ON THE PERIODIC ORBITS OF THE CONTINUOUS-DISCONTINUOUS PIECEWISE DIFFERENTIAL SYSTEMS WITH THREE PIECES SEPARATED BY TWO PARALLEL STRAIGHT LINES

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Abstract. During these last twenty years many papers have been published about the piecewise differential systems in the plane. The increasing interest on these kind of differential systems mainly is due to their big number of applications for modelling many natural phenomena. As usual one of the main difficulties for understanding the dynamics of the differential systems in the plane consists in controlling their periodic orbits and mainly their limit cycles. Therefore there are a big number of papers studying the existence or non-existence of periodic orbits of the continuous and discontinuous piecewise differential systems, but as far as we know this paper will be one of the first papers studying the periodic orbits of a class of piecewise differential systems such that in a part of the line of separation between the two differential systems forming the piecewise differential system is continuous and in the other part it is discontinuous.

Thus we shall study the piecewise differential systems separated by two parallel straight lines, in one of these straight lines the piecewise differential system is continuous and in the other discontinuous, moreover in the three pieces of these piecewise differential systems we put arbitrary Hamiltonian systems of degree one. We obtain that such kind of continuous-discontinuous piecewise differential systems cannot have limit cycles, but they can have a continuum of periodic orbits.

1. Introduction and results

Around 1920's started the first studies on the piecewise differential systems, see the book [2] of Andronov, Vitt and Khaikin. Nowdays the piecewise differential systems continue receiving the attention of many researchers because they model many phenomena appearing in mechanics, electronics, economy, etc., see the books of di Bernardo et al. [3] and Simpson [15], and also the survey of Makarenkov and Lamb [14], and the hundreds of references mentioned in these last references.

The simplest continuous piecewise differential systems are the ones formed by two pieces separated by one straight line in the plane $\mathbb{R}^2$ and having in each piece a linear differential systems. In 1991 Lum and Chua [12, 13] conjectured that such piecewise differential systems have at most one limit cycle. We recall that a limit cycle is an isolated periodic orbit in the set of all periodic orbits of a differential

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system. In 1998 Ihe mentioned conjecture was proved by Freire, Ponce, Rodrigo and Torres [7]. In 2013 a different and easier proof was given by Llibre, Ordóñez and Ponce [10], and in 2021 a new proof has been given by Carmona, Fernández-Sánchez and Novaes [5].

Of course, the easiest discontinuous piecewise differential systems are the ones formed by two pieces separated by a straight line in the plane $\mathbb{R}^2$ and having in each piece a linear differential systems. The behaviour of the discontinuous piecewise systems on the line of discontinuity, where the two differential systems live together, is defined in the book of Filippov [6]. There are more than thirty papers published studying the limit cycles of this class of discontinuous piecewise differential systems, for a survey on these papers see [8]. For this class of piecewise differential systems it remains open the following question: Is 3 the maximum number of limit cycles that a discontinuous piecewise differential system with a straight line of separation and formed by two linear differential systems can have?

We did not find in the literature papers which study the limit cycles of the piecewise differential systems such that in a part of the line of separation the system is continuous and in the other part it is discontinuous. Probably the easiest of such piecewise differential systems are the ones having the separation line formed by two parallel straight lines, in one of these straight lines the piecewise differential system is continuous and in the other discontinuous, moreover in the three pieces of these piecewise differential systems we put arbitrary Hamiltonian systems of degree one. This kind of piecewise differential systems will be called *continuous-discontinuous piecewise differential systems* in what follows.

Doing a rescaling and a rotation, if necessary, we can assume without loss of generality that the two parallel straigth lines are $x = \pm 1$. Thus we shall study the continuous-discontinuous piecewise differential systems of the form

$$(\dot{x}, \dot{y}) = \mathbf{X}(x,y) = \begin{cases} \mathbf{X}_1(x,y) = \left(-\dfrac{\partial H_1}{\partial y}, \dfrac{\partial H_1}{\partial x}\right) & \text{if } x \geq 1, \\[2ex] \mathbf{X}_2(x,y) = \left(-\dfrac{\partial H_2}{\partial y}, \dfrac{\partial H_2}{\partial x}\right) & \text{if } -1 \leq x \leq 1, \\[2ex] \mathbf{X}_3(x,y) = \left(-\dfrac{\partial H_3}{\partial y}, \dfrac{\partial H_3}{\partial x}\right) & \text{if } x \leq -1, \end{cases} \tag{1}$$

where $H_i = H_i(x,y)$ is an arbitrary polynomial of degree 2 for $i = 1,2,3$. Moreover we assume that these piecewise differential systems are continuous on the straight line $x = 1$ and discontinuous on the straight line $x = -1$. The objective of this paper is to determine the existence or non-existence of limit cycles that such kind of continuous-discontinuous piecewise differential systems can exhibit. And in the case of the existence of limit cycles which is the maximum number of them.

Note that the three differential systems which form the continuous-discontinuous piecewise differential systems (1) are polynomial Hamiltonian systems of degree 1.

Our main result is the following.

Theorem 1. *A continuous-discontinuous piecewise differential system* (1) *has no limit cycles. Moreorver, there are continuous-discontinuous piecewise differential system* (1) *having a continuum of periodic orbits, see Figure* 1.

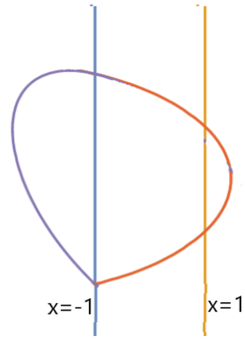

Figure 1. One of the periodic orbits of the continuum of periodic orbits of the continuous-discontinuous piecewise differential system formed by the Hamiltonian systems defined by the Hamiltonians (13).

The proof of Theorem 1 is proved in section 2.

In general, for a given class of differential systems is not easy to know if they have or not limit cycles. Here the non-existence of limit cycles for the class of continuous-discontinuous piecewise differential systems studied follows using the Gröbner basis.

In view of Theorem 1 is natural to wait that the continuous piecewise differential system (1), i.e. systems (1) which are continuous on both straight lines $x = \pm 1$ have also no limit cycles. This is the case as it is proved in the next result.

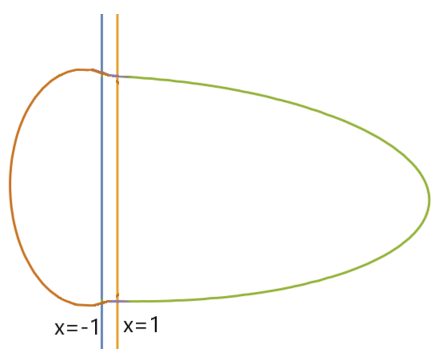

Figure 2. One of the periodic orbits of the continuum of periodic orbits of the continuous piecewise differential system formed by the Hamiltonian systems defined by the Hamiltonians (17).

Theorem 2. *A continuous piecewise differential system* (1) *has no limit cycles. Moreorver, there are continuous piecewise differential system* (1) *having a continuum of periodic orbits, see Figure* 2.

The proof of Theorem 2 is proved in section 3.

Second in 2018 in the paper [11] it is proved that there are discontinuous piecewise differential systems separated by two parallel straight lines and formed by three linear center having one limit cycle. Of course the linear centers are particular linear Hamiltonian systems of degree 1. Recently it has been improved this result in the following theorem proved in [9].

Theorem 3. *Discontinuous piecewise differential systems with three pieces separated by two parallel straight lines and formed by three arbitrary linear Hamiltonian systems has at most one limit cycle. Besides there are systems in this class having one limit cycle.*

2. Proof of Theorem 1

We shall use the following two lemmas in the proof of Theorem 1.

Lemma 4. *A continuous piecewise differential system separated by one straight line and formed by two linear Hamiltonian systems has no limit cycles.*

Proof. Doing an affine transformation, we can assume without loss of generality that, the straight line of separation of the two linear Hamiltonian systems forming the continuous piecewise differential system is $x = 0$.

We denote the two Hamiltonians which appear in the continuous piecewise differential system by

$$\begin{aligned} H_1(x,y) &= a_1x + a_2y + a_3x^2 + a_4xy + a_5y^2, \\ H_2(x,y) &= b_1x + b_2y + b_3x^2 + b_4xy + b_5y^2. \end{aligned} \tag{2}$$

So the corresponding Hamiltonian systems are

$$\begin{aligned} \dot{x} &= -a_2 - a_4x - 2a_5y, \qquad \dot{y} = a_1 + 2a_3x + a_4y; \\ \dot{x} &= -b_2 - b_4x - 2b_5y, \qquad \dot{y} = b_1 + 2b_3x + b_4y; \end{aligned} \tag{3}$$

respectively.

Now we impose that the Hamiltonian systems associated to the Hamiltonians H_1 and H_2 coincide on the straight line $x = 0$, and then the two Hamiltonians systems of (3) become

$$\begin{aligned} \dot{x} &= -a_2 - a_4x - 2a_5y, \qquad \dot{y} = a_1 + 2a_3x + a_4y; \\ \dot{x} &= -a_2 - a_4x - 2a_5y, \qquad \dot{y} = a_1 + 2b_3x + a_4y; \end{aligned}$$

respectively, and the corresponding Hamiltonians are

$$\begin{aligned} H_1(x,y) &= a_1x + a_2y + a_3x^2 + a_4xy + a_5y^2, \\ H_2(x,y) &= a_1x + a_2y + b_3x^2 + a_4xy + a_5y^2. \end{aligned}$$

Since in the half-planes $x \geq 0$ and $x \leq 0$ the piecewise differential system is Hamiltonian with a polynomial Hamiltonian, in these two half-planes there are no limit cycles. Then if there are limit cycles these must intersect the straight line $x = 0$ in two points, that we denote by $(0, y_1)$ and $(0, y_2)$ with $y_1 \neq y_2$. Therefore, since the functions H_1 and H_2 are first integrals in the half-planes $x \geq 0$ and $x \leq 0$ respectively, y_1 and y_2 must satisfy the following system of equations

$$\begin{aligned} H_1(1, y_1) - H_1(1, y_2) &= (y_1 - y_2)(a_2 + a_4 + a_5y_1 + a_5y_2) = 0, \\ H_2(1, y_2) - H_2(1, y_1) &= (y_1 - y_2)(a_2 + a_4 + a_5y_1 + a_5y_2) = 0 \end{aligned}$$

So this system has a continuum of solutions with $y_1 \neq y_2$. Consequently, if these solutions provide periodic orbits of the continuous piecewise differential system, there is a continuum of periodic orbits, but there are no limit cycles. □

Lemma 5. *A discontinuous piecewise differential system separated by one straight line and formed by two linear Hamiltonian systems has no limit cycles.*

Proof. As in the proof of Lemma 4 we can assume that the straight line of separation of the two linear Hamiltonian systems forming the discontinuous piecewise differential system is $x = 0$.

We denote the two Hamiltonians which appear in the discontinuous piecewise differential system as in (2). So the corresponding Hamiltonian systems are as in (3). Again since in the half-planes $x \geq 0$ and $x \leq 0$ the piecewise differential system is Hamiltonian with a polynomial Hamiltonian, in these two half-planes there are no limit cycles. Then if there are limit cycles these must intersect the straight line $x = 0$ in two points, that we denote by $(0, y_1)$ and $(0, y_2)$ with $y_1 \neq y_2$. Therefore, since the functions H_1 and H_2 are first integrals in the half-planes $x \geq 0$ and $x \leq 0$ respectively, y_1 and y_2 must satisfy the following system of equations

$$
\begin{aligned}
H_1(1, y_1) - H_1(1, y_2) &= (y_1 - y_2)(a_2 + a_4 + a_5 y_1 + a_5 y_2) = 0,\\
H_2(1, y_2) - H_2(1, y_1) &= (y_1 - y_2)(b_2 + b_4 + b_5 y_1 + b_5 y_2) = 0.
\end{aligned}
$$

Omitting the common factor $y_1 - y_2$ in the previous system which cannot be zero, the linear system which remains has determinant zero, so such linear system either has no solutions, or it has a continuum of solutions. As in the proof of Lemma 1 the discontinuous piecewise differential system cannot have no limit cycles. □

Proof of Theorem 1. We denote the three Hamiltonians which appear in the continuous-discontinuous piecewise differential system (1) as follows

$$
\begin{aligned}
H_1(x, y) &= a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2,\\
H_2(x, y) &= b_1 x + b_2 y + b_3 x^2 + b_4 xy + b_5 y^2,\\
H_3(x, y) &= c_1 x + c_2 y + c_3 x^2 + c_4 xy + c_5 y^2.
\end{aligned} \tag{4}
$$

So the corresponding Hamiltonian systems are

$$
\begin{aligned}
\dot{x} &= -a_2 - a_4 x - 2a_5 y, \qquad \dot{y} = a_1 + 2a_3 x + a_4 y;\\
\dot{x} &= -b_2 - b_4 x - 2b_5 y, \qquad \dot{y} = b_1 + 2b_3 x + b_4 y;\\
\dot{x} &= -c_2 - c_4 x - 2c_5 y, \qquad \dot{y} = c_1 + 2c_3 x + c_4 y;
\end{aligned} \tag{5}
$$

respectively.

Now we impose that the Hamiltonian systems associated to the Hamiltonians H_1 and H_2 coincide on the straight line $x = 1$, and then the three Hamiltonians systems of (5) become

$$
\begin{aligned}
\dot{x} &= -a_2 - a_4 x - 2a_5 y, \qquad \dot{y} = a_1 + 2a_3 x + a_4 y;\\
\dot{x} &= -a_2 - a_4 x - 2a_5 y, \qquad \dot{y} = b_1 + (a_1 + 2a_3 - b_1)x + a_4 y;\\
\dot{x} &= -c_2 - c_4 x - 2c_5 y, \qquad \dot{y} = c_1 + 2c_3 x + c_4 y;
\end{aligned} \tag{6}
$$

respectively, and the corresponding Hamiltonians are

$$
\begin{aligned}
H_1(x,y) &= a_1x + a_2y + a_3x^2 + a_4xy + a_5y^2,\\
H_2(x,y) &= b_1x + a_2y + \frac{1}{2}(a_1 + 2a_3 - b_1)x^2 + a_4xy + a_5y^2,\\
H_3(x,y) &= c_1x + c_2y + c_3x^2 + c_4xy + c_5y^2.
\end{aligned} \tag{7}
$$

Due to Lemmas 4 and 5 if a continuous-discontinuous piecewise differential system has a limit cycle it must intersect the two straight lines $x = \pm 1$. Therefore assuming that the four intersection points of a limit cycle with the straight lines $x = \pm 1$ are $(1, y_1)$, $(1, y_2)$, $(-1, y_2)$ and $(-1, y_4)$, with $y_1 < y_2$ and $y_3 > y_4$, and since the functions $H_i(x, y)$ are first integrals it follows that y_1, y_2, y_3 and y_4 must satisfy the following four equations

$$
\begin{aligned}
&H_1(1,y_1) - H_1(1,y_2) = (y_1 - y_2)(a_2 + a_4 + a_5y_1 + a_5y_2) = 0,\\
&H_2(1,y_2) - H_2(-1,y_3) = 2b_1 + a_2y_2 + a_4y_2 + a_5y_2^2 - a_2y_3 + a_4y_3 - a_5y_3^2 = 0,\\
&H_3(-1,y_3) - H_3(-1,y_4) = (y_3 - y_4)(c_2 - c_4 + c_5y_3 + c_5y_4) = 0,\\
&H_2(-1,y_4) - H_2(1,y_1) = -2b_1 - a_2y_1 - a_4y_1 - a_5y_1^2 + a_2y_4 - a_4y_4 + a_5y_4^2 = 0.
\end{aligned} \tag{8}
$$

Since $y_1 < y_2$ and $y_3 > y_4$ the polynomial system (8) reduces to

$$
\begin{aligned}
P_1(y_1,y_2,y_3,y_4) &= a_2 + a_4 + a_5y_1 + a_5y_2 = 0,\\
P_2(y_1,y_2,y_3,y_4) &= 2b_1 + a_2y_2 + a_4y_2 + a_5y_2^2 - a_2y_3 + a_4y_3 - a_5y_3^2 = 0,\\
P_3(y_1,y_2,y_3,y_4) &= c_2 - c_4 + c_5y_3 + c_5y_4 = 0,\\
P_4(y_1,y_2,y_3,y_4) &= -2b_1 - a_2y_1 - a_4y_1 - a_5y_1^2 + a_2y_4 - a_4y_4 + a_5y_4^2 = 0.
\end{aligned} \tag{9}
$$

For solving the polynomial system (9) with four equations and four unknowns y_1, y_2, y_3 and y_4 we shall use the Gröbner basis, see for more details on this basis [1, 4]. Thus we compute the Gröbner basis of the polynomials P_i with respect to the variables y_i, for $i = 1, 2, 3, 4$ and we obtain the following polynomial system with thirteen polynomial equations whose solutions are the same than the solutions of the polynomial system (9), but in general they are easier to solve. The thirteen equations of the Gröbner basis are

(10)

$$
\begin{aligned}
&(c_2 - a_5c_4 - a_2c_5 + a_4c_5)(c_2 - c_4 + 2c_5y_4) = 0,\\
&c_2 - c_4 + c_5y_3 + c_5y_4 = 0,\\
&a_2 - a_4 + a_5y_3 + a_5y_4 = 0,\\
&a_2 + a_4 + a_5y_1 + a_5y_2 = 0,\\
&b_1y_1 + b_1y_2 + a_4y_1y_2 - b_1y_3 - b_1y_4 - a_4y_3y_4 = 0,\\
&a_2y_1 - a_4y_1 + a_2y_2 - a_4y_2 - a_2y_3 - a_4y_3 - a_2y_4 - a_4y_4 = 0,\\
&2b_1 + a_2y_2 + a_4y_2 + a_5y_2^2 - a_2y_4 + a_4y_4 - a_5y_4^2 = 0,\\
&2b_1y_1 + 2b_1y_2 + a_2y_1y_2 + a_4y_1y_2 - a_2y_1y_4 + a_4y_1y_4 - a_2y_2y_4 + a_4y_2y_4 + a_2y_4^2\\
&+a_4y_4^2 = 0,\\
&a_2c_2 - a_4c_2 - a_2c_4 + a_4c_4 + a_5c_2y_3 - a_5c_4y_3 - a_5c_2y_4 + a_5c_4y_4 + 2a_2c_5y_4\\
&-2a_4c_5y_4 = 0,\\
&a_2y_2^2 - a_4y_2^2 - 2b_1y_3 - a_2y_2y_3 - a_4y_2y_3 - 2b_1y_4 - a_2y_2y_4 - a_4y_2y_4 + a_2y_3y_4\\
&-a_4y_3y_4 = 0,\\
&a_2c_2y_1 - a_4c_2y_1 - a_2c_4y_1 + a_4c_4y_1 + a_2c_2y_2 - a_4c_2y_2 - a_2c_4y_2 + a_4c_4y_2\\
&-a_2c_2y_3a_5 - a_4c_2y_3 + a_2c_4y_3 + a_4c_4y_3 + a_2c_2y_4 + a_4c_2y_4 - a_2c_4y_4 - a_4c_4y_4\\
&+2a_2c_5y_1y_4 - 2a_4c_5y_1y_4 + 2a_2c_5y_2y_4 - 2a_4c_5y_2y_4 = 0,\\
&b_1c_2y_1 - b_1c_4y_1 + b_1c_2y_2 - b_1c_4y_2 + a_4c_2y_1y_2 - a_4c_4y_1y_2 - b_1c_2y_3 + b_1c_4y_3\\
&+b_1c_2y_4 - b_1c_4y_4 + 2b_1c_5y_1y_4 + 2b_1c_5y_2y_4 + 2a_4c_5y_1y_2y_4 - a_4c_2y_3y_4\\
&+a_4c_4y_3y_4 + 2a_4c_2y_4^2 - 2a_4c_4y_4^2 + 2a_4c_5y_4^3 = 0,\\
&a_2c_2y_2^2 - a_4c_2y_2^2 - a_2c_4y_2^2 + a_4c_4y_2^2 - 2b_1c_2y_3 + 2b_1c_4y_3 - a_2c_2y_2y_3 - a_4c_2y_2y_3\\
&+a_2c_4y_2y_3 + a_4c_4y_2y_3 + 2b_1c_2y_4 - 2b_1c_4y_4 + a_2c_2y_2y_4 + a_4c_2y_2y_4 - a_2c_4y_2y_4\\
&-a_4c_4y_2y_4 + 2a_2c_5y_2^2y_4 - 2a_4c_5y_2^2y_4 + a_2c_2y_3y_4 - a_4c_2y_3y_4 - a_2c_4y_3y_4\\
&+a_4c_4y_3y_4 - 2a_2c_2y_4^2 + 2a_4c_2y_4^2 + 2a_2c_4y_4^2 - 2a_4c_4y_4^2 - 2a_2c_5y_4^3 + 2a_4c_5y_4^3 = 0.
\end{aligned}
$$

Case 1: Assume that $a_5c_2 - a_5c_4 - a_2c_5 + a_4c_5 \neq 0$ and $c_5 \neq 0$. Then, from the first equation of the Gröbner basis it follows that $y_4 = (c_4 - c_2)/(2c_5)$. Substituting y_4 in the second equation of the Gröbner basis we obtain the equation $c_2 - c_4 + 2c_5y_3 = 0$. So $y_3 = y_4$, and the continuous-discontinuous piecewise differential system in this case has no limit cycles.

Case 2: Assume that $a_5c_2 - a_5c_4 - a_2c_5 + a_4c_5 \neq 0$ and $c_5 = 0$. Then from the first equation of the Gröbner basis it follows that $c_4 = c_2$, and consequently $a_5c_2 - a_5c_4 - a_2c_5 + a_4c_5 = 0$, a contradiction with the assumptions of this case.

Case 3: Assume that $a_5c_2 - a_5c_4 - a_2c_5 + a_4c_5 = 0$ and $c_5 \neq 0$. Then we have that $a_4 = (a_5c_4 - a_5c_2 + a_2c_5)/c_5$. Substituting the value of a_4 into the polynomial system obtained from the Gröbner basis this polynomial system becomes

$$
\begin{aligned}
&c_2 - c_4 + c_5 y_3 + c_5 y_4 = 0,\\
&a_5 c_4 - a_5 c_2 + 2a_2 c_5 + a_5 c_5 y_1 + a_5 c_5 y_2 = 0,\\
&b_1 c_5 y_1 + b_1 c_5 y_2 - a_5 c_2 y_1 y_2 + a_5 c_4 y_1 y_2 + a_2 c_5 y_1 y_2 - b_1 c_5 y_3 - b_1 c_5 y_4\\
&+a_5 c_2 y_3 y_4 - a_5 c_4 y_3 y_4 - a_2 c_5 y_3 y_4 = 0,\\
&a_5 c_2 y_1 - a_5 c_4 y_1 + a_5 c_2 y_2 - a_5 c_4 y_2 + a_5 c_2 y_3 - a_5 c_4 y_3 - 2a_2 c_5 y_3\\
&+a_5 c_2 y_4 - a_5 c_4 y_4 - 2a_2 c_5 y_4 = 0,\\
&2b_1 c_5 - a_5 c_2 y_2 + a_5 c_4 y_2 + 2a_2 c_5 y_2 + a_5 c_5 y_2^2 - a_5 c_2 y_4 + a_5 c_4 y_4\\
&-a_5 c_5 y_4^2 = 0,\\
&2b_1 c_5 y_1 + 2b_1 c_5 y_2 - a_5 c_2 y_1 y_2 + a_5 c_4 y_1 y_2 + 2a_2 c_5 y_1 y_2 - a_5 c_2 y_1 y_4\\
&+a_5 c_4 y_1 y_4 - a_5 c_2 y_2 y_4 + a_5 c_4 y_2 y_4 - a_5 c_2 y_4^2 + a_5 c_4 y_4^2 + 2a_2 c_5 y_4^2 = 0,\\
&a_5 c_2 y_2^2 - a_5 c_4 y_2^2 - 2b_1 c_5 y_3 + a_5 c_2 y_2 y_3 - a_5 c_4 y_2 y_3 - 2a_2 c_5 y_2 y_3\\
&-2b_1 c_5 y_4 + a_5 c_2 y_2 y_4 - a_5 c_4 y_2 y_4 - 2a_2 c_5 y_2 y_4 + a_5 c_2 y_3 y_4 - a_5 c_4 y_3 y_4 = 0,\\
&(c_4 - c_2)(a_5 c_4 y_1 - a_5 c_2 y_1 - a_5 c_2 y_2 + a_5 c_4 y_2 - a_5 c_2 y_3 + a_5 c_4 y_3\\
&+2a_2 c_5 y_3 + a_5 c_2 y_4 - a_5 c_4 y_4 - 2a_2 c_5 y_4 - 2a_5 c_5 y_1 y_4 - 2a_5 c_5 y_2 y_4) = 0,\\
&b_1 c_2 c_5 y_1 - b_1 c_4 c_5 y_1 + b_1 c_2 c_5 y_2 - b_1 c_4 c_5 y_2 - a_5 c_2^2 y_1 y_2 + 2a_5 c_2 c_4 y_1 y_2\\
&-a_5 c_4^2 y_1 y_2 + a_2 c_2 c_5 y_1 y_2 - a_2 c_4 c_5 y_1 y_2 - b_1 c_2 c_5 y_3 + b_1 c_4 c_5 y_3 + b_1 c_2 c_5 y_4\\
&-b_1 c_4 c_5 y_4 + 2b_1 c_5^2 y_1 y_4 + 2b_1 c_5^2 y_2 y_4 - 2a_5 c_2 c_5 y_1 y_2 y_4 + 2a_5 c_4 c_5 y_1 y_2 y_4\\
&+2a_2 c_5^2 y_1 y_2 y_4 + a_5 c_2^2 y_3 y_4 - 2a_5 c_2 c_4 y_3 y_4 + a_5 c_4^2 y_3 y_4 - a_2 c_2 c_5 y_3 y_4\\
&+a_2 c_4 c_5 y_3 y_4 - 2a_5 c_2^2 y_4^2 + 4a_5 c_2 c_4 y_4^2 - 2a_5 c_4^2 y_4^2 + 2a_2 c_2 c_5 y_4^2 - 2a_2 c_4 c_5 y_4^2\\
&-2a_5 c_2 c_5 y_4^3 + 2a_5 c_4 c_5 y_4^3 + 2a_2 c_5^2 y_4^3 = 0,\\
&(c_4 - c_2)(-a_5 c_2 y_2^2 + a_5 c_4 y_2^2 + 2b_1 c_5 y_3 - a_5 c_2 y_2 y_3 + a_5 c_4 y_2 y_3 + 2a_2 c_5 y_2 y_3\\
&-2b_1 c_5 y_4 + a_5 c_2 y_2 y_4 - a_5 c_4 y_2 y_4 - 2a_2 c_5 y_2 y_4 - 2a_5 c_5 y_2^2 y_4 - a_5 c_2 y_3 y_4\\
&+a_5 c_4 y_3 y_4 + 2a_5 c_2 y_4^2 - 2a_5 c_4 y_4^2 + 2a_5 c_5 y_4^3) = 0.
\end{aligned}
\tag{11}
$$

Subcase 3.1: $a_5 \neq 0$. Then, after some computations, the polynomial system obtained from the Gröbner basis has the following two solutions if $S > 0$, only one solution if $S = 0$, and no solutions if $S < 0$:

$$y_1 = \frac{S + a_5 c_2 - a_5 c_4 - 2a_2 c_5}{2a_5 c_5},\ y_2 = \frac{S - a_5 c_2 + a_5 c_4 + 2a_2 c_5}{2a_5 c_5},\ y_3 = \frac{c_4 - c_2 - c_5 y_4}{c_5},$$

and

$$y_1 = \frac{S - a_5 c_2 + a_5 c_4 + 2a_2 c_5}{2a_5 c_5},\ y_2 = \frac{S + a_5 c_2 - a_5 c_4 - 2a_2 c_5}{2a_5 c_5},\ y_3 = \frac{c_4 - c_2 - c_5 y_4}{c_5},$$

where $S = \sqrt{a_5(-c_2 + c_4) + 2a_2 c_5)^2 + 4a_5 c_5(-2b_1 c_5 + a_5 y_4(c_2 - c_4 + c_5 y_4)}$. But every one of these solutions, when it exists, is formed by a continuum of solutions because the variable y_4 can take arbitrary values. So in this case again the continuous-discontinuos piecewise differential system cannot have limit cycles, at most a continuum of periodic orbits.

Subcase 3.2: $a_5 = 0$. Then from the second equation of (11) we obtain that $a_2 = 0$. After from the fifth equation of (11) we get that $b_1 = 0$. Hence only remains the first equation of (11), which reduces to $c_2 - c_4 + c_5(y_3 + y_4) = 0$. So the system has a continuum of solutions, and consequently the continuous-discontinuos piecewise differential system has no limit cycles.

Case 4: Assume that $a_5c_2 - a_5c_4 - a_2c_5 + a_4c_5 = 0$ and $c_5 = 0$. Then from the second equation of (10) it follows that $c_4 = c_2$. Therefore the polynomials of the Gröbner basis reduces to the following polynomial system with seven equations

$$
(12) \qquad
\begin{aligned}
&a_2 - a_4 + a_5y_3 + a_5y_4 = 0,\\
&a_2 + a_4 + a_5y_1 + a_5y_2 = 0,\\
&b_1y_1 + b_1y_2 + a_4y_1y_2 - b_1y_3 - b_1y_4 - a_4y_3y_4 = 0,\\
&a_2y_1 - a_4y_1 + a_2y_2 - a_4y_2 - a_2y_3 - a_4y_3 - a_2y_4 - a_4y_4 = 0,\\
&2b_1 + a_2y_2 + a_4y_2 + a_5y_2^2 - a_2y_4 + a_4y_4 - a_5y_4^2 = 0,\\
&2b_1y_1 + 2b_1y_2 + a_2y_1y_2 + a_4y_1y_2 - a_2y_1y_4 + a_4y_1y_4 - a_2y_2y_4 + a_4y_2y_4\\
&+a_2y_4^2 + a_4y_4^2 = 0,\\
&a_2y_2^2 - a_4y_2^2 - 2b_1y_3 - a_2y_2y_3 - a_4y_2y_3 - 2b_1y_4 - a_2y_2y_4 - a_4y_2y_4\\
&+a_2y_3y_4 - a_4y_3y_4 = 0,
\end{aligned}
$$

where we have eliminate from the previous system some factors $y_3 - y_4$ which are non-zero.

Case 4.1: $a_5 \neq 0$. Then system (12) has the following two solutions if $R > 0$, only one solution if $R = 0$, and no solutions if $R < 0$:

$$
y_1 = -\frac{a_2 + a_4 - R}{2a_5}, \quad y_2 = -\frac{a_2 - a_4 + R}{2a_5}, \quad y_3 = \frac{a_2 - a_4 + a_5y_4}{a_5},
$$

and

$$
y_1 = -\frac{a_2 - a_4 + R}{2a_5}, \quad y_2 = -\frac{a_2 + a_4 - R}{2a_5}, \quad y_3 = \frac{a_2 - a_4 + a_5y_4}{a_5},
$$

where $R = \sqrt{(a_2 + a_4)^2 - 8a_5b_1 + 4a_5y_4(a_2 - a_4 + a_5y_4)}$. But every one of these solutions, when it exists, is formed by a continuum of solutions because the variable y_4 can take arbitrary values. So in this case the continuous-discontinuous piecewise differential system cannot have limit cycles, at most a continuum of periodic orbits.

Case 4.2: $a_5 = 0$. Then from the first two equations of (12) we obtain that $a_2 = a_4 = 0$. So from the six equation of (12) we get that $b_1 = 0$, and the polynomial system (9) becomes identically zero, hence again a continuum of solutions.

In summary, we have proved that the continuous-discontinuous piecewise differential systems (1) have no limit cycles, at most they have a continuum of periodic orbits. It remains to prove that there are continuous-discontinuous piecewise differential systems (1) having a continuum of periodic orbits.

Consider the three Hamiltonians

$$
(13) \qquad
\begin{aligned}
&H_1(x,y) = x^2 + y^2,\\
&H_2(x,y) = \tfrac{3}{2}x + \frac{1}{4}x^2 + y^2,\\
&H_3(x,y) = 10x + 15y + 14x^2 + 15xy + 12y^2.
\end{aligned}
$$

System (8) for these particular Hamiltonians becomes

$$\begin{aligned} &H_1(1,y_1) - H_1(1,y_2) = (y_1-y_2)(y_1+y_2) = 0,\\ &H_2(1,y_2) - H_2(-1,y_3) = 3 + y_2^2 - y_3^2 = 0,\\ &H_3(-1,y_3) - H_3(-1,y_4) = 12(y_3-y_4)(y_3+y_4) = 0,\\ &H_2(-1,y_4) - H_2(1,y_1) = -3 - y_1^2 + y_4^2 = 0. \end{aligned}$$

The solutions of this system with $y_1 \neq y_2$ and $y_3 \neq y_4$ are

$$y_2 = -y_1, \quad y_3 = \sqrt{3+y_1^2}, \quad y_4 = -\sqrt{3+y_1^2}.$$

Since in the previous solutions the variable y_1 can take arbitrary values, there is a continuum of solutions, which in this case provide a continuum of periodic orbits. For instance for the particular solution $(y_1, y_2, y_3, y_4) = (-1, 1, 2, -2)$ we get the periodic orbit of Figure 1. This periodic orbit has been drawn computing the four pieces of the orbits of the three Hamiltonian systems passing for the four points $(1,-1)$, $(1,1)$, $(-1,2)$ and $(-1,-2)$. Moving the value of y_1 in a neigborhood of $y_1 = -1$ we obtain a continuum of periodic orbits. This completes the proof of the theorem. □

3. Proof of Theorem 2

We denote the three Hamiltonians which appear in the continuous piecewise differential system (1) as in (4), and their associated Hamiltonian systems as in (5).

Due to Lemma 4 if a continuous piecewise differential system has a limit cycle it must intersect the two straight lines $x = \pm 1$. We know from the proof of Theorem 1 that the continuity of the piecewise differential systems (1) on the straight line $x = 1$ imply that the Hamiltonians (4) reduce to the Hamiltonians (7).

Now we shall impose that the Hamiltonian systems (6) defined by the Hamiltonians H_2 and H_3 of (7) coincide on the straight line $x = -1$, and then these Hamiltonian systems become together with the Hamiltonian system defined by the Hamiltonian H_1 as follow

$$\begin{aligned} &\dot{x} = -a_2 - a_4 x - 2a_5 y, \qquad \dot{y} = a_1 + 2a_3 x + a_4 y;\\ &\dot{x} = -a_2 - a_4 x - 2a_5 y, \qquad \dot{y} = b_1 + (a_1 + 2a_3 - b_1)x + a_4 y;\\ &\dot{x} = -a_2 - a_4 x - 2a_5 y, \qquad \dot{y} = c_1 + (a_1 + 2a_3 - 2b_1 + c_1)x + a_4 y; \end{aligned} \tag{14}$$

and the corresponding Hamiltonians are

$$\begin{aligned} &H_1(x,y) = a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2,\\ &H_2(x,y) = b_1 x + a_2 y + \frac{1}{2}(a_1 + 2a_3 - b_1)x^2 + a_4 xy + a_5 y^2,\\ &H_3(x,y) = c_1 x + a_2 y + \frac{1}{2}(a_1 + 2a_3 - 2b_1 + c_1)x^2 + a_4 xy + a_5 y^2. \end{aligned} \tag{15}$$

Therefore we assume that the four intersection points of a limit cycle with the straight lines $x = 1$ are $(1, y_1)$, $(1, y_2)$, $(-1, y_2)$ and $(-1, y_4)$, with $y_1 < y_2$ and

$y_3 > y_4$, and since the functions $H_i(x,y)$ are first integrals it follows that y_1, y_2, y_3 and y_4 must satisfy the following four equations
(16)

$$\begin{aligned}
&H_1(1,y_1) - H_1(1,y_2) = (y_1 - y_2)(a_2 + a_4 + a_5 y_1 + a_5 y_2) = 0,\\
&H_2(1,y_2) - H_2(-1,y_3) = 2b_1 + a_2 y_2 + a_4 y_2 + a_5 y_2^2 - a_2 y_3 + a_4 y_3 - a_5 y_3^2 = 0,\\
&H_3(-1,y_3) - H_3(-1,y_4) = (y_3 - y_4)(c_2 - c_4 + c_5 y_3 + c_5 y_4) = 0,\\
&H_2(-1,y_4) - H_2(1,y_1) = -2b_1 - a_2 y_1 - a_4 y_1 - a_5 y_1^2 + a_2 y_4 - a_4 y_4 + a_5 y_4^2 = 0.
\end{aligned}$$

Since from the previous four equations we have that the second and fourth equations coincide, we have a system of three equations and four unknowns y_1, y_2, y_3 and y_4, so we have a continuum solutions. As before this means that if there are periodic orbits the continuous piecewise differential system has a continuum of periodic orbits, but it does not have limit cycles.

To end the proof of this theorem we provide an example of a continuous piecewise differential system (**??**) with a continuum set of periodic orbits. Consider the three Hamiltonians

$$\begin{aligned}
H_1(x,y) &= -\frac{4650118361}{4487643500}x - \frac{1}{2}y - \frac{22749782749}{22438217500}x^2 - \frac{9}{5}xy - 75y^2,\\
H_2(x,y) &= -45x - \frac{1}{2}y + \frac{311270139}{14845000}x^2 - \frac{9}{5}xy - 75y^2,\\
H_3(x,y) &= -128x - \frac{y}{2} - \frac{304797361}{14845000}x^2 - \frac{9}{5}xy - 75y^2.
\end{aligned} \tag{17}$$

System (8) for these particular Hamiltonians becomes

$$\begin{aligned}
H_1(1,y_1) - H_1(1,y_2) &= -\frac{1}{10}(y_1 - y_2)(23 + 750y_1 + 750y_2) = 0,\\
H_2(1,y_2) - H_2(-1,y_3) &= \frac{1}{10}(900 - 23y_2 - 750y_2^2 - 13y_3 + 750y_3^2) = 0,\\
H_3(-1,y_3) - H_3(-1,y_4) &= -\frac{1}{10}(y_3 - y_4)(-13 + 750y_3 + 750y_4) = 0,\\
H_2(-1,y_4) - H_2(1,y_1) &= \frac{1}{10}(-900 + 23y_1 + 750y_1^2 + 13y_4 - 750y_4^2) = 0.
\end{aligned}$$

The solutions of this system with $y_1 \neq y_2$ and $y_3 \neq y_4$ are

$$y_2 = \frac{-23 - 750y_1}{750}, \quad y_3 = \frac{13 + R}{1500}, \quad y_4 = \frac{13 - R}{1500},$$

where $R = \sqrt{-2699831 + 69000y_1 + 2250000y_1^2}$ Since in the previous solutions the variable y_1 can take arbitrary values, there is a continuum of solutions, which in this case provide a continuum of periodic orbits. For instance for the particular solution

$$(y_1, y_2, y_3, y_4) = \left(-\frac{24}{5}, \frac{3577}{750}, \frac{13 + \sqrt{54208969}}{1500}, \frac{13 - \sqrt{54208969}}{1500}\right)$$

we get the periodic orbit of Figure 2. Moving the value of y_1 in a neigborhood of $y_1 = -24/5$ we obtain a continuum of periodic orbits. This completes the proof of the theorem.

Declaration of competing interest

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Author contributions

Both authors have contributed equally to this paper.

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