Z. Angew. Math. Phys. (2021) 72:15

https://doi.org/10.1007/s00033-020-01450-1

Zeitschrift für angewandte Mathematik und Physik ZAMP

Dynamics of the FitzHugh–Nagumo system having invariant algebraic surfaces

Jaume Llibre and Yuzhou Tian

Abstract. In this paper, we study the dynamics of the FitzHugh–Nagumo system $\dot{x} = z,\ \dot{y} = b\,(x - dy)\,,\ \dot{z} = x\,(x-1)\,(x-a) + y + cz$ having invariant algebraic surfaces. This system has four different types of invariant algebraic surfaces. The dynamics of the FitzHugh–Nagumo system having two of these classes of invariant algebraic surfaces have been characterized in Valls (J Nonlinear Math Phys 26:569–578, 2019). Using the quasi-homogeneous directional blow-up and the Poincaré compactification, we describe the dynamics of the FitzHugh–Nagumo system having the two remaining classes of invariant algebraic surfaces. Moreover, for these FitzHugh–Nagumo systems we prove that they do not have limit cycles.

Mathematics Subject Classification. Primary 37C10; Secondary 34C05, 37C70.

Keywords. Global dynamics, FitzHugh–Nagumo system, Invariant algebraic surface, Poincaré compactification.

1. Introduction

The FitzHugh–Nagumo system is given by the partial differential system

$$u_t = u_{xx} - f\,(u) - v, \quad v_t = \varepsilon\,(u - \gamma v)\,, \tag{1}$$

where $f\,(u) = u\,(u-1)\,(u-a)$ and $0 < a < 1/2$, $\varepsilon > 0$, $\gamma > 0$ are parameters. We say that a bounded solution $(u,v)\,(x,t)$ of the FitzHugh–Nagumo system (1) with $x, t \in \mathbb{R}$ is a *traveling wave* if $(u,v)\,(x,t) = (u,v)\,(\xi)$, where $\xi = x + ct$ and c is the constant denoting the wave speed. Substituting $u = u\,(\xi)\,, v = v\,(\xi)$ into (1), we obtain the ordinary differential system

$$\begin{aligned} \dot{x} &= z = P\,(x,y,z)\,,\\ \dot{y} &= b\,(x - dy) = Q\,(x,y,z)\,,\\ \dot{z} &= x\,(x-1)\,(x-a) + y + cz = R\,(x,y,z)\,. \end{aligned} \tag{2}$$

Here, the dot denotes derivative with respect to ξ, $x = u$, $y = v$, $z = \dot{u}$, $b = \varepsilon/c$ and $d = \gamma$; see for more details [11].

The FitzHugh–Nagumo system (1) is a classical differential system introduced independently by FitzHugh [8] and Nagumo et al. [19]. It is an important model for describing the excitation of neural membranes and the propagation of nerve impulses along an axon. Besides its biological interest, the FitzHugh–Nagumo system has gained wide investigation from the mathematical point of view, such as the existence, uniqueness and stability of its traveling wave solutions; see, for instance, [2,9,12–14,19], etc.

In recent years, the FitzHugh–Nagumo system (2) has been investigated from the points of view of its dynamics and integrability. The analytical integrability of the FitzHugh–Nagumo system (2) has been studied by Llibre and Valls in [17]. The Liouvillian integrability of the planar FitzHugh–Nagumo

Published online: 11 January 2021