

# LIMIT CYCLES IN FILIPPOV SYSTEMS HAVING A CIRCLE AS SWITCHING MANIFOLD

JAUME LLIBRE<sup>1</sup> AND MARCO ANTONIO TEIXEIRA<sup>2</sup>

ABSTRACT. It is known that planar discontinuous piecewise linear differential systems separated by a straight line have no limit cycles when both linear differential systems are centers.

Here we study the limit cycles of the planar discontinuous piecewise linear differential systems separated by a circle, when both linear differential systems are centers. Our main results show that such discontinuous piecewise differential systems can have 0, 1 or 2 limit cycles, but no more limit cycles than 2.

## 1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

**1.1. Historical facts.** The problem of existence of limit cycles has been extensively treated in the literature since the early days of celestial mechanics. More recently, much work has been done on the rigorous mathematical foundation of nonsmooth dynamical systems problems, in particular, in the search of typical minimal sets that there are no counterparts in the smooth universe. It is worth to mention that, some existing smooth techniques are useful in solving many nonsmooth problems.

Some of the orbits of planar differential systems are difficult to study, this is the case of the limit cycles. Recall that a *limit cycle* of a differential system (S) is a periodic solution of (S) which is isolated in the set of all periodic solutions of (S). Concerning the nonsmooth universe and in the 2-dimensional case one can find many results on the existence of limit cycles when the switching set is an imbedded curve in  $\mathbb{R}^2$ , see [1–15, 18–29].

One of the main properties of smooth integrable systems in the plane  $\mathbb{R}^2$  is that their periodic orbits usually appear in continuous one-parameter families, in contrast to the periodic orbits of piecewise nonsmooth integrable systems which typically are limit cycles, see [2–6, 18–28].

Andronov, Vitt and Khaikin [1] started the study of the discontinuous piecewise linear differential systems in the plane, mainly motivated for their applications to some mechanical problems. Recently the interest for this kind of differential systems increased due mainly to the fact that these differential systems model many processes appearing in mechanics, electronics, economy,... See for these applications the survey of Makarenkov and Lamb [27], and the books of Simpson [30] and of

---

2010 *Mathematics Subject Classification.* 34C29, 34C25, 47H11.

*Key words and phrases.* limit cycles, discontinuous piecewise differential systems, linear differential centers.

di Bernardo, Budd, Champneys and Kowalczyk [7], together with the hundreds of references quoted in these works.

In the plane the discontinuous piecewise linear differential systems separated by a straight line are the easiest possible discontinuous piecewise linear differential systems. For these systems a *crossing limit cycle* is a limit cycle which has exactly two points on the line of discontinuity. But for these easy systems there are open questions as the following: *Is three the maximum number of crossing limit cycles that a discontinuous piecewise linear differential systems separated by a straight line can have?* See for instance [6, 10–12, 14, 20, 23, 23].

**1.2. Setting the problem.** The main goal of this article is to discuss the existence of limit cycles of nonsmooth piecewise integrable systems in the plane  $\mathbb{R}^2$  where the switching set is concentrated on a closed curve. Recently, the mentioned great interest in non-smooth dynamics leads us the need to explore new challenges in this subject. In the classical smooth case we find in the literature an endless number of papers involving the problem on the existence of limit cycles in the plane, see for instance the books [31, 32] and the hundreds of references therein. Moreover, problems involving the existence of limit cycles bifurcating from a smooth center are extensively studied in the last decades, see the book [8] and the references quoted there.

We recall that a *center* of a planar differential system is a singular point  $p$  of the system for which there is a neighborhood  $U$  such that all the orbits of  $U \setminus \{p\}$  are periodic.

It is interesting to note that if we consider discontinuous piecewise linear differential systems formed by two centers separated by a straight line, then such systems have no crossing limit cycles, for a proof see Theorem 4 of [20].

In this paper we deal with discontinuous piecewise linear differential systems ( $C$ ) formed by two centers separated by a circle, that without loss of generality we can assume that is the circle

$$\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

A *crossing periodic orbit* for a system ( $C$ ) is a periodic orbit which has exactly two points on the circle  $\mathbb{S}^1$  of discontinuity, and of course a *crossing limit cycle* for a system ( $S$ ) in ( $C$ ) is a crossing periodic orbit isolated in the set of all crossing periodic orbits of system ( $S$ ).

To provide a sharp upper bound on the number of limit cycles for a class of planar differential systems is in general a very difficult problem, consider for instance the 16-th Hilbert problem. Thus, in particular, for the class of quadratic polynomial differential systems we have systems with four limit cycles, but we do not know if there is a uniform bound for the maximum number of limit cycles for all the systems of this class. See for more details [17, 18].

We shall provide an upper bound for the number of limit cycles of the planar discontinuous piecewise linear differential systems separated by a circle, when both linear differential systems are centers. We stress that the respective proof is non-trivial although the expressions of the systems are very friendly. It should be noted

that, as far as the authors know, there are few developed techniques or tools within the theory of discontinuous piecewise linear differential systems. So new techniques in this regard are, of course, notoriously welcomed.

**1.3. Statement of the main results.** The objective of this paper is to study the number of crossing limit cycles that systems in the class  $(C)$  can have, and to present an exotic center. Our main result is the following.

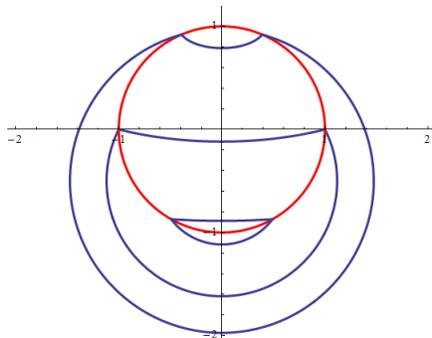


Figure 1: Three periodic solutions of the center  $(0, -1)$  of the discontinuous piecewise linear differential system formed by the centers (1) and (2).

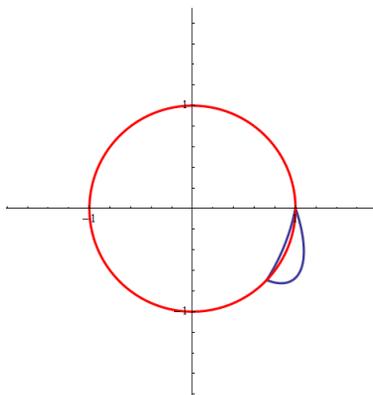


Figure 2: The unique limit cycle of the discontinuous piecewise linear differential system formed by the centers (3) and (4).

**Theorem 1.** *Let  $(C)$  be the class of planar discontinuous piecewise linear differential systems formed by two centers separated by the circle  $\mathbb{S}^1$ . Then the following statement hold.*

- (a) *There are systems in  $(C)$  with infinitely many crossing periodic orbits, in fact with a continuum of crossing periodic orbits forming a center, see Figure 1.*

- (b) *There are systems in  $(C)$  without crossing periodic orbits.*
- (c) *There are systems in  $(C)$  having exactly one crossing limit cycle, see Figure 2.*
- (d) *There are systems in  $(C)$  having exactly two crossing limit cycles, see Figure 3.*
- (e) *Every system in  $(C)$  has at most two crossing limit cycles.*

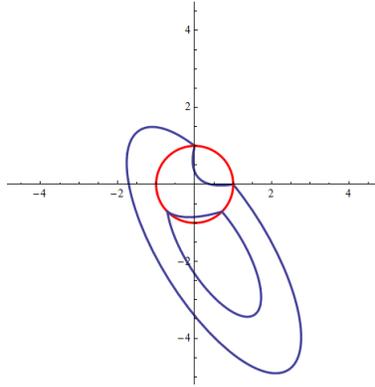


Figure 3: The two limit cycles of the discontinuous piecewise linear differential system formed by the centers (6) and (7).

Theorem 1 is proved in section 2.

The results of Theorem 1 show once again that the shape of the discontinuity line plays a main role on the number of crossing limit cycles that discontinuous planar piecewise linear differential systems can exhibit, see also [4, 5, 26, 28].

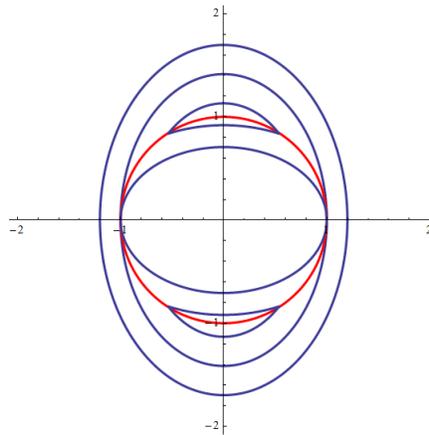


Figure 4: An exotic center.

We exhibit in section 3 a planar discontinuous piecewise linear differential systems formed by two centers separated by the circle  $\mathbb{S}^1$  possessing an unusual center (exotic center). That means that its phase portrait contains the origin as a canonical smooth center and there are on the switching set two 2-fold singularities of the center type, see Figure 4.

## 2. PROOF OF THEOREM 1

*Proof of statement (a) of Theorem 1.* We consider in the bounded region limited by the circle  $\mathbb{S}^1$  the linear differential center

$$(1) \quad \dot{x} = 1 - y, \quad \dot{y} = \frac{1}{4}x.$$

In all this paper the dot denotes derivative with respect to the time  $t$ , except if we say explicitly another thing. This system has the first integral

$$H_1(x, y) = \frac{1}{4}x^2 + y^2 - 2y.$$

In the unbounded region of  $\mathbb{R}^2$  with boundary the circle  $\mathbb{S}^1$  we have the linear differential center

$$(2) \quad \dot{x} = -\frac{1}{2} - y, \quad \dot{y} = x.$$

This system has the first integral

$$H_2(x, y) = x^2 + y^2 + y.$$

For all  $\alpha \in [0, 2\pi) \setminus \{\pm\pi/2\}$  the two ellipses

$$H_1(x, y) = H_1(\cos \alpha, \sin \alpha) \quad \text{and} \quad H_2(x, y) = H_2(\cos \alpha, \sin \alpha),$$

intersect exactly with the circle  $\mathbb{S}^1$  in the two points  $(\pm \cos \alpha, \sin \alpha)$ . Therefore the discontinuous piecewise linear differential system formed by the centers (1) and (2) has a continuum of crossing periodic solutions, which intersect the circle at the two points  $(\pm \cos \alpha, \sin \alpha)$  for all  $\alpha \in [0, 2\pi) \setminus \{\pm\pi/2\}$ . See three crossing periodic solutions of this continuum of crossing periodic solutions corresponding to the values of  $\alpha$  equal to 0, 1.0572619009906146.. and 5.114273122257285.. in Figure 1.  $\square$

*Proof of statement (b) of Theorem 1.* This statement has an easy proof it is sufficient to consider the discontinuous piecewise linear differential system having the linear center  $\dot{x} = -y, \dot{y} = x$  in the bounded region limited by the circle  $\mathbb{S}^1$ , and an arbitrary center different from the previous one in the unbounded region limited by  $\mathbb{S}^1$ . Clearly a such discontinuous piecewise linear differential system has no crossing periodic orbits.  $\square$

*Proof of statement (c) of Theorem 1.* In the unbounded region limited by the circle  $\mathbb{S}^1$  we consider the linear differential center

$$(3) \quad \dot{x} = \frac{1}{3} - \frac{x}{2} - \frac{y}{2}, \quad \dot{y} = x + \frac{y}{2} - \frac{1}{2}.$$

This system has the first integral

$$H_1(x, y) = 4 \left( x + \frac{y}{2} \right)^2 - 8 \left( \frac{x}{2} + \frac{y}{3} \right) + y^2.$$

In the bounded region of  $\mathbb{R}^2$  with boundary the circle  $\mathbb{S}^1$  we have the linear differential center

$$(4) \quad \dot{x} = \frac{2}{5} - y, \quad \dot{y} = 1 + x.$$

This system has the first integral

$$H_2(x, y) = (x + 1)^2 + \left( \frac{2}{5} - y \right)^2.$$

This discontinuous piecewise differential system formed by the linear differential systems (3) and (4) has exactly one crossing limit cycle, because the unique real solutions  $(\alpha, \beta, \gamma, \delta)$  of the system

$$(5) \quad \begin{aligned} H_1(\alpha, \beta) &= H_1(\gamma, \delta), \\ H_2(\alpha, \beta) &= H_2(\gamma, \delta), \\ \alpha^2 + \beta^2 &= 1, \\ \gamma^2 + \delta^2 &= 1, \end{aligned}$$

where  $(\alpha, \beta, \gamma, \delta)$  satisfying  $(\alpha, \beta) \neq (\gamma, \delta)$  are

$$\left( \frac{10(232 - 7\sqrt{29})}{2697}, \frac{-928 - 175\sqrt{29}}{2697}, \frac{10(232 + 7\sqrt{29})}{2697}, \frac{175\sqrt{29} - 928}{2697} \right),$$

and the one interchanging  $(\alpha, \beta)$  by  $(\gamma, \delta)$ , which defines the same crossing limit cycle. See this crossing limit cycle in Figure 2.  $\square$

*Proof of statement (d) of Theorem 1.* In the unbounded region limited by the circle  $\mathbb{S}^1$  we consider the linear differential center

$$(6) \quad \dot{x} = -x - y - \frac{1}{8} (9 + 5\sqrt{3}), \quad \dot{y} = 2x + y + \frac{5}{8} (1 + \sqrt{3}).$$

This system has the first integral

$$H_1(x, y) = 2x^2 + x(2y + \sqrt{2}) + y(y + \sqrt{2} + 1).$$

In the bounded region of  $\mathbb{R}^2$  with boundary the circle  $\mathbb{S}^1$  we have the linear differential center

$$(7) \quad \dot{x} = x - \frac{5y}{4} + \frac{1}{\sqrt{2}} + \frac{1}{8}, \quad \dot{y} = x - y - \frac{1}{\sqrt{2}}.$$

This system has the first integral

$$H_2(x, y) = 4x^2 - 4x(2y + \sqrt{2}) + y(5y - 4\sqrt{2} - 1).$$

This discontinuous piecewise differential system formed by the linear differential systems (6) and (7) has exactly two crossing limit cycles, because the unique real solutions  $(\alpha, \beta, \gamma, \delta)$  of the system (10) are the four solutions  $(1, 0, 0, 1)$ ,  $(0, 1, 1, 0)$ ,  $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$  and  $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ . Hence, the pairs  $(1, 0)$ ,

$(0, 1)$  and  $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$  are the intersection points of the two crossing limit cycles with the circle  $\mathbb{S}^1$ . See these two crossing limit cycles in Figure 3.  $\square$

We shall use the next result which is well known, see for instance [24, 26].

**Lemma 2.** *Doing a rescaling of the independent variable and a linear change of variables every linear differential system in  $\mathbb{R}^2$  having a center can be written as*

$$(8) \quad \dot{x} = -bx - \frac{4b^2 + \omega^2}{4a}y + d, \quad \dot{y} = ax + by + c,$$

with  $a > 0$  and  $\omega > 0$ .

*Proof of statement (e) of Theorem 1.* In the bounded region limited by the circle  $\mathbb{S}^1$  we consider the arbitrary linear differential center (8), which has the first integral

$$H_1(x, y) = 4(ax + by)^2 + 8a(cx - dy) + y^2\omega^2.$$

In the unbounded region limited by the circle  $\mathbb{S}^1$  we consider the arbitrary linear differential center

$$(9) \quad \dot{x} = -Bx - \frac{4B^2 + \Omega^2}{4A}y + dD, \quad \dot{y} = Ax + By + C,$$

with  $A > 0$  and  $\Omega > 0$ . This system has the first integral

$$H_2(x, y) = 4(Ax + By)^2 + 8A(Cx - Dy) + y^2\Omega^2.$$

If this discontinuous piecewise differential system formed by the linear differential systems (8) and (9) has a crossing limit cycle which intersect the circle  $x^2 + y^2 = 1$  in the pair of points  $(\alpha, \beta)$  and  $(\gamma, \delta)$ , then these points must satisfy the system

$$\begin{aligned} H_1(\alpha, \beta) &= H_1(\gamma, \delta), \\ H_2(\alpha, \beta) &= H_2(\gamma, \delta), \\ \alpha^2 + \beta^2 &= 1, \\ \gamma^2 + \delta^2 &= 1, \end{aligned}$$

with  $(\alpha, \beta) \neq (\gamma, \delta)$ , or equivalently the system

$$(10) \quad \begin{aligned} H_1(\alpha, \beta) &= h_1, \\ H_2(\alpha, \beta) &= h_2, \\ H_1(\gamma, \delta) &= h_1, \\ H_2(\gamma, \delta) &= h_2, \\ \alpha^2 + \beta^2 &= 1, \\ \gamma^2 + \delta^2 &= 1. \end{aligned}$$

The two first equations of system (10) has at most four real solutions for  $(\alpha, \beta)$ , because these solutions are the intersection of two ellipses. These four solutions are also the four solutions of the third and fourth equations of system (10). Since these four solutions on the circle  $x^2 + y^2 = 1$  must be the intersection points of the crossing limit cycles with the circle, we can have at most two crossing limit cycles.  $\square$

This completes the proof of Theorem 1.

## 3. EXOTIC CENTER

Let  $f(x, y) = x^2 + y^2 - 1$ ,  $H_0(x, y) = x^2 + \frac{y^2}{2}$  and

$$H_1(x, y; \theta) = (x \cos \theta + y \sin \theta)^2 + \frac{(-x \sin \theta + y \cos \theta)^2}{2}.$$

Notice that the Hamiltonian vector field  $X_0$  originated by  $H_0$  has four tangential contacts with  $\Sigma = f^{-1}(0)$  at  $p_1 = (0, 1)$ ,  $p_2 = (1, 0)$ ,  $p_3 = (0, -1)$  and  $p_4 = (-1, 0)$ . Since the level curves of  $H_1$  are the same level curves of  $H_0$  after a rotation of angle  $\theta$ , we obtain that the Hamiltonian vector field  $Y_\theta$  generated by  $H_1$  has also four contact points at  $q_1(\theta) = (\sin \theta, \cos \theta)$ ,  $q_2(\theta) = (\cos \theta, -\sin \theta)$ ,  $q_3(\theta) = (-\sin \theta, -\cos \theta)$ , and  $q_4(\theta) = (-\cos \theta, \sin \theta)$ .

Of course the orbits of  $X_0$  (resp.  $Y_\theta$ ) are symmetric with respect to the lines containing the segments  $\overline{p_1 p_2}$  and  $\overline{p_3 p_4}$  (resp.  $\overline{q_1 q_2}$  and  $\overline{q_3 q_4}$ ).

If  $\theta = \pi/2$  or  $\theta = 3\pi/2$ , we have that all the tangency points of the Hamiltonian systems  $X_0$  and  $Y_\theta$  coincide generating four fold–fold points. Since the phase portrait of this discontinuous piecewise linear differential system is symmetric with respect to the  $x$ -axis and the  $y$ -axis, we have that  $p_1$  and  $p_3$  are parabolic fold–fold points surrounded by crossing periodic orbits presenting a behavior similar to a singularity of type center. Also, the singularities  $p_2$  and  $p_4$  are connected by four orbits giving rise to pseudo–separatrices for the discontinuous piecewise linear system formed by  $X_0$  and  $Y_\theta$ .

If  $\theta = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$ , then it follows from symmetry that the orbits of  $Y_\theta$  give rise to two connections between the fold–regular points  $p_1$  and  $p_2$ .

If  $\theta$  is different from the values mentioned in items 1 and 2 the two previous paragraphs, then we have that the system presents a polycycle having two parabolic fold–fold singularities.

## ACKNOWLEDGMENTS

The first author is supported by the Ministerio de Economía, Industria y competitividad, Agencia Estatal de Investigación grant MTM2016-77278-P (FEDER), the Agència de Gestió d'Ajuts Universitaris i de Recerca grant 2017SGR1617, and the H2020 European Research Council grant MSCA-RISE-2017-777911. The second author is supported by ????????

## REFERENCES

- [1] A. ANDRONOV, A. VITT AND S. KHAIKIN, *Theory of oscillations*, Pergamon Press, Oxford, 1966.
- [2] J.C. ARTÉS, J. LLIBRE, J.C. MEDRADO AND M.A. TEIXEIRA, *Piecewise linear differential systems with two real saddles*, Math. Comp. Sim. **95** (2013), 13–22.
- [3] D.C. BRAGA AND L.F. MELLO, *Limit cycles in a family of discontinuous piecewise linear differential systems with two zones in the plane*, Nonlinear Dyn. **73** (2013), 1283–1288.
- [4] D.C. BRAGA AND L.F. MELLO, *More than three limit cycles in discontinuous piecewise linear differential systems with two zones in the plane*, Int. J. Bifurcation and Chaos **24** (2014), 145005610 pp.

- [5] D.C. BRAGA AND L.F. MELLO, *Arbitrary number of limit cycles for planar discontinuous piecewise linear differential systems with two zones*, Electron. J. Differential Equations **2015** (2015), No. 228, 12 pp.
- [6] C. BUZZI, C. PESSOA AND J. TORREGROSA, *Piecewise linear perturbations of a linear center*, Discrete Contin. Dyn. Syst. **33** (2013), 3915–3936.
- [7] M. DI BERNARDO, C.J. BUDD, A.R. CHAMPNEYS AND P. KOWALCZYK, *Piecewise-smooth dynamical systems: theory and applications*, Appl. Math. Sci. Series 163, Springer-Verlag, London, 2008.
- [8] C. CHRISTOPHER AND C. LI, *Limit cycles of differential equations*, Advanced Courses in Mathematics, CRM Barcelona, Birkhuser Verlag, Basel, 2007.
- [9] R.D. EUZÉBIO AND J. LLIBRE, *On the number of limit cycles in discontinuous piecewise linear differential systems with two pieces separated by a straight line*, J. Math. Anal. Appl. **424** (2015), 475–486.
- [10] E. FREIRE, E. PONCE AND F. TORRES, *Canonical discontinuous planar piecewise linear systems*, SIAM J. Appl. Dyn. Syst. **11** (2012), 181–211.
- [11] E. FREIRE, E. PONCE AND F. TORRES, *The discontinuous matching of two planar linear foci can have three nested crossing limit cycles*, Publ. Mat. Vol. extra (2014), 221–253.
- [12] E. FREIRE, E. PONCE AND F. TORRES, *A general mechanism to generate three limit cycles in planar Filippov systems with two zones*, Nonlinear Dyn. **78** (2014), 251–263.
- [13] M. HAN AND W. ZHANG, *On Hopf bifurcation in non-smooth planar systems*, J. of Differential Equations **248** (2010), 2399–2416.
- [14] S.M. HUAN AND X.S. YANG, *On the number of limit cycles in general planar piecewise linear systems*, Disc. Contin. Dyn. Sys.-Ser. A **32** (2012), 2147–2164.
- [15] S.M. HUAN AND X.S. YANG, *Existence of limit cycles in general planar piecewise linear systems of saddle–saddle dynamics*, Nonlinear Anal. **92** (2013), 82–95.
- [16] S.M. HUAN AND X.S. YANG, *On the number of limit cycles in general planar piecewise linear systems of node–node types*, J. Math. Anal. Appl. **411** (2013), 340–353.
- [17] YU. ILYASHENKO, *Centennial history of Hilbert’s 16th problem*, Bull. (New Series) Amer. Math. Soc. **39** (2002), 301–354.
- [18] J. LI, *Hilbert’s 16th problem and bifurcations of planar polynomial vector fields*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **13** (2003), 47–106.
- [19] J. LLIBRE, D.D. NOVAES AND M.A. TEIXEIRA, *Limit cycles bifurcating from the periodic orbits of a discontinuous piecewise linear differential center with two zones*, Int. J. of Bifurcation and Chaos **25** (2015), 1550144–pp 11.
- [20] J. LLIBRE, D.D. NOVAES AND M.A. TEIXEIRA, *Maximum number of limit cycles for certain piecewise linear dynamical systems*, Nonlinear Dyn. **82** (2015), 1159–1175.
- [21] J. LLIBRE, M. ORDÓÑEZ AND E. PONCE, *On the existence and uniqueness of limit cycles in a planar piecewise linear systems without symmetry*, Nonlinear Anal. Series B: Real World Appl. **14** (2013), 2002–2012.
- [22] J. LLIBRE AND E. PONCE, *Piecewise linear feedback systems with arbitrary number of limit cycles* Int. J. of Bifurcation and Chaos **13** (2003), 895–904.
- [23] J. LLIBRE AND E. PONCE, *Three nested limit cycles in discontinuous piecewise linear differential systems with two zones*, Dynamics of Continuous, Discrete & Impulsive Systems, Series B **19** (2012), 325–335.
- [24] J. LLIBRE AND M.A. TEIXEIRA, *Piecewise linear differential systems with only centers can create limit cycles?*, Nonlinear Dyn. **91** (2018), 249–255.
- [25] J. LLIBRE, M.A. TEIXEIRA AND J. TORREGROSA, *Lower bounds for the maximum number of limit cycles of discontinuous piecewise linear differential systems with a straight line of separation*, Int. J. Bifurcation and Chaos **23** (2013), 1350066, pp. 10.
- [26] J. LLIBRE AND X. ZHANG, *Limit cycles for discontinuous planar piecewise linear differential systems separated by an algebraic curve*, to appear in Int. J. Bifurcation and Chaos, (2018).
- [27] O. MAKARENKO AND J.S.W. LAMB, *Dynamics and bifurcations of nonsmooth systems: A survey*, Physica D **241** (2012), 1826–1844.
- [28] D.D. NOVAES AND E. PONCE, *A simple solution to the Braga–Mello conjecture*, Int. J. Bifurcation and Chaos **25** (2015), 15500097 pp.
- [29] S. SHUI, X. ZHANG AND J. LI, *The qualitative analysis of a class of planar Filippov systems*, Nonlinear Anal. **73** (2010), 1277–1288.

- [30] D.J.W. SIMPSON, *Bifurcations in piecewise-smooth continuous systems*, World Scientific Series on Nonlinear Science A, vol **69**, World Scientific, Singapore, 2010.
- [31] YE YANQIAN, *Theory of Limit Cycles*, Translations of Math. Monographs, Vol. **66**, Amer. Math. Soc, Providence, 1986.
- [32] Z. ZHANG, T. DING, W. HUANG, AND Z. DONG, *Qualitative theory of differential equations*, Vol. **101** of Trans. of Mathematical Monographs, Amer. Math. Soc., Providence, RI, 1992.

<sup>1</sup> DEPARTAMENT DE MATEMÀTIQUES, UNIVERSITAT AUTÒNOMA DE BARCELONA, 08193 BELLATERRA, BARCELONA, CATALONIA, SPAIN

*E-mail address:* `jllibre@mat.uab.cat`

<sup>2</sup> DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE ESTADUAL DE CAMPINAS, CAIXA POSTAL 6065, 13083-970, CAMPINAS, SP, BRAZIL

*E-mail address:* `teixeira@ime.unicamp.br`