q_1(x)(s_1(x)+c_1) = R_1(x)R_2(x)(R_3(x)R_4(x)+c_1), \\ p_3(x) &= q_2(x)(s_2(x)+c_2) = R_2(x)R_4(x)(R_1(x)R_3(x)+c_2). \end{aligned}$$

We consider two cases.

2.2.1*: $R_1(x)$ and $R_4(x)$ are coprime.* We have

$$\begin{aligned} B(x) &= R_1(x)R_2(x)(R_1(x)R_2(x))'R_3(x)R_4(x)(R_3(x)R_4(x)+c_1) \\ &= R_2(x)R_2'(x)R_1(x)R_4(x)(R_3(x)R_4(x)+c_1)(R_1(x)R_3(x)+c_2), \end{aligned}$$

and so

$$(R_1(x)R_2(x))'R_3(x) = R_2'(x)(R_1(x)R_3(x)+c_2),$$

which yields

$$R_1'(x)R_2(x)R_3(x) = c_2R_2'(x).$$

This is not possible because the right hand side has less dimension than the left hand side.

2.2.2: *$R_1(x)$ and $R_4(x)$ are not coprime.* We write

$$R_1(x) = R(x)\hat{R}_1(x), \quad R_4(x) = R(x)\hat{R}_4(x)$$

where $R(x), \hat{R}_1(x), \hat{R}_4(x) \in \mathbb{R}[x]$ with $R(x) \notin \mathbb{R}$. Note that

$$\begin{aligned} p_1(x) &= q_1(x)s_1(x) = R^2(x)\hat{R}_1(x)R_2(x)R_3(x)\hat{R}_4(x), \\ p_2(x) &= q_1(x)(s_1(x)+c_1) = R(x)\hat{R}_1(x)R_2(x)(R_3(x)R(x)\hat{R}_4(x)+c_1), \\ p_3(x) &= q_2(x)(s_2(x)+c_2) = R_2(x)R(x)\hat{R}_4(x)(R(x)\hat{R}_1(x)R_3(x)+c_2). \end{aligned}$$

Then

$$\begin{aligned} B(x) &= (R(x)\hat{R}_1(x)R_2(x))'R(x)\hat{R}_1(x)R_2(x)R(x)R_3(x)\hat{R}_4(x)(R_3(x)R(x)\hat{R}_4(x)+c_1) \\ &= R(x)R_2(x)(R(x)R_2(x))'\hat{R}_1(x)\hat{R}_4(x)(R_3(x)R_4(x)+c_1)(R(x)\hat{R}_1(x)R_3(x)+c_2), \end{aligned}$$

and so

$$(R(x)\hat{R}_1(x)R_2(x))'R(x)R_3(x) = (R(x)R_2(x))'(R(x)\hat{R}_1(x)R_3(x)+c_2),$$

that is

$$\begin{aligned} &(R(x)R_2(x))'\hat{R}_1(x)R(x)R_3(x) + (R(x)R_2(x))\hat{R}_1'(x)R(x)R_3(x) \\ &= (R(x)R_2(x))'(R(x)\hat{R}_1(x)R_3(x) + c_2(R(x)R_2(x))', \end{aligned}$$

which yields

$$R(x)R_2(x)(\hat{R}_1(x))'R(x)R_3(x) = c_2(R(x)R_2(x))'.$$

This is not possible because the right hand side has less dimension than the left hand side. So subcase 2.2 is not possible.

In short from (22) there are at most three rational limit cycles and if they exist then there must exist a polynomial $S(x) = s_1(x)$ and two different constants $c_1, c_2 \in \mathbb{R} \setminus \{0\}$ such that from Lemma 3 we have $S(0) = S(1)$, and $S(x) \neq 0$, $S(x)+c_1 \neq 0$ and $S(x)+c_2 \neq 0$ for $x \in [0,1]$, and $B(x)$ is as in (2). Moreover the three limit cycles can be taken to be

$$\begin{aligned} p_1(x) &= S(x)(S(x)+c_1), \\ p_2(x) &= (S(x)+c_1)(S(x)+c_2), \\ p_3(x) &= S(x)(S(x)+c_2), \end{aligned}$$

which are the ones given in (3) in the statement of the theorem. Note that $y_i(x) = 1/p_i(x)$ for $i = 1,2,3$, satisfy (4) with $c = 1$ and that $p_i(0) = p_i(1)$ and $p_i(x) \neq 0$ for $x \in [0,1]$. Hence, in view of Lemma 3, the three solutions y_1, y_2, y_3 are three periodic solutions of (1). Moreover they are isolated and so they are limit cycles of (1). This concludes the proof of the theorem. $\square$

Now we provide an Abel equation (1) with three rational limit cycles. Indeed, taking $S(x) = x^2 - x + 1$, $c_1 = 1$ and $c_2 = 2$ we construct the Abel equation (1) with

$$A(x) = -3(-1+2x) - 3(-1+2x)(1-x+x^2),$$
$$B(x) = (-1+2x)(1-x+x^2)(2-x+x^2)(3-x+x^2).$$

Then system (1) has the three rational solutions $y_i(x) = 1/p_i(x)$ for $i = 1, 2, 3$ with

$$p_1(x) = (x^2-x+1)(x^2-x+2),$$
$$p_2(x) = (x^2-x+2)(x^2-x+3),$$
$$p_3(x) = (x^2-x+1)(x^2-x+3).$$

Note that $S(0) = S(1)$ and $S(x) \neq 0$ for $x \in [0,1]$. Moreover, $S(x)+1 = x^2-x+2 \neq 0$ for $x \in [0,1]$, and $S(x)+2 = x^2-x+3 \neq 0$ for $x \in [0,1]$. The Abel system that we have constructed has three rational limit cycles.

3. Proof of Theorem 2

To decide whether a periodic solution is a hyperbolic limit cycle we need the following lemma, whose proof can be found in [17].

Lemma 5. *If $y = 1/p(x)$ is a periodic solution of system* (1) *then it is a hyperbolic limit cycle if and only if $\int_0^1 (B(x)/p^2(x))\,dx \neq 0$.*

Now we can prove Theorem 2. In view of Theorem 1 the three limit cycles are $y_i = 1/p_i(x)$ with p_i given in (3) for $i = 1, 2, 3$. We will prove first that y_0 is not a hyperbolic limit cycle. In view of (3) since $p_1(x) = S(x)(S(x)+c_1)$, we have that

$$\frac{B(x)}{p_1^2(x)} = \frac{S'(x)(S(x)+c_2)}{S(x)(S(x)+c_1)}.$$

Note that we can write

$$\frac{S'(x)(S(x)+c_2)}{S(x)(S(x)+c_1)} = \frac{c_2}{c_1}\frac{S'(x)}{S(x)} + \frac{c_1-c_2}{c_1}\frac{S'(x)}{S(x)+c_1}.$$

Hence

$$\int_0^1 \frac{B(x)}{p_1^2(x)}\,dx = \frac{c_2}{c_1}\log S(x)\Big|_0^1 + \frac{c_1-c_2}{c_1}\log(S(x)+c_1)\Big|_0^1 = 0,$$

because $S(0) = S(1)$. So the periodic solution $y = 1/p_1(x)$ is not a hyperbolic limit cycle.

For $y = 1/p_2(x)$ with $p_2(x) = S(x)(S(x)+c_2)$, we have

$$\frac{B(x)}{p_2^2(x)} = \frac{S'(x)(S(x)+c_1)}{S(x)(S(x)+c_2)}$$

and the proof is the same as for $1/p_1(x)$ interchanging the roles of c_1 and c_2.

Finally, for the case of $y = 1/p_3(x)$ since $p_3(x) = (S(x)+c_1)(S(x)+c_2)$ it follows that
$$\frac{B(x)}{p_2^2(x)} = \frac{S'(x)S(x)}{(S(x)+c_1)(S(x)+c_2)}.$$
Note that we can write
$$\frac{S'(x)S(x)}{(S(x)+c_1)(S(x)+c_2)} = \frac{c_1}{c_1-c_2}\frac{S'(x)}{S(x)+c_1} - \frac{c_2}{c_1-c_2}\frac{S'(x)}{S(x)+c_2}.$$
Hence
$$\int_0^1 \frac{B(x)}{p_2^2(x)}\,dx = \frac{c_1}{c_1-c_2}\log(S(x)+c_1)\big|_0^1 - \frac{c_2}{c_1-c_2}\log(S(x)+c_2)\big|_0^1 = 0$$
because $S(0) = S(1)$. So, the periodic solution $y = 1/p_3(x)$ is not a hyperbolic limit cycle. This concludes the proof of the theorem.

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