CONFIGURATIONS OF THE TOPOLOGICAL INDICES OF THE PLANAR POLYNOMIAL DIFFERENTIAL SYSTEMS OF DEGREE $(2, m)$

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Abstract. Using the Euler-Jacobi formula there is a relation between the singular points of a polynomial vector field and their topological indices. Using this formula we obtain the configuration of the singular points together with their topological indices for the polynomial differential systems $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ with degree of P equal to 2 and degree of Q equal to m when these systems have $2m$ finite singular points.

1. Introduction and statement of the main results

Consider in $\mathbb{R}^2$ the polynomial differential system

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{1}$$

where $P(x, y)$ and $Q(x, y)$ are real polynomials of degrees 2 and m, respectively, or simply of degree $(2, m)$.

The motivation of our paper comes from the fact that for the planar quadratic polynomial differential systems (i.e. the ones of degree $(2, 2)$) the characterization of all configurations of the indices of the singular points of all systems that have four singular points is the well-known Berlinskii's Theorem proved in [2, 6] and reproved in [4] using the Euler-Jacobi formula. More precisely, the Berlinskii's Theorem can be stated as follows: *Assume that a real quadratic system has exactly four real singular points. In this case if the quadrilateral formed by these points is convex, then two opposite singular points are anti-saddles (i.e. nodes, foci or centers) and the other two are saddles. If this quadrilateral is not convex, then either the three exterior vertices are saddles and the interior vertex is an anti-saddle or the exterior vertices are anti-saddles and the interior vertex is a saddle.*

We want to extend the Berlinskii's Theorem from degree $(2, 2)$ to degree $(2, 2m)$ for all $m \geq 2$, i.e., we shall obtain all configurations of the polynomial differential system of degree $(2, m)$.

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Assuming that the differential system (1) has $2m$ finite singular points, then using the Euler-Jacobi formula we obtain a relation between the finite singular points of the polynomial differential system (1) and the topological indices of their singular points.

In the complex projective plane, and taking into account all the multiplicities of the singular points of a polynomial differential system, if the number of singular points is finite, then it is at most $2m$, see for more details the Bézout's Theorem (see [7] for a proof of this theorem). When all the singular points have multiplicity one and are located in the finite part of the projective space, we can apply the Euler-Jacobi formula (see [1] for a proof of such formula). For system (1) if the set of zeroes contains exactly $2m$ elements, then the Jacobian determinant

$$J = \begin{vmatrix} \partial P/\partial x & \partial P/\partial y \\ \partial Q/\partial x & \partial Q/\partial y \end{vmatrix}$$

evaluated at each zero does not vanish, and for any polynomial R of degree less than or equal to $m-1$ we have

$$\sum_{a \in A} \frac{R(a)}{J(a)} = 0. \tag{2}$$

We denote by A the set of finite singular points of system (1). Given a finite subset B of $\mathbb{R}^2$, we denote by $\hat{B}$ its convex hull, by ∂B its boundary, and by $\#B$ its cardinal.

Set $A_0 = A$ and for $i \geq 1$ $A_i = A_{i-1} \setminus (A_{i-1} \cap \partial \hat{A}_{i-1})$. There is an integer q such that $A_{q+1} = \emptyset$.

We say that A has the *configuration* $(K_0; K_1; K_2; \ldots; K_q)$ where $K_i = \#(A_i \cap \partial \hat{A}_i)$.

We are also interested in the study of the (topological) indices of the singular points of system (1). We say that the singular points of system (1) which belong to $A_i \cap \partial \hat{A}_i$ are on the *i-th level*.

We recall that if we assume that $\#A = 2m$ then the Jacobian determinant J is non-zero at any singular point of system (1), consequently the topological indices of the singular points are ± 1, and then the number K_i of the i-th level is substituted by the vector $(n_i^1+, n_i^2-, \ldots, n_i^{m_i-1}+, n_i^{m_i}-)$, where n_i^j are positive integers such that $\sum_j n_i^j = K_i$. More precisely, since $A_i \cap \partial \hat{A}_i$ is a convex polygon, these numbers take into account the number of consecutive points with positive and negative indices, viewing the ith level oriented counterclockwise: n_i^1 corresponds to the string with largest number of consecutive points with positive indices. If there are several strings with the same number of points we choose one such that the next string (that has points with negative indices) is as large as possible. We continue the process with n_i^2 and so on. Furthermore, when $A_i \cap \partial \hat{A}_i$ is a segment, the numbers take into account the number of consecutive points with positive indices, beginning at one of its endpoints.

We denote by $i(a)$ the index of a singular point $a \in A$ of system (1). To introduce the main results of the paper we introduce some notations

$$
\begin{aligned}
&4^{(k)} = (4;\ldots;4), \quad \Delta_- = (+,2-), \quad \Delta_+ = (2+,-), \quad \Gamma = (2+,2-),\\
&R^k = (+,-,\ldots,+,-), \quad O_+^k = (2+,-,+,-,+,\ldots,+,-),\\
&O_-^k = (+,2-,+,-,+,\ldots,+,-),
\end{aligned}
$$

where k is the length of the strings $4^{(k)}$, R^k and $O_\pm^k$. We take the convention that $O_-^1 = -$ and $O_+^1 = +$.

In our notation Berlinskii's Theorem corresponds to the case $m = 2$ and can be stated as follows.

Theorem 1 (Berlinskii's Theorem)**.** *For planar quadratic polynomial differential systems such that $\#A = 4$ the following statements hold:*

(a) $\sum_{a\in A} i(a) = 0$ *or* $|\sum_{a\in A} i(a)| = 2$.
(b) *If* $\sum_{a\in A} i(a) = 0$, *only the configuration* $(4) = R^4$ *is possible and there are examples of such configuration.*
(c) *If* $|\sum_{a\in A} i(a)| = 2$, *only the configurations* $(3) = (3+, O_-^1)$ *and* $(3) = (3-, O_+^1)$ *are possible and there are examples of such configurations.*

With the notation introduced above we can state the first main theorem in the paper that deals with $m \geq 3$ odd.

Theorem 2. *For planar polynomial differential systems of degree $(2, m)$ such that $\#A = 2m$ with $m \geq 3$ odd, the following statements hold.*

(a) $\sum_{a\in A} i(a) = 0$.
(b) *Only the following configurations are possible*
  (i) $(2m) = R^{2m}$;
  (ii) $(4^{(j)}; 2m-4j) = (\Gamma_1;\ldots;\Gamma_j; R^{2m-4j})$ *for* $j = 1,\ldots,(m-3)/2$ *where* $\Gamma_k = \Gamma$ *for* $k = 1,\ldots,j$;
  (iii) $(4^{(j)}; 3; 2m-4j-3) = (\Gamma_1;\ldots;\Gamma_j; \Delta_+, R^{2m-4j-3})$ *for j even and* $(4^{(j)}; 3; 2m-4j-3) = (\Gamma_1;\ldots;\Gamma_j; \Delta_-, R^{2m-4j-3})$ *for j odd, where* $j = 0,1,\ldots,(m-3)/2$ *and* $\Gamma_k = \Gamma$ *for* $k = 1,\ldots,j$.

There exist examples of all these configurations.

The proof of Theorem 2 is given in section 2. Note that the particular case $m = 3$ was proved in [4].

Now we introduce the second main result of the paper which deals with the case $m \geq 2$ even.

Theorem 3. *For planar polynomial differential systems of degree $(2, m)$ such that $\#A = 2m$ with $m \geq 2$ even, the following statements hold.*

(a) $\sum_{a\in A} i(a) = 0$ *or* $|\sum_{a\in A} i(a)| = 2$.

(b) *If* $\sum_{a\in A} i(a) = 0$*, only the following configurations are possible*

(i) $(2m) = R^{2m}$*;*

(ii) $(4^{(j)}; 2m-4j) = (R^4_1; \dots; R^4_j; R^{2m-4j})$ *for* $j = 1, \dots, (m-3)/2$ *where* $R^4_k = R$ *for* $k = 1, \dots, j$.

There exist examples of such configurations.

(c) *If* $|\sum_{a\in A} i(a)| = 2$*, only the following configurations are possible*

(iii) $(4^{(j)}; 3; 2m-4j-3) = (S_1+; S_2-; \dots; S_j\delta_j; 3(-\delta_j); O^{2m-4j-3}_-)$ *or* $(4^{(j)}; 3; 2m-4j-3) = (S_1-; S_2+; \dots; S_j(-\delta_j); 3\delta_j; O^{2m-4j-3}_+)$ *for* $j = 1, \dots, (m-3)/2$*, where* $S_k = 4$ *for* $k = 1, \dots, j$ *and* $\delta_j = +$ *if* j *is odd and* $\delta_j = -$ *if* j *is even.*

There exist examples of such configurations.

The proof of Theorem 3 is given in section 3. The particular cases $m = 2$ is the well-known Berinskii's Theorem and $m = 4$ was proved in [9].

2. Proof of Theorem 2

First of all we observe that if a configuration exists for a polynomial vector field X with degrees $(2, m)$ and $\#A = 2m$, then it is possible to construct the same configuration but interchanging the indices of the singular point, i.e. the points with index $+1$ become with index -1 and vice versa. For doing that it is enough to take $Y = (-P, Q)$ instead of $X = (P, Q)$.

In the proof of Theorem 2 we will denote by $\{p_1, \dots, p_{2m}\}$ the set of points of A if there is no information about their indices, by $\{p^+_1, \dots, p^+_k\}$ the set of points of A with positive index, and by $\{p^-_1, \dots, p^-_{2m-k}\}$ the set of points of A with negative index. Also we will denote by $L_{i,j}$ the straight line $L_{i,j}(x, y) = 0$ through the points p_i and p_j where $i \in \{1, \dots, 2m\}$, and by L_i a straight line through a point $p_i \in \partial\hat{A}$ such that for all $q \in A$ we have $L_i(A) \neq 0$ and it is zero only at q.

We will also use the following auxiliary result proved in [3].

Lemma 4. *Let* $X = (P, Q)$ *be a polynomial vector field with* $\max(\deg P, \deg Q) = n$*. If* X *has* n *singular points on a straight line* $L(x, y) = 0$*, this line is an isocline. If* X *has* $n+1$ *singular points on* $L(x, y) = 0$ *then this line is full of singular points.*

2.1. Proof of statement (a) of Theorem 2.

It was proved in [8] that in the case of polynomial vector fields of degree $(2, m)$ with m odd it holds that $\sum_{a\in A} i_X(a) = 0$. This proves statement (a).

2.2. Proof of statement (b) of Theorem 2.

By statement (a) we have m points with positive index and m points with negative one. Note that configurations $(2m-1; 1)$, $(2m-2; 2)$, $(2m-3; 3)$, $(2m-4; 4)$, $(2m-4; 3; 1)$,

$(2m-5,*), \dots, (5;*)$ are not possible because any convex hull of $2m-1, 2m-2, \dots, 5$ points on a conic has at most four points in the boundary of the convex hull except for the ellipse, but in the case of the ellipse cannot be points in the 1–level. Furthermore, the unique possible configurations of the form $(3;*)$ are $(3;2m-3)$. Indeed, since the polynomial P has degree 2, $P(x,y)=0$ is a conic and the $2m$ finite singular points of system (1) are on this conic. Therefore any real conic (ellipse, parabola, hyperbola, two parallel straight lines, two straight lines intersecting in a real point, one double straight line or one point) do not allow the configuration of the form $(3;*;*)$. In a similar manner a configuration of the form $(4;5;*)$, or $(4;4;5;*)$ or $(4;3;*;*)$. Clearly configurations of the form $(2;*)$ cannot occur because the $2m$ singular points would be on a straight line, and by Lemma 4 this straight line will be full of singular points, a contradiction. Moreover, configurations of the form $(1;*)$ have no meaning. In short, the unique possible configurations are $(2m)$, $(4^{(j)};2m-4j-4)$, $(3;2m-3)$ and $(4^{(j)};3;2m-4j-3)$ for $j=1,\dots,(m-3)/2$. Moreover we will show that each configuration mentioned above is realizable.

We will study each configuration separately.

Configuration $(2m)$: Assume that the subscripts of the points in A are in such a way that $p_1,\dots,p_{2m}$ are ordered in $\partial\hat{A}$ in counterclockwise sense. Also we consider the subscripts in $\mathbb{Z}/2m\mathbb{Z}$. Take

$$C_i(x,y)=\prod_{j=0}^{m-2} L_{i+2j,i+2j+1}, \quad \text{for } j=1,2,\dots,2m.$$

Then the Euler Jacobi formula applied to C_i yields

$$\frac{C_i(p_{i+2m})}{J(p_{i+2m})}+\frac{C_i(p_{i+2m+1})}{J(p_{i+2m+1})}=0.$$

Since all the points $p_1,\dots,p_{2m}$ are in an ellipse, the polynomial $C_i(x,y)$ has the same sign on the two points p_{i+2m} and p_{i+2m+1}. So $J(p_{i+2m})J(p_{i+2m+1})<0$ for all i. Hence the indices of p_i and p_{i+1} are different and the configuration of A must be $(2m)=(+,-,\dots,+,-)=R^{2m}$.

The configuration R^{2m} can be realized intersecting an ellipse with m parallel straight lines each one having two points in the ellipse.

Configurations $(4^{(j)};2m-4j-4)$ *for* $j=0,1,\dots,(m-3)/2$. We will only prove the cases $j=0$ and $j=1$ since the other cases are done in a similar manner. So, we study the cases $(4;2m-4)$ and $(4;4;2m-8)$.

Configuration $(4;2m-4)$. Clearly the configuration $(4;2m-4)$ only can be realizable being $P(x,y)=0$ a hyperbola or a conic formed by two straight lines which intersect.

First we prove it for the case when $P(x,y)=0$ is a hyperbola. Since all the singular points lie in a hyperbola and in the 1st-level of A we must have $2m-4$ points, it is clear that two points are in one branch of the hyperbola and the other $2m-2$ points in the other branch of the hyperbola. Denote by p_1, p_2 the points in one branch of the hyperbola (ordered in clockwise sense) and by $p_3, \ldots, p_{2m}$ the remaining points which are in the other branch of the hyperbola and ordered in counterclockwise sense. Applying the Euler-Jacobi formula to

$$C(x,y) = \prod_{j=0}^{m-2} L_{3+j,2m},$$

and taking into account the convex hull of p_1, p_2, p_3, p_4 and the convex hull of $p_5, \ldots, p_{2m}$ we get that p_1 and p_2 have different indices. Now applying again the Euler-Jacobi formula successively to

$$C_k(x,y) = L_{1,2} \prod_{j=0, j\neq k}^{m-2} L_{3+j,2m-j},$$

with $k=0,\ldots,m-2$ we get that p_{3+k} and p_{2m-k} have different indices for $k=0,\ldots,m-2$. Without loss of generality we can assume that $p_3=p_3^+$ and $p_{2m}=p_{2m}^-$. Moreover, applying the Euler-Jacobi formula to

$$C(x,y) = L_{2,2m} \prod_{j=1}^{m-2} L_{3+j,2m-j},$$

we get that p_1 and p_3 have the same index (because the number of straight lines is $m-2$ which is odd) and so $p_1=p_1^+$ and $p_2=p_2^-$. Finally, applying the Euler-Jacobi formula to

$$C_k(x,y) = L_{1,2} \prod_{j=0}^{k-1} L_{3+j,2m-j} \prod_{j=0}^{m-k-3} L_{5+j+k,2m-j-k},$$

for $k=0,\ldots,m-3$ we get that $p_k=p_k^+$ for k odd and $p_k=p_k^-$ for k even. Hence the unique possible configuration is $(2+,2-;+,-,\ldots,+,-)=(\Gamma, R^{2m-4})$.

If the conic is the product of two intersecting straight lines at the real point p, by Lemma 4 on each straight line we must have exactly m points, and the point p on every straight line separates 1 point from the other $m-1$ points of the same straight line. In this configuration we can repeat the arguments done on the hyperbola for obtaining the configuration (Γ, R^{2m-4}).

The configuration (Γ, R^{2m-4}) can be realized intersecting the hyperbola $x^2-y^2=1$ by the $m-1$ straight lines $x=-m, 2, 3, \ldots, m$.

Configuration $(4;4;2m-8)$. The configuration $(4;4;2m-8)$ only can be realized being $P(x,y)=0$ a hyperbola or a conic formed by two straight lines which intersect. Since the proof in both cases follow the same lines, we only do it for the hyperbola. In that case, observe that since all the singular points lie in a hyperbola and in the 1st-level and 2nd level of A we must have four points, it is clear that four points are in one branch of the hyperbola and the other $2m-4$ points in the other branch of the hyperbola. Denote by p_1, p_2, p_3, p_4 the points in one branch of the hyperbola ordered in clockwise sense and by $p_5, \ldots, p_{2m}$ the remaining points which are in the other branch of the hyperbola and ordered in counterclockwise sense. Applying the Euler-Jacobi formula to

$$C(x,y) = L_{1,4} \prod_{j=0}^{m-3} L_{5+j,2m-j} \text{ and } C(x,y) = L_{2,3} \prod_{j=0}^{m-3} L_{5+j,2m-j},$$

we get that p_2 and p_3 have different indices and p_1 and p_4 have different indices. Moreover, applying the Euler-Jacobi formula to

$$C(x,y) = L_{1,4} L_{2,3} \prod_{j=0, j\neq k}^{m-3} L_{5+j,2m-j}$$

with $k = 0, \ldots, m-3$ we get that p_{5+k} and p_{2m-k} have different indices. Without loss of generality we can assume that $p_5 = p_5^+$ and $p_{2m} = p_{2m}^-$. Applying the Euler-Jacobi formula to

$$C(x,y) = L_{4,2m} L_{2,3} \prod_{j=1}^{m-3} L_{5+j,2m-j},$$

we get that p_1 and p_5 have the same index (because the number of straight lines is $m-2$ which is odd). In short $p_1 = p_1^+$ and $p_4 = p_4^-$. Now we apply the Euler-Jacobi formula to

$$C(x,y) = L_{4,2m} L_{3,2m-1} \prod_{j=1}^{m-3} L_{5+j,2m-j-1},$$

with $L_{m+2,m+2}$ being a straight line through the point p_{m+2} which separates the points $\{p_1, p_2, p_3, p_4\}$ from the points $\{p_5, \ldots, p_{m+1}, p_{m+3}, \ldots, p_{2m}\}$. Doing so, we get that p_2 and p_5 must have different indices (because the number of straight lines is $m-3$ which is even) and so $p_2 = p_2^-$ and $p_3 = p_3^+$. Finally, applying the Euler-Jacobi formula to

$$C_k(x,y) = L_{1,4} L_{2,3} \prod_{j=0}^{k-1} L_{5+j,2m-j} \prod_{j=0}^{m-k-4} L_{7+j+k,2m-j-k},$$

for $k = 0, \ldots, m-4$ we get that $p_k = p_k^+$ for k odd and $p_k = p_k^-$ for k even. Hence, the unique possible configuration is $(2+, 2-; 2+, 2-; +, -, \ldots, +, -) = (\Gamma; \Gamma; R^{2m-8})$.

The configuration $(\Gamma; \Gamma; R^{2m-8})$ can be realized intersecting the hyperbola $x^2 - y^2 = 1$ with the straight lines $x = -3, -2, 2, 3, \ldots, m-1$.

Configurations $(4^{(j)}; 3; 2m-4j-3)$ *for* $j = 0, 1, \dots, (m-3)/2$. We will only prove the cases $j = 0$, $j = 1$ and $j = 2$ since the other cases are done in a similar manner. So, we study the cases $(3; 2m-3)$, $(4; 3; 2m-7)$ and $(4, 4, 3, 2m-11)$.

Configuration $(3; 2m-3)$. In this case it is clear that this configuration only can be realized being $P(x, y) = 0$ a hyperbola. Since all the singular points lie in a hyperbola and in the 1st-level of A we must have $2m-3$ points, it is clear that one point is in one branch of the hyperbola and the other $2m-1$ points in the other branch of the hyperbola. Denote by p_1 the point alone in one branch of the hyperbola and by $p_2, \dots, p_{2m}$ the remaining points which are in the other branch of the hyperbola ordered in counterclockwise sense. Applying the Euler-Jacobi formula to

$$C(x, y) = \prod_{j=0}^{m-2} L_{2+j,2m-j}$$

we get that p_1 and p_{m+1} have different indices. Now applying again the Euler-Jacobi formula successively to

$$C_k(x, y) = L_{1,m+1} \prod_{j=0, j\neq k}^{m-2} L_{2+j,2m-j},$$

with $k = 0, \dots, m-2$ we get that p_{2+k} and p_{2m-k} have the same index for $k = 0, \dots, m-2$. Without loss of generality we can assume that $p_2 = p_2^+$ and so $p_{2m} = p_{2m}^+$. Now applying the Euler-Jacobi formula to

$$C(x, y) = L_{m+1,m+2} \prod_{j=1}^{m-2} L_{2+j,2m-j+1},$$

we get that p_1 and p_2 have opposite indices (because the number of straight lines is $m-1$ which is even). So $p_1 = p_1^-$ and $p_{m+1} = p_{m+1}^+$. Now applying the Euler-Jacobi formula to

$$C_k(x, y) = L_{1,m+1} \prod_{j=0}^{k-1} L_{2+j,2m-j} \prod_{j=0}^{m-k-2} L_{4+j+k,2m-j-k},$$

for $k = 0, \dots, m-2$, where $L_{m+2,m+2}$ is a straight line through the point p_{m+2} that separates the point p_1 from $\{p_2, \dots, p_{m+1}, p_{m+3}, \dots, p_{2m}\}$, we get that $p_{2+k} = p_{2+k}^+$ and $p_{2m-k} = p_{2m-k}^+$ for k even and $p_{2+k} = p_{2+k}^-$ and $p_{2m-k} = p_{2m-k}^-$ for k odd. Hence, the unique possible configuration is $(2+, -; +, -, \dots, +, -) = (\Delta_+, R^{2m-3})$.

The configuration (Δ_+, R^{2m-3}) can be realized intersecting the hyperbola $x^2 - y^2 = 1$ with the straight lines $x = -1, 1, 2, \dots, m-1$.

Configuration $(4; 3; 2m-7)$. In this case it is clear that the configuration $(4; 3; 2m-7)$ only can be realized being $P(x, y) = 0$ a hyperbola. Since all the singular points lie in a hyperbola and in the 0-level of A we must have four points and in the first level we must have three points, it is clear that three points are in one branch of the hyperbola and the other $2m-3$ points in the other branch of the hyperbola. Denote by p_1, p_2, p_3 the points in one branch of the hyperbola ordered in clockwise sense and by $p_4, \ldots, p_{2m}$ the remaining points which are in the other branch of the hyperbola and ordered in counterclockwise sense. Applying the Euler-Jacobi formula to

$$C(x,y) = L_{1,3} \prod_{j=0}^{m-3} L_{4+j,2m-j},$$

we get that p_2 and p_{m+2} have different indices. Now applying again the Euler-Jacobi formula to

$$C(x,y) = L_{2,m+2} \prod_{j=0}^{m-3} L_{4+j,2m-j} \text{ and } C_k(x,y) = L_{2,m+2} L_{1,3} \prod_{j=0, j\neq k}^{m-3} L_{4+j,2m-j},$$

with $k = 0, \ldots, m-3$ we get that p_1 and p_3 have the same index and p_{4+k} and p_{2m-k} have the same indices for $k = 0, \ldots, m-3$. Without loss of generality we can assume that $p_4 = p_4^+$ and so $p_{2m} = p_{2m}^+$. Moreover, applying the Euler-Jacobi formula to

$$C(x,y) = L_{2,m+2} L_{3,2m} \prod_{j=1}^{m-3} L_{4+j,2m-j},$$

we get that p_1 and p_4 have opposite indices (because the number of straight lines is $m-1$ which is even). So $p_1 = p_1^-$ and $p_3 = p_3^-$. Now we apply successively the Euler-Jacobi formula to

$$C_k(x,y) = L_{1,3} L_{2,m+3} \prod_{j=0}^{k-1} L_{4+j,2m-j} \prod_{j=0}^{m-k-4} L_{6+j+k,2m-j-k},$$

for $k = 0, \ldots, m-4$ and we get that $p_k = p_k^-$ for k odd with $k \neq m+2$, and $p_k = p_k^+$ for k even with $k \neq 2$. Finally, applying the Euler-Jacobi formula to

$$C(x,y) = L_{1,3} L_{2,m+1} \prod_{j=0}^{m-4} L_{4+j,2m-j},$$

we get that $p_{m+2} = p_{m+2}^-$ and $p_2 = p_2^+$. In short the configuration is $(2+, 2-; +, 2-; +, -, \ldots, +, -) = (\Gamma, \Delta_-, R^{2m-7})$.

The configuration $(\Gamma, \Delta_-, R^{2m-7})$ can be realized intersecting the hyperbola $x^2 - y^2 = 1$ with the straight lines $x = -2, -1, 1, 2, \ldots, m-2$.

Configuration $(4; 4; 3; 2m-11)$. As before, in this case it is clear that the configuration $(4; 4; 3; 2m-11)$ only can be realized being $P(x,y) = 0$ a hyperbola. Since all the singular points lie in a hyperbola and in the 0-level and 1-st level of A we must have four points and in the 2nd level of A we must have three points, it is clear that five points are in one branch of the hyperbola and the other $2m-5$ points in the other branch of the hyperbola. Denote by p_1, p_2, p_3, p_4, p_5 the points in one branch of the hyperbola ordered in clockwise sense and by $p_6, \ldots, p_{2m}$ the remaining points which are in the other branch of the hyperbola and ordered in counterclockwise sense. Applying the Euler-Jacobi formula to

$$C(x,y) = L_{1,5}L_{2,4}\prod_{j=0}^{m-4} L_{6+j,2m-j},$$

we get that p_3 and p_{m+3} have different indices. Now applying again the Euler-Jacobi formula to

$$C(x,y) = L_{3,m+3}L_{1,5}\prod_{j=0}^{m-4} L_{6+j,2m-j}, \quad C(x,y) = L_{3,m+3}L_{2,4}\prod_{j=0}^{m-4} L_{6+j,2m-j}$$

and

$$C_k(x,y) = L_{3,m+3}L_{1,5}L_{2,4}\prod_{j=0, j\neq k}^{m-4} L_{6+j,2m-j},$$

with $k = 0, \ldots, m-4$ we get that p_2 and p_4 have the same indices, p_1 and p_5 have the same indices and p_{6+k} and p_{2m-k} have the same indices for $k = 0, \ldots, m-4$. Without loss of generality we can assume that $p_6 = p_6^+$ and so $p_{2m} = p_{2m}^+$. Moreover, applying the Euler-Jacobi formula to

$$C(x,y) = L_{3,m+3}L_{5,2m}L_{2,4}\prod_{j=1}^{m-4} L_{6+j,2m-j},$$

we get that p_1 and p_6 have opposite indices (because the number of straight lines is $m-3$ which is even). So $p_1 = p_1^-$ and $p_5 = p_5^-$. Now we apply the Euler-Jacobi formula to

$$C(x,y) = L_{5,2m}L_{4,2m-1}L_{3,m+2}\prod_{j=0}^{m-5} L_{6+j,2m-j-2},$$

and we get that p_1 and p_2 have opposite indices. So $p_2 = p_2^+$ and $p_4 = p_4^+$. Now we apply the Euler-Jacobi formula to

$$C(x,y) = L_{1,5}L_{2,4}L_{3,m+4}\prod_{j=0}^{k-1} L_{6+j,2m-j}\prod_{j=0}^{m-k-5} L_{8+j+k,2m-j-k},$$

and we get that $p_3 = p_3^-$, $p_k = p_k^-$ for $k \geq 7$ odd with $k \neq m+4$ and $p_k = p_k^+$ for $k \geq 6$ even. Finally, applying the Euler-Jacobi formula to

$$C(x,y) = L_{1,5}L_{2,4}L_{3,m+2}\prod_{j=0}^{m-5} L_{6+j,2m-j},$$

we obtain that p_{m+4} and p_{m+3} have opposite indices, and so $p_{m+4} = p^-_{m+4}$. In short the unique possible configuration is

$$(2+, 2-; 2+, 2-; 2+, -; +, -, \dots, +, -) = (\Gamma, \Gamma, \Delta_+, R^{2m-11}).$$

The configuration $(\Gamma, \Gamma, \Delta_+, R^{2m-11})$ can be realized intersecting the hyperbola $x^2 - y^2 = 1$ with the straight lines $x = -3, -2, -1, 1, 2, \dots, m-3$.

3. Proof of Theorem 3

3.1. **Proof of statement (a) of Theorem 3.** It was proved in [8] that in the case of polynomial vector fields of degree $(2, m)$ with m even it holds that $\sum_{a\in A} i_X(a) = 0$ or $|\sum_{a\in A} i_X(a)| = 2$. This proves statement (a).

3.2. **Proof of statements (b) and (c) of Theorem 3.** Proceeding as in the same manner as in the proof of Theorem 2 we have that the only possible configurations are $(2m)$, $(4^{(j)}; 2m-4j-4)$, $(3; 2m-3)$ and $(4^{(j)}; 3; 2m-4j-3)$ for $j = 1, \dots, (m-3)/2$.

We will study each one of the configurations separately.

Configuration $(2m)$: Proceeding exactly in the same way as in the proof of Theorem 2 we get that the unique possible configuration is $(+, -, \dots, +, -) = R^{2m}$. Again, the configuration R^{2m} can be realized intersecting an ellipse with m parallel straight lines each one having two points in the ellipse.

Configurations $(4^{(j)}; 2m-4j)$ *for* $j = 1, \dots, (m-3)/2$. We will only prove the cases $j = 1$ and $j = 2$ since the other cases are done in a similar manner.

Configuration $(4; 2m-4)$. We take the same notation as in the proof of the configuration $(4; 2m-4)$ in Theorem 2 and proceeding exactly as in that proof we get that p_1 and p_2 have different indices and p_{3+j} and p_{2m-j} have also different indices for $j = 0, \dots, m-2$. Without loss of generality we can assume that $p_3 = p_3^+$ and $p_{2m} = p^-_{2m}$. Moreover, applying the Euler-Jacobi formula to

$$C(x, y) = L_{2,2m} \prod_{j=1}^{m-2} L_{3+j,2m-j},$$

we get that p_1 and p_3 have opposite index (because the number of straight lines is $m-2$ which is even) and so $p_1 = p_1^-$ and $p_2 = p_2^+$. Now proceeding as in the final part of the proof of Theorem 2 we get that $p_j = p_j^+$ for j odd and $p_j = p_j^-$ for j even. Hence, the unique possible configuration is $(R^4; +, -, \dots, +, -) = (R^4, R^{2m-4})$.

Note that here $\sum_{a\in A} i(a) = 0$. The configuration (R^4, R^{2m-4}) can be realized intersecting the hyperbola $x^2 - y^2 = 1$ by the $m-1$ straight lines $x = -m, 2, 3, \dots, m$.

Configuration $(4;4;2m-8)$. We take the same notation as the proof of the configuration $(4;4;2m-8)$ in Theorem 2. Proceeding as there we get that p_1 and p_4 have different indices, p_2 and p_3 have different indices and p_{5+j} and p_{2m-j} have different indices. Without loss of generality we can assume that $p_5=p_5^+$ and $p_{2m}=p_{2m}^-$. Applying the Euler-Jacobi formula to

$$C(x,y)=L_{4,2m}L_{2,3}\prod_{j=1}^{m-3}L_{5+j,2m-j},$$

we get that p_1 and p_5 have opposite index (because the number of straight lines is $m-2$ which is even). In short $p_1=p_1^-$ and $p_4=p_4^+$. Now applying the Euler-Jacobi formula to

$$C(x,y)=L_{4,2m}L_{3,2m-1}\prod_{j=0}^{m-4}L_{5+j,2m-j-2},$$

we get that p_1 and p_2 must have opposite indices and so $p_2=p_2^+$ and $p_3=p_3^-$. Now proceeding as in the last part of the proof of this configuration in Theorem 2 we get that $p_j=p_j^+$ for j odd and $p_j=p_j^-$ for j even. Hence, the unique possible configuration is $(R^4;R^4;+,-,\dots,+,-)=(R^4;R^4;R^{2m-8})$.

Note that here $\sum_{a\in A}i(a)=0$. The configuration $(R^4;R^4;R^{2m-8})$ can be realized intersecting the hyperbola $x^2-y^2=1$ with the straight lines $x=-3,-2,2,3,\dots,m-1$.

This proves statement (b) of the Theorem.

Configurations $(4^{(j)};3;2m-4j-3)$ *for* $j=0,1,\dots,(m-3)/2$. We will only prove the cases $j=0$, $j=1$ and $j=2$ since the other cases are done in a similar manner. So, we study the cases $(3;2m-3)$, $(4;3;2m-7)$ and $(4,4,3,2m-11)$.

Configuration $(3;2m-3)$: We take the same notation as in the proof of the configuration $(3;2m-3)$ in Theorem 2 and proceeding as in the first part of that proof we get that p_1 and p_{m+1} have different index, and p_{2+j} and p_{2m-j} have the same index for $j=0,\dots,m-2$. Without loss of generality we can assume that $p_2=p_2^+$ and so $p_{2m}=p_{2m}^+$. Moreover, applying the Euler-Jacobi formula to

$$C(x,y)=L_{m+1,m+2}\prod_{j=1}^{m-2}L_{2+j,2m-j+1},$$

we get that p_1 and p_2 have the same indices (because the number of straight lines is $m-1$ which is odd). So $p_1=p_1^+$ and $p_{m+1}=p_{m+1}^-$. Now proceeding as in the last part of the proof of Theorem 2 we get that $p_{2+j}=p_{2+j}^+$ and $p_{2m-j}=p_{2m-j}^+$ for j even and $p_{2+j}=p_{2+j}^-$ and $p_{2m-j}=p_{2m-j}^-$ for j odd. Hence, the unique possible configuration is $(3+;+,2-,+,-,\dots,+,-)=(3+;O_-^{2m-3})$.

Note that here $\sum_{a\in A} i(a) = 2$, so when $\sum_{a\in A} i(a) = -2$ we would have the configuration $(3-;O_{+}^{2m-3})$. The configuration $(3+;O_{-}^{2m-3})$ can be realized intersecting the hyperbola $x^2 - y^2 = 1$ with the straight lines $x = -1, 1, 2, \dots, m-1$.

Configuration $(4;3;2m-7)$. We use the notation in the configuration $(4;3;2m-7)$ in the proof of Theorem 2. Proceeding as there we get that p_2 and p_{m+2} have different indices, p_1 and p_3 have the same index and p_{4+j} and p_{2m-j} have the same index for $j = 0, \dots, m-2$. Without loss of generality we can assume that $p_4 = p_4^+$ and so $p_{2m} = p_{2m}^+$. Moreover, applying the Euler-Jacobi formula to

$$C(x,y) = L_{2,m+2}L_{3,2m}\prod_{j=1}^{m-3} L_{4+j,2m-j},$$

we get that p_1 and p_4 have the same indices (because the number of straight lines is $m-1$ which is odd). So $p_1 = p_1^+$ and $p_3 = p_3^+$. Now proceeding as in the last part of the proof of this configuration in Theorem 2 we get that $p_{4+j} = p_{4+j}^+$ and $p_{2m-j} = p_{2m-j}^+$ for j even and $p_{4+j} = p_{4+j}^-$ and $p_{2m-j} = p_{2m-j}^-$ for j odd. In short the configuration is $(4+;3-;2+,-,+,-,\dots,+,-) = (4+,3-,O_{+}^{2m-7})$.

Note that here $\sum_{a\in A} i(a) = 2$, so when $\sum_{a\in A} i(a) = -2$ we would have the configuration $(4-,3+,O_{-}^{2m-7})$. The configuration $(4+,3-,O_{+}^{2m-7})$ can be realized intersecting the hyperbola $x^2 - y^2 = 1$ with the straight lines $x = -2,-1,1,2,\dots,m-2$.

Configuration $(4;4;3;2m-11)$. We use the notation in the configuration $(4;3;2m-7)$ in the proof of Theorem 2. Proceeding as there we get that p_3 and p_{m+3} have different indices, p_1 and p_5 have the same index, p_2 and p_4 have the same index and p_{6+j} and p_{2m-j} have the same index for $j = 0, \dots, m-2$. Without loss of generality we can assume that $p_6 = p_6^+$ and so $p_{2m} = p_{2m}^+$. Moreover, applying the Euler-Jacobi formula to

$$C(x,y) = L_{3,m+3}L_{5,2m}L_{2,4}\prod_{j=1}^{m-4} L_{6+j,2m-j},$$

we get that p_1 and p_6 have the same index (because the number of straight lines is $m-2$ which is even). So $p_1 = p_1^+$ and $p_5 = p_5^+$. Now, proceeding as in the last part of the proof of configuration $(4;4;3;2m-11)$ in Theorem 2 we get that $p_2 = p_2^-$, $p_4 = p_4^-$, $p_{6+j} = p_{6+j}^+$ and $p_{2m-j} = p_{2m-j}^+$ for j even and $p_{6+j} = p_{6+j}^-$ and $p_{2m-j} = p_{2m-j}^-$ for j odd, $p_3 =_3^+$ and $p_{m+3} = p_{m+3}^-$. In short the unique possible configuration is $(4+;4-;3+;O_{-}^{2m-11})$.

Note that here $\sum_{a\in A} i(a) = 2$, so when $\sum_{a\in A} i(a) = -2$ we would have the configuration $(4-;4+;3-;O_{+}^{2m-11})$. Moreover, the configuration

$(4+;4-;3+;O_+^{2m-11})$ can be realized intersecting the hyperbola $x^2-y^2=1$ with the straight lines $x=-3,-2,-1,1,2,\ldots,m-3$.

This proves statement (c) of the theorem and concludes the proof of it.

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