# WEIERSTRASS INTEGRABILITY OF COMPLEX DIFFERENTIAL EQUATIONS

JAUME LLIBRE[1] AND CLÀUDIA VALLS[2]

Abstract. We characterize the complex differential equations of the form

$$\frac{dy}{dx} = a_n(x)y^n + a_{n-1}(x)y^{n-1} + \cdots + a_1(x)y + a_0(x),$$

where $a_j(x)$ are meromorphic functions in the variable $x$ for $j = 0, \ldots, n$ that admit either a Weierstrass first integral or a Weierstrass inverse integrating factor.

## 1. Introduction and statement of the main results

Let $x$ and $y$ be complex variables. In this paper we study the differential equations of the form

$$\frac{dy}{dx} = a_n(x)y^n + a_{n-1}(x)y^{n-1} + \cdots + a_1(x)y + a_0(x) \quad \text{with} \quad a_n(0) \not\equiv 0, \tag{1}$$

where $a_j(x)$ are meromorphic functions of $x$ for $j = 0, \ldots, n$. In particular, the differential equation (1) contains the well-known *Abel differential equations* when $n = 3$, the *Riccati differential equations* when $n = 2$, and the *linear differential equations* when $n = 1$.

In what follows instead of working with the differential equation (1) we shall work with the equivalent differential system

$$\dot{x} = 1, \quad \dot{y} = a_n(x)y^n + a_{n-1}(x)y^{n-1} + \cdots + a_1(x)y + a_0(x) \quad \text{with} \quad a_n(0) \not\equiv 0, \tag{2}$$

in $\mathbb{C}^2$, where the dot denotes derivative with respect to the time $t$, real or complex.

The objective of this paper is to study the integrability of the differential equations (1) restricted to a special kind of first integrals. For such systems the notion of integrability is based on the existence of a first integral, and we want to characterize when the differential equations (1) have either a Weierstrass first integral or a Weierstrass inverse integrating factor.

When one studies the integrability of a differential system, the first class of functions to look for first integral is the polynomial functions. Then, one can go a step further and try to look for analytic first integrals. Usually, this is a very hard task and instead of this, one studies the first integrals that can be described by formal series. The use of formal series in the study of differential equations is a classical tool (see for instance [4], where the author used formal series to prove the Dulac's conjecture). Here, guided by the fact that the equations in (1) are polynomial in the variable $y$, we study the first integrals that are polynomials in the variable $y$ and formal series in the variable $x$, called Weierstrass first integrals.

---

2010 *Mathematics Subject Classification.* Primary 34C05, 34A34, 34C14.

*Key words and phrases.* Weierstrass first integrals, Weierstrass inverse integrating factor, complex differential equations.

The integrability of the Abel differential equations (i.e. of equation (1) with $n = 3$) has been studied by several authors, see [2, 3, 6, 7] to cite just a few. For instance in [2], [3] and [6], the authors provide a list of the known integrable Abel differential equations with $a_j(x)$ for $j = 0, 1, 2, 3$ rational functions, and in [8] the Weierstrass integrability of the Abel differential equations has been characterized.

As usual $\mathbb{C}[[x]]$ is the ring of formal power series in the variable $x$ with coefficients in $\mathbb{C}$, and $\mathbb{C}[y]$ is the ring of polynomials in the variable $y$ with coefficients in $\mathbb{C}$. A function of the form

$$\sum_{i=0}^{n} w_i(x)y^i \in \mathbb{C}[[x]][y], \tag{3}$$

is called a *formal Weierstrass polynomial* in $y$ of degree $n$ if and only if $w_n(x) = 1$ and $w_i(0) = 0$ for $i < n$. A formal Weierstrass polynomial whose coefficients are convergent is called *Weierstrass polynomial*, see [1].

Let $V : W \to \mathbb{C}$ be a function satisfying

$$\begin{aligned}&\frac{\partial V}{\partial x} + \frac{\partial V}{\partial y}(a_n(x)y^n + a_{n-1}(x)y^{n-1} + \cdots + a_1(x)y + a_0(x))\\ &= (na_n(x)y^{n-1} + (n-1)a_{n-1}(x)y^{n-2} + \cdots + a_1(x))V.\end{aligned}$$

Then $V$ is an *inverse integrating factor* of system (2), and it is known that there exists a first integral $H$ such that

$$\frac{1}{V} = \frac{\partial H}{\partial y}, \quad \frac{a_n(x)y^n + a_{n-1}(x)y^{n-1} + \cdots + a_1(x)y + a_0(x)}{V} = -\frac{\partial H}{\partial x}.$$

We say that a differential system (2) is *Weierstrass integrable* if it admits a first integral or an inverse integrating factor which is a Weierstrass polynomial. We recall that an analytic first integral $H : U \to \mathbb{C}$ of system (2) where $U$ is an open subset of $\mathbb{C}^2$ is a non-locally constant analytic function such that it is constant on the solution of system (2) contained in $U$. In [5] the definition of Weierstrass integrability is given in a more general context.

The main objective of this paper is to provide the differential equations (2) that have Weierstrass first integrals, or Weierstrass integrating factors. More precisely: *How to recognize functions $a_j(x)$, for $j = 0, 1, \ldots, n$ for which the differential equation* (1) *is Weierstrass integrable*?

Our first result is the following.

**Proposition 1.** *System* (2) *has no Weierstrass first integrals.*

The proof of Proposition 1 is given in section 2.

We look for inverse integrating factors given by the Weierstrass polynomial

$$V = y^s + V_{s-1}(x)y^{s-1} + \cdots + V_1(x)y + V_0(x) = \sum_{i=0}^{s} V_i(x)y^i \tag{4}$$

with $V_s(x) = 1$. Then our main result is the following.

**Theorem 2.** *System* (2) *admits a Weierstrass inverse integrating factor of the form* (4) *if and only if*

$$a_{n-k}(0) = 0 \quad \textit{for} \quad k = 1, \ldots, n-1,$$

$$a_0(0) = \frac{1}{n a_n(0)^2}(a_{n-1}(0)a_n'(0) - a_{n-1}'(0)a_n(0)),$$

$$a_{n-k}(x) = \frac{\prod_{l=2}^{k}(n-l+1)}{n^{k-1}k!}\frac{a_{n-1}(x)^k}{a_n(x)^{k-1}}, \quad \textit{for } k = 2, \dots, n-1,$$

*and*

$$a_0(x) = -\frac{1}{n}\frac{d}{dx}\frac{a_{n-1}(x)}{a_n(x)} + \frac{\prod_{l=2}^{n-1}(n-l+1)}{n^{n-1}n!}\frac{a_{n-1}(x)^n}{a_n(x)^{n-1}}.$$

*In this case the inverse Weierstrass integrating factor is*

$$V = y^n + \sum_{i=0}^{n-1} V_i(x)y^i$$

*with*

$$V_i(x) = \frac{a_i(x)}{a_n(x)} \quad \textit{for} \quad i = 1, \dots, n-1$$

*and*

$$V_0(x) = \frac{a_0(x)}{a_n(x)} + \frac{1}{n a_n(x)}\frac{d}{dx}\frac{a_{n-1}(x)}{a_n(x)}.$$

The proof of Theorem 2 is given in section 2.

## 2. Proof of the results

*Proof of Proposition 1.* Imposing that system (2) has a first integral given by the Weierstrass polynomial

$$H = y^s + H_{s-1}(x)y^{s-1} + \cdots + H_1(x)y + H_0(x) = \sum_{i=0}^{s} H_i(x)y^i$$

(with $H_s(x) = 1$) we obtain a polynomial in $y$ whose coefficients must be zero. Hence we get that

$$\sum_{i=0}^{s-1} H_i'(x)y^i + \sum_{i=0}^{s} iH_i(x)y^{i-1}(a_n(x)y^n + a_{n-1}(x)y^{n-1} + \cdots + a_1(x)y + a_0(x)) = 0. \tag{5}$$

The highest power in (5) of the variable $y$ is $s+n-1$ and its coefficient is $sa_n(x) = 0$. Since $a_n(x) \neq 0$ we get $s = 0$. So, from the definition of Weierstrass polynomial, $H = H_0(x) = 1$ in contradiction with the fact that a first integral $H$ cannot be constant. This completes the proof. □

Now we shall give some preliminary results that we shall need for proving Theorem 2.

Imposing that system (2) has an inverse integrating factor of the form (4) we obtain a polynomial in $y$ whose coefficients must be zero. Hence we get that

$$\begin{aligned} &\sum_{i=0}^{s} V_i'(x)y^i + \sum_{i=0}^{s} iV_i(x)y^{i-1}(a_n(x)y^n + a_{n-1}(x)y^{n-1} + \cdots + a_1(x)y + a_0(x)) \\ &= (na_n(x)y^{n-1} + (n-1)a_{n-1}(x)y^{n-2} + \cdots + 2a_2(x)y + a_1(x))\Big(\sum_{i=0}^{s} V_i(x)y^i\Big). \end{aligned} \tag{6}$$

Now, computing the coefficients in (6) of $y^{s+n-1}$ we get $sV_s(x)a_n(x) = nV_s(x)a_n(x)$. So we have that $sa_n(0) = na_n(0)$. Since $a_n(0) \neq 0$, it follows that $s = n$.

**Lemma 3.** *Equation* (6) *can be written as*

$$\sum_{i=0}^{n-1} V_i'(x)y^i + a_0(x)\sum_{i=0}^{n-1}(i+1)V_{i+1}(x)y^i + \sum_{l=0}^{2n-2} y^l \sum_{i=\max\{0,l+1-n\}}^{\min\{l,n\}} (2i-1-l)V_i(x)a_{l-i+1}(x). \tag{7}$$

*Proof.* Since $V_n(x) = 1$ and $s = n$, equation (6) can be written as

$$\begin{aligned}
0 &= \sum_{i=0}^{n-1} V_i'(x)y^i + a_0(x)\sum_{i=1}^{n} iV_i(x)y^{i-1} \\
&\quad + \sum_{i=0}^{n} iV_i(x)y^{i-1}(a_n(x)y^n + a_{n-1}(x)y^{n-1} + \cdots + a_1(x)y) \\
&\quad - (na_n(x)y^{n-1} + (n-1)a_{n-1}(x)y^{n-2} + \cdots + 2a_2(x)y + a_1(x))\Big(\sum_{i=0}^{n} V_i(x)y^i\Big) \\
&= \sum_{i=0}^{n-1} V_i'(x)y^i + a_0(x)\sum_{i=0}^{n-1}(i+1)V_{i+1}(x)y^i \\
&\quad + \sum_{i=0}^{n} iV_i(x)y^{i-1}(a_n(x)y^n + a_{n-1}(x)y^{n-1} + \cdots + a_1(x)y) \\
&\quad - \sum_{i=0}^{n} V_i(x)y^{i-1}(na_n(x)y^n + (n-1)a_{n-1}(x)y^{n-1} + \cdots + a_1(x)y) \\
&= \sum_{i=0}^{n-1} V_i'(x)y^i + a_0(x)\sum_{i=0}^{n-1}(i+1)V_{i+1}(x)y^i \\
&\quad + \sum_{i=0}^{n} V_i(x)y^{i-1}((i-n)a_n(x)y^n + (i-n+1)a_{n-1}(x)y^{n-1} + \cdots + (i-1)a_1(x)y).
\end{aligned} \tag{8}$$

We can write the last sum in (8) as

$$\begin{aligned}
&\sum_{i=0}^{n} V_i(x)y^{i-1}((i-n)a_n(x)y^n + (i-n+1)a_{n-1}(x)y^{n-1} + \cdots + (i-1)a_1(x)y) \\
&= \sum_{j=0}^{n-1}\sum_{i=0}^{n}(i-n+j)V_i(x)a_{n-j}(x)y^{n+i-1-j} \\
&= \sum_{l=0}^{2n-2} y^l \sum_{i=\max\{0,l+1-n\}}^{\min\{l,n\}} (2i-1-l)V_i(x)a_{l-i+1}(x).
\end{aligned} \tag{9}$$

Now, the proof follows immediately from (8) and (9). $\square$

Let

$$S_{n,l}(x) = \sum_{j=1}^{l-1}(l-2j)a_{n-j}(x)a_{n-l+j}(x). \tag{10}$$

Note that $l \le n+1$.

**Lemma 4.** *We have* $S_{n,l}(x) = 0$.

*Proof.* We consider two cases.

*Case 1: $l$ even.* In this case $l = 2m$ and $S_{n,l}(x) = S_{n,2m}(x)$ becomes

$$\begin{aligned}
&2\sum_{j=1}^{2m-1}(m-j)a_{n-j}(x)a_{n-2m+j}(x)\\
&= 2\sum_{j=1}^{m-1}(m-j)a_{n-j}(x)a_{n-2m+j}(x) + 2\sum_{j=m+1}^{2m-1}(m-j)a_{n-j}(x)a_{n-2m+j}(x)\\
&= 2\sum_{l=m+1}^{2m-1}(l-m)a_{n-2m+l}(x)a_{n-l}(x) + 2\sum_{j=m+1}^{2m-1}(m-j)a_{n-j}(x)a_{n-2m+j}(x)\\
&= 0.
\end{aligned}$$

*Case 2: $l$ odd.* In this case $l = 2m+1$ and $S_{n,l}(x) = S_{n,2m+1}(x)$ becomes

$$\begin{aligned}
&\sum_{j=1}^{2m}(2m+1-2j)a_{n-j}(x)a_{n-2m-1+j}(x)\\
&= \sum_{j=1}^{m}(2m+1-2j)a_{n-j}(x)a_{n-2m-1+j}(x)\\
&\quad + \sum_{j=m+1}^{2m}(2m+1-2j)a_{n-j}(x)a_{n-2m-1+j}(x)\\
&= -\sum_{l=m+1}^{2m}(2m+1-2l)a_{n-2m-1+l}(x)a_{n-l}(x)\\
&\quad + \sum_{j=m+1}^{2m}(2m+1-2j)a_{n-j}(x)a_{n-2m-1+j}(x) = 0.
\end{aligned}$$

This concludes the proof of the lemma. □

**Lemma 5.** *For $k = 1, \dots, n-1$ we have that*

$$V_{n-k}(x) = \frac{a_{n-k}(x)}{a_n(x)}. \tag{11}$$

*Proof.* We compute in (6) with $s = n$ the coefficients of $y^l$ for $l = n, \dots, 2n-2$. Note that in this case, by Lemma 3 (see (7)) for $l = n, \dots, 2n-2$ we have

$$\sum_{i=l+1-n}^{n}(2i-1-l)V_i(x)a_{l-i+1}(x) = 0. \tag{12}$$

For $k = 1, \dots, n-1$ we write $l = 2n-1-k$ and (12) becomes

$$0 = \sum_{i=n-k}^{n}(2i-2n+k)V_i(x)a_{2n-k-i}(x) = \sum_{j=0}^{k}(k-2j)V_{n-j}(x)a_{n-k+j}(x), \tag{13}$$

where in the last sum we have taken $j = n - i$. We rewrite (13) as (recall that $V_n(x) \equiv 1$)

$$\begin{aligned} 0 &= kV_n(x)a_{n-k}(x) - kV_{n-k}(x)a_n(x) + \sum_{j=1}^{k-1}(k-2j)V_{n-j}(x)a_{n-k+j}(x) \\ &= k\big(V_n(x)a_{n-k}(x) - V_{n-k}(x)a_n(x)\big) + \sum_{j=1}^{k-1}(k-2j)V_{n-j}(x)a_{n-k+j}(x) \\ &= k\big(a_{n-k}(x) - V_{n-k}(x)a_n(x)\big) + \sum_{j=1}^{k-1}(k-2j)V_{n-j}(x)a_{n-k+j}(x). \end{aligned} \tag{14}$$

Now we proceed by backwards induction. For $k = 1$ in (14) we have

$$a_{n-1}(x) - V_{n-1}(x)a_n(x) = 0 \quad \text{that is} \quad V_{n-1}(x) = \frac{a_{n-1}(x)}{a_n(x)},$$

which proves (11) for $k = 1$. Now assume that (11) is true for $k = 1, \ldots, l-1$ and we shall prove it for $k = l$. Thus, using the induction hypothesis in (14) we get

$$\begin{aligned} 0 &= l\big(a_{n-l}(x) - V_{n-l}(x)a_n(x)\big) + \sum_{j=1}^{l-1}(l-2j)V_{n-j}(x)a_{n-l+j}(x) \\ &= l\big(a_{n-l}(x) - V_{n-l}(x)a_n(x)\big) + \sum_{j=1}^{l-1}(l-2j)\frac{a_{n-j}(x)}{a_n(x)}a_{n-l+j}(x) \\ &= l\big(a_{n-l}(x) - V_{n-l}(x)a_n(x)\big) + \frac{1}{a_n(x)}\sum_{j=1}^{l-1}(l-2j)a_{n-j}(x)a_{n-l+j}(x) \\ &= l\big(a_{n-l}(x) - V_{n-l}(x)a_n(x)\big) + \frac{1}{a_n(x)}S_{n,l}(x), \end{aligned} \tag{15}$$

where $S_{n,l}(x)$ we introduced in (10). By Lemma 4, $S_{n,l}(x) = 0$, and thus (15) becomes

$$l\big(a_{n-l}(x) - V_{n-l}(x)a_n(x)\big) = 0 \quad \text{that is} \quad V_{n-l}(x) = \frac{a_{n-l}(x)}{a_n(x)}.$$

This concludes the proof of the lemma. □

**Lemma 6.** *We have that*

$$V_0(x) = \frac{a_0(x)}{a_n(x)} + \frac{1}{na_n(x)}\frac{d}{dx}\frac{a_{n-1}(x)}{a_n(x)}.$$

*Proof.* We compute in (6) with $s = n$ the coefficients of $y^{n-1}$. By Lemma 3 with $l = n-1 < n$ we get

$$V'_{n-1}(x) + na_0(x)V_n(x) + \sum_{i=0}^{n-1}(2i-n)V_i(x)a_{n-i}(x) = 0.$$

Now, using Lemma 5 we get

$$\begin{aligned}
0 &= V'_{n-1}(x) + na_0(x)V_n(x) + \sum_{i=0}^{n-1}(2i-n)V_i(x)a_{n-i}(x)\\
&= V'_{n-1}(x) + na_0(x)V_n(x) - nV_0(x)a_n(x) + \frac{1}{a_n(x)}\sum_{i=1}^{n-1}(2i-n)a_i(x)a_{n-i}(x)\\
&= V'_{n-1}(x) + na_0(x) - na_n(x)V_0(x) - \frac{S_{n,n}(x)}{a_n(x)}\\
&= V'_{n-1}(x) + na_0(x) - na_n(x)V_0(x),
\end{aligned}$$

where in the last equality we have used (10) and Lemma 4. Therefore

$$V_0(x) = \frac{1}{na_n(x)}\big(na_0(x) + V'_{n-1}(x)\big) = \frac{a_0(x)}{a_n(x)} + \frac{1}{na_n(x)}\frac{d}{dx}\frac{a_{n-1}(x)}{a_n(x)}.$$

This concludes the proof of the lemma. □

**Lemma 7.** *For $k = 1, \ldots, n-1$ we have that $a_{n-k}(0) = 0$, and*

$$a_0(0) = \frac{1}{na_n(0)^2}(a_{n-1}(0)a'_n(0) - a'_{n-1}(0)a_n(0)).$$

*Proof.* From the definition of Weierstrass polynomial, $V_k(0) = 0$ for $k = 0, \ldots, n-1$. Then, and since by assumption $a_n(0) \neq 0$, it follows from Lemma 5 that $a_{n-k}(0) = 0$ for $k = 1, \ldots, n-1$.

Now using that $V_0(0) = 0$ it follows from Lemma 6 that

$$0 = \frac{a_0(0)}{a_n(0)} + \frac{1}{na_n(0)^3}(a'_{n-1}(0)a_n(0) - a_{n-1}(0)a'_n(0)),$$

which clearly concludes the proof of the lemma. □

**Lemma 8.** *We have*

$$a_{n-k}(x) = \frac{\prod_{l=2}^{k}(n-l+1)}{n^{k-1}k!}\frac{a_{n-1}(x)^k}{a_n(x)^{k-1}}, \quad \textit{for } k = 2, \ldots, n-1. \tag{16}$$

*Proof.* We compute in (6) with $s = n$ the coefficients of $y^{n-k}$ with $k = 2, \ldots, n-1$. By Lemma 3 with $l = n-k < n$ we get

$$V'_{n-k}(x) + (n-k+1)a_0(x)V_{n-k+1}(x) + \sum_{i=0}^{n-k}(2i-1-n+k)V_i(x)a_{n-k-i+1}(x) = 0. \tag{17}$$

Using Lemma 5 we get that (17) becomes

$$
\begin{aligned}
0 &= \frac{d}{dx}\frac{a_{n-k}(x)}{a_n(x)} + (n-k+1)a_0(x)\frac{a_{n-k+1}(x)}{a_n(x)} + \sum_{i=0}^{n-k}(2i-1-n+k)V_i(x)a_{n-k-i+1}(x)\\
0 &= \frac{d}{dx}\frac{a_{n-k}(x)}{a_n(x)} + (n-k+1)a_0(x)\frac{a_{n-k+1}(x)}{a_n(x)} - (n-k+1)V_0(x)a_{n-k+1}(x)\\
&\quad + \frac{1}{a_n(x)}\sum_{i=1}^{n-k}(2i-1-n+k)a_i(x)a_{n-k-i+1}(x)\\
&= \frac{d}{dx}\frac{a_{n-k}(x)}{a_n(x)} + (n-k+1)a_0(x)\frac{a_{n-k+1}(x)}{a_n(x)} - (n-k+1)V_0(x)a_{n-k+1}(x)\\
&\quad - \frac{S_{n-k+1,n-k+1}(x)}{a_n(x)}.
\end{aligned}
$$

Now using Lemma 4 we get

$$
\frac{d}{dx}\frac{a_{n-k}(x)}{a_n(x)} + (n-k+1)a_0(x)\frac{a_{n-k+1}(x)}{a_n(x)} - (n-k+1)V_0(x)a_{n-k+1}(x) = 0,
$$

and by Lemma 6,

$$
\begin{aligned}
0 &= \frac{d}{dx}\frac{a_{n-k}(x)}{a_n(x)} + (n-k+1)a_0(x)\frac{a_{n-k+1}(x)}{a_n(x)} - (n-k+1)\frac{a_0(x)}{a_n(x)}a_{n-k+1}(x)\\
&\quad - \frac{n-k+1}{na_n(x)}a_{n-k+1}(x)\frac{d}{dx}\frac{a_{n-1}(x)}{a_n(x)}\\
&= \frac{d}{dx}\frac{a_{n-k}(x)}{a_n(x)} - \frac{n-k+1}{na_n(x)}a_{n-k+1}(x)\frac{d}{dx}\frac{a_{n-1}(x)}{a_n(x)}.
\end{aligned}
$$

That is

$$
\frac{d}{dx}\frac{a_{n-k}(x)}{a_n(x)} = \frac{n-k+1}{na_n(x)}a_{n-k+1}(x)\frac{d}{dx}\frac{a_{n-1}(x)}{a_n(x)}. \tag{18}
$$

Now we proceed by induction on $k$.

For $k = 2$ we have

$$
\begin{aligned}
\frac{d}{dx}\frac{a_{n-2}(x)}{a_n(x)} &= \frac{n-1}{n}\frac{a_{n-1}(x)}{a_n(x)}\frac{d}{dx}\frac{a_{n-1}(x)}{a_n(x)}\\
&= \frac{n-1}{2n}\frac{d}{dx}\Big(\frac{a_{n-1}(x)}{a_n(x)}\Big)^2.
\end{aligned}
$$

Solving this equation we get

$$
\frac{a_{n-2}(x)}{a_n(x)} = K_2 + \frac{n-1}{2n}\Big(\frac{a_{n-1}(x)}{a_n(x)}\Big)^2.
$$

Using Lemma 7 we get that $K_2 = 0$ and thus

$$
a_{n-2}(x) = \frac{n-1}{2n}\frac{a_{n-1}(x)^2}{a_n(x)},
$$

which proves (16) for $k = 2$.

Now we assume that (16) holds for $k = 2, \dots, j-1$ and we shall prove it for $k = j$. Thus, using the induction hypothesis, it follows from (18) with $k = j$ that

$$\begin{aligned}
\frac{d}{dx}\frac{a_{n-j}(x)}{a_n(x)} &= \frac{n-j+1}{na_n(x)} a_{n-(j-1}(x)\frac{d}{dx}\frac{a_{n-1}(x)}{a_n(x)} \\
&= \frac{n-j+1}{na_n(x)}\frac{\prod_{l=2}^{j-1}(n-l+1)}{n^{j-2}(j-1)!}\frac{a_{n-1}(x)^{j-1}}{a_n(x)^{j-2}}\frac{d}{dx}\frac{a_{n-1}(x)}{a_n(x)} \\
&= \frac{\prod_{l=2}^{j}(n-l+1)}{n^{j-1}(j-1)!}\frac{a_{n-1}(x)^{j-1}}{a_n(x)^{j-1}}\frac{d}{dx}\frac{a_{n-1}(x)}{a_n(x)} \\
&= \frac{\prod_{l=2}^{j}(n-l+1)}{n^{j-1}j!}\frac{d}{dx}\Big(\frac{a_{n-1}(x)}{a_n(x)}\Big)^j.
\end{aligned}$$

Solving this equation we get

$$\frac{a_{n-j}(x)}{a_n(x)} = K_j + \frac{\prod_{l=2}^{j}(n-l+1)}{n^{j-1}j!}\Big(\frac{a_{n-1}(x)}{a_n(x)}\Big)^j.$$

Using Lemma 7 we get that $K_j = 0$ and thus

$$a_{n-j}(x) = \frac{\prod_{l=2}^{j}(n-l+1)}{n^{j-1}j!}\frac{a_{n-1}(x)^j}{a_n(x)^{j-1}},$$

which proves (16) for $k = j$. This concludes the proof of the lemma. □

**Lemma 9.** *We have*

$$a_0(x) = -\frac{1}{n}\frac{d}{dx}\frac{a_{n-1}(x)}{a_n(x)} + \frac{\prod_{l=2}^{n-1}(n-l+1)}{n^{n-1}n!}\frac{a_{n-1}(x)^n}{a_n(x)^{n-1}}.$$

*Proof.* We compute in (6) with $s = n$ the coefficients of $y^0$. By Lemma 3 with $l = 0$ we get

$$V_0'(x) + a_0(x)V_1(x) - a_1(x)V_0(x) = 0. \tag{19}$$

Using Lemmas 5 and 6 we get that (19) can be written as

$$\begin{aligned}
0 &= \frac{d}{dx}V_0(x) + \frac{a_0(x)a_1(x)}{a_n(x)} - a_1(x)\Big(\frac{a_0(x)}{a_n(x)} + \frac{1}{na_n(x)}\frac{d}{dx}\frac{a_{n-1}(x)}{a_n(x)}\Big) \\
&= \frac{d}{dx}V_0(x) - \frac{a_1(x)}{na_n(x)}\frac{d}{dx}\frac{a_{n-1}(x)}{a_n(x)}.
\end{aligned}$$

Now using Lemma 8 with $k = n-1$ we get

$$\begin{aligned}
\frac{d}{dx}V_0(x) &= \frac{a_1(x)}{na_n(x)}\frac{d}{dx}\frac{a_{n-1}(x)}{a_n(x)} \\
&= \frac{\prod_{l=2}^{n-1}(n-l+1)}{n^{n-1}(n-1)!}\frac{a_{n-1}(x)^{n-1}}{a_n(x)^{n-1}}\frac{d}{dx}\frac{a_{n-1}(x)}{a_n(x)} \\
&= \frac{\prod_{l=2}^{n-1}(n-l+1)}{n^{n-1}n!}\frac{d}{dx}\Big(\frac{a_{n-1}(x)}{a_n(x)}\Big)^n.
\end{aligned}$$

Therefore, integrating it we get

$$V_0(x) = K_0 + \frac{\prod_{l=2}^{n-1}(n-l+1)}{n^{n-1}n!}\Big(\frac{a_{n-1}(x)}{a_n(x)}\Big)^n.$$

Using Lemma 7 (i.e. that $V_0(0) = 0$) we get that $K_0 = 0$ and thus

$$V_0(x) = \frac{\prod_{l=2}^{n-1}(n-l+1)}{n^{n-1}n!}\Big(\frac{a_{n-1}(x)}{a_n(x)}\Big)^n.$$

Now, using Lemma 6 we have

$$\frac{a_0(x)}{a_n(x)} + \frac{1}{na_n(x)}\frac{d}{dx}\frac{a_{n-1}(x)}{a_n(x)} = \frac{\prod_{l=2}^{n-1}(n-l+1)}{n^{n-1}n!}\Big(\frac{a_{n-1}(x)}{a_n(x)}\Big)^n,$$

which yields

$$a_0(x) = -\frac{1}{n}\frac{d}{dx}\frac{a_{n-1}(x)}{a_n(x)} + \frac{\prod_{l=2}^{n-1}(n-l+1)}{n^{n-1}n!}\frac{a_{n-1}(x)^n}{a_n(x)^{n-1}},$$

as we wanted to prove. □

*Proof of Theorem 2.* It follows readily from Lemmas 5, 6, 7, 8 and 9. □

## Acknowledgements

The first author is partially supported by a MCYT/FEDER grant MTM2008-03437, a CIRIT grant number 2009SGR-410, an ICREA Academia, and two grants FP7-PEOPLE-2012-IRSES 316338 and 318999. The second author has been supported by FCT (grant PTDC/MAT/117106/2010 and through CAMGSD).

## References

[1] E Casas-Alvero, *Singularities of Plane Curves*, London Math. Soc. Lecture Note Ser., vol. 276, Cambridge University Press, Cambridge, 2000.

[2] E.S. Cheb-Terrab and A.D. Roche, *Abel ODE's: equivalence and integrable classes*, Comput. Phys. Comm. **130** (2000), 204–231.

[3] E.S. Cheb-Terrab and A.D. Roche, *An Abel ordinary differential equation class generalizing known integrable classes*, Eur. J. Appl. Math. **14** (2003), 217–229.

[4] J. Écalle, *Introduction deux fonctions analysables et preuve constructive de la conjecture de Dulac*, Hermann, 1992.

[5] J. Giné and M. Grau, *Weierstrass integrability of differential equations*, Appl. Math. Lett. **23** (2010), 523–526.

[6] J. Giné and J. Llibre, *On the integrable rational Abel differential equations*, Z. Angew. Math. Phys. **61** (2010), 33–39.

[7] E. Kamke, *Differentialgleichungen "losungsmethoden und losungen"*, Col. Mathematik und ihre anwendungen vol. **18**, Akademische Verlagsgesellschaft Becker und Erler Kom-Ges., Leipzig (1943).

[8] J. Llibre and C. Valls, *Generalized Weierstrass integrability of the Abel differential equations*, to appear in Mediterr. J. Math.

[1] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain
*E-mail address*: jllibre@mat.uab.cat

[3] Departamento de Matemática, Instituto Superior Técnico, Universidade Técnica de Lisboa, Av. Rovisco Pais 1049–001, Lisboa, Portugal
*E-mail address*: cvalls@math.ist.utl.pt