ENTROPY AND PERIODS FOR CONTINUOUS GRAPH MAPS

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Abstract. For continuous maps from a topological graph into itself we provide new relationships between their topological entropy, their homology and their periods.

1. Introduction and statement of the main results

A *topological graph* or simpliy a *graph* G is a compact connected space having a finite set of points V such that $G \setminus V$ consists of finitely many connected components each of them homeomorphic to an open interval. Some graphs are homotopic to particular cases of wedge sums of circles, which we shall define later on. However not all graphs can be obtained in this way, e.g. the interval, in general any tree, the topological space with the shape of the capital letter sigma. We say that a *graph is trivial* if it is homotopic to a circle or to a point, e.g. intervals and trees are trivial graphs.

Given topological spaces X and Y with chosen points $x_0 \in X$ and $y_0 \in Y$, then the *wedge sum* $X \vee Y$ is the quotient of the disjoint union X and Y obtained by identifying x_0 and y_0 to a single point (for details, see [9, pp. 10]). The wedge sum is also known as "one point union". For example, $\mathbb{S}^1 \vee \mathbb{S}^1$ is homeomorphic to the figure of shpe "8", two circles touching at a point. Some graphs can be obtained as particular cases of wedge sums of $\mathbb{S}^1$, and a compact connected graph X such that $\dim(H_1(X, \mathbb{Q})) = s$ is homotopic to $\mathbb{S}^1 \vee \overset{s-\text{times}}{\cdots} \vee \mathbb{S}^1$, as usual here $H_1(X, \mathbb{Q})$ denotes the first homology group of the topological space X with coefficients in $\mathbb{Q}$. These spaces are also called *bouquet of circles*, we denote by $G_s := \mathbb{S}^1 \vee \overset{s-\text{times}}{\cdots} \vee \mathbb{S}^1$.

Since our techniques rely on homology, we shall mainly consider bouquet of circles, i.e. graphs of the type G_s, for some integer $s > 1$.

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We have that the homology spaces for G_s are: $H_0(G_s,\mathbb{Q}) = \mathbb{Q}$, since G_s is connected; and $H_1(G_s,\mathbb{Q}) = \underbrace{\mathbb{Q}\oplus\cdots\oplus\mathbb{Q}}_{s}$, by elementary properties of the wedge sum of spaces ([9, pp. 126]). For trivial graphs their $H_1(G_s,\mathbb{Q})$ is trivial (when there are homotopic to a point) or $\mathbb{Q}$ (when there are homotopic to a circle).

The *spectral radius* sp(T) of a linear transformation $T : U \to U$ on a finite dimensional vector space U is defined as the maximun of the norm of its eigenvalues, i.e.

$$\mathrm{sp}(T) := \max\{|\,\lambda| \,:\, \lambda \text{ is an eigenvalue of } T\}\,.$$

A continuous map from a graph into itself is called a *graph map*.

In the present article we explore some relationships between the topological entropy of a graph map, the induced map on homology and its periodic structure. In particular we give sufficient conditions on a graph map in order to have positive topological entropy (Theorem 4). In Proposition 5, we show that these conditions are not necessary conditions. Theorems 8 and 10 and Corollaries 9 and 11 show how the first s Lefschetz numbers of a graph map on G_s determine its periodic structure. In section 3 we extend the construction given in Proposition 5, so that we can have maps on G_s with positive topological entropy, all possible periods and the characteristic polynomial of the induced map on homology consists of products of cyclotomic polynomials of total degree s. Furthermore we present some open questions related to these matters.

1.1. **Topological entropy of graph maps.**

For a definition of topological entropy of a continuous map of a topological space into itself see for instance [1, 2, 14].

A well known result that relates the topological entropy $h(f)$ of a continuous map $f : X \to X$ with the homology of X is the following one due to Manning ([18]).

Theorem 1 (Manning)**.** *Let f be a C^0 self-map on a compact manifold. Then $h(f) \geq \log(\mathrm{sp}(f_{*1}))$.*

However for working on graphs-maps we cannot use Theorem 1 since it is valid for compact manifolds. There are particular results that deals the entropy for maps on graphs and the spectra of the induced maps on homology.

The *Lefschetz number* of a graph map f is defined as

$$L(f) := 1 - \mathrm{trace}(f_{*1}). \tag{1}$$

We note that $f_{*1} : H_1(G_s, \mathbb{Q}) \to H_1(G_s, \mathbb{Q})$ is a linear transformation that can be represented by an $s \times s$ matrix with integer entries.

The Lefschetz Fixed Point Theorem states that if $L(f) \neq 0$ then f has a fixed point (*cf.* [4] or [13]).

From (1) we get

$$L(f^m) = 1 - \mathrm{trace}(f_{*1}^m) = 1 - (\lambda_1^m + \cdots + \lambda_s^m),$$

where $\lambda_1, \ldots, \lambda_s$ are the eigenvalues of f_{*1}.

The *asymptotic Lefschetz number* $L^\infty(f)$ is defined to be the growth rate of the Lefschetz number of the iterates of f, i.e.

$$L^\infty(f) := \max\{1, \limsup_{n\to\infty} |L(f^m)|^{1/m}\}.$$

The asymptotic Lefschetz number allows to obtain a lower bound for the topological entropy of a continuous graph map.

Theorem 2. *Let $f : G \to G$ be a graph map.*

(a) $L^\infty(f) = \max\{1, \mathrm{sp}(f_{*1})\}$.
(b) *The topological entropy of f satisfies $h(f) \geq \log L^\infty(f)$.*

Statement (a) of Theorem 2 is proved in [8], and statement (b) is due to Jiang [10, 11].

The following result is well known, but since its proof is easy we shall provide it in section 2.

Corollary 3. *If the topological entropy of a graph map f is zero, then all the roots of the characteristic polynomial of f_{*1} are zero or roots of unity. Moreover $\mathrm{sp}(f_{*1})$ is either 0, or 1.*

Related with Theorem 2 and Corollary 3 we have the following question.

Open question 1. *Are there some conditions on f_{*1} with $\mathrm{sp}(f_{*1}) = 1$, which force that $h(f) > 0$?*

The *Lefschetz zeta function* of f is defined as

$$\zeta_f(t) := \exp\left(\sum_{m\geq 1} \frac{L(f^m)}{m} t^m\right).$$

Since $\zeta_f(t)$ is the generating function of all the Lefschetz numbers, $L(f^m)$, it keeps the information of the Lefschetz number for all the iterates of f. There is an alternative way to compute the Lefschetz zeta function of a graph map

$$\zeta_f(t) = \frac{\det(Id - tf_{*k1})}{1-t}, \tag{2}$$

where Id is the identity map on $H_1(G,\mathbb{Q})$ (*cf.* [5]).

Our first main result is the following one.

Theorem 4. *Let G be a graph, homotopic to G_s, for some $s>1$, and $f : G \to G$ be a continuous map.*

(a) *Let $p(t) = t^s - a_1 t^{s-1} + \cdots + (-1)^s a_s$ be the characteristic polynomial of f_{*1}. If $|a_k| > \binom{s}{k}$ for some $1 \le k \le s$, then the entropy of the map f is positive.*
(b) *If $L(f^m) > 1+s$ for some $m \ge 1$, then $h(f) > 0$.*
(c) *Let $\zeta_f(t)$ be the Lefschetz zeta function of f. Then $\zeta_f(t) = q(t)(1-t)^{-1}$, where $q(t) = b_0 t^{s-r} + \cdots + b_{s-r}$, where r is the multiplicity of 0 as eigenvalue of f_{*1}. If $|b_k| > \binom{s-r}{s-r-k}$ for some $0 \le k \le s-r$, then $h(f) > 0$.*

Theorem 4 is proved in section 2.

The next result shows that there are continuous maps of G_s with positive entropy, spectral radius equal to 1, and having all periods

Proposition 5. *Let G be a graph homotopic to G_s being s an arbitrary positive integer. Then there are continuous maps $f : G \to G$ such that $h(f) > 0$, $sp(f_{*1}) = 1$, $L(f^k) = 0$ for all positive integer k and $\mathrm{Per}(f) = \mathbb{N}$, being $\mathbb{N}$ the set of all positive integers.*

Proposition 5 is proved in section 2.

In [7] was proved the following result, however stated in a different manner:

Theorem 6 ([7])**.** *Let G be a graph, homotopic to G_s and $f : G \to G$ be a continuous map with f_{*1} the induced map on homology and $p(t)$ the characteristic polynomial of f_{*1}.*

(a) *If s is odd and the number of roots of $p(t)$ equal to 0 or ± 1, considering their multiplicity is even, then $h(f) > 0$.*
(b) *If s is even and the number of roots of $p(t)$ equal to 0 or ± 1, considering their multiplicity is odd, then $h(f) > 0$.*

The proof of this theorem is based in the fact the cyclotomic polynomials have even degree (with the exception of the first two), so the hypotheses force that the characteristic polynomial of f_{*1} has odd degree, so it has an eigenvalue of norm greater than 1.

Proposition 5 shows that the conditions given in Theorems 4 and 6 are not necessary conditions in order that a map has positive topological entropy.

1.2. **Periods of graphs maps.** In this subsection we provide some results which relate the periodic structure of a graph map with its homology and its topological entropy.

If $f : G \to G$ is a graph map. A point $x \in G$ is *periodic* of *period* k if $f^k(x) = x$ and $f^j(x) \neq x$ for $j = 1, \ldots, k-1$. We denote by $\mathrm{Per}(f)$ the set of periods of all periodic points of f.

First we recall a well-known result that relates the topological entropy and the periods of the map. This theorem was in proved [3] and [15].

Theorem 7. *Let $f : G \to G$ be a graph map. Then the entropy of f is positive if and only if there is an $m \in \mathbb{N}$ such that $\{km \,|\, k \in \mathbb{N}\} \subset \mathrm{Per}(f)$.*

From its proof it is not clear how the number m is related with the homology of the graph G, more precisely we have the following question.

Open question 2. *Let $f : G_s \to G_s$ be a graph map with positive entropy. We wonder if there is a relationship between the values $h(f)$, m, s (as in the statement of Theorem 7) and f_{*1}.*

In [17] it was shown the following relation between the periods of f and its homology on the graph G.

Theorem 8 ([17]). *Let G be a graph, homotopic to G_s and $f : G \to G$ be a continuous map such that it does not have periodic points of period k for $1 \leq k \leq n$, with $n \geq 2$. Assume that the induced map in the first homology space f_{*1} is invertible. Then $n < s$.*

A map f is *Lefschetz periodic point free* if $L(f^k) = 0$ for all positive integers k. From Theorem 8 it follows the next result.

Corollary 9. *Let G be a graph, homotopic to G_s and $f : G \to G$ be a continuous map. If f_{*1} is invertible, then f is not Lefschetz periodic point free.*

Corollary 9 is proved in section 2.

The following result is a refinement of Theorem 8, and it is proved in section 2 using the same tool: Newton's formulae for symmetric polynomials.

Theorem 10. *Let G be a graph, homotopic to G_s and $f : G \to G$ be a continuous map such that the characteristic polynomial of f_{*1} is $t^s - a_1 t^{s-1} + \cdots + (-1)^s a_s$. Then $L(f) = \cdots = L(f^j) = 0$ if and only if $a_j = 0$ with $j \in \{2, \ldots, s\}$.*

Corollary 11. *Let G be a graph homotopic to G_s and $f : G \to G$ be a continuous map. If $L(f) = \cdots = L(f^s) = 0$ then $L(f^k) = 0$ for all positive integer k.*

Corollary 11 is proved in section 2.

Some other results on the periods of graph maps can be found in [2, 6, 16].

In section 3 we show how to get graph maps of G_s with positive entropy, an infinite set $\mathrm{Per}(f)$, spectral radius equal to 1 and with a characteristic polynomial $p(t)$ of f_{*1} formed by an arbitrary product of cyclotonic polynomials and the factor t^r, where r is any non-negative integer. Without loss of generality we shall take $s = 2$, the arguments extend to any positive integer s.

2. Proofs of the results

Proof of Corollary 3. If all the eigenvalues of f_{*1} are zero, then $\mathrm{sp}(f_{*1}) = 0$, and the corollary is proved.

Let $\lambda_1, \ldots, \lambda_r$ with $1 \le r \le s$ be the non-zero eigenvalues of f_{*1}. By Theorem 2 we get that $|\lambda_j| \le 1$ for $j = 1, \ldots, r$. Then the characteristic polynomial of f_{*1} is of the form

$$t^{s-r} \prod_{j=1}^{r} (t - \lambda_j) = t^{s-r}(t^r - a_1 t^{r-1} + \cdots + (-1)^r a_r).$$

Since $a_r = \prod_{j=1}^{r} \lambda_j$ is a non-zero integer and $|\lambda_j| \le 1$ for $j = 1, \ldots, r$, we have that $|a_r| = \prod_{j=1}^{r} |\lambda_j| = 1$. Hence $|\lambda_j| = 1$ for $j = 1, \ldots, r$.

It is known that if a polynomial has integer coefficients, constant term equal to one and all its roots have modulus one, then all its roots are roots of unity, see for instance [20]. Therefore, clearly $\mathrm{sp}(f_{*1}) = 1$ and the corollary is proved. $\square$

Proof of Theorem 4. Let $\lambda_1, \cdots, \lambda_s$ be the eigenvalues of f_{*1}, i.e.

$$p(t) = \prod_{i=1}^{s}(t - \lambda_i) = t^s - a_1 t^{s-1} + \cdots + (-1)^s a_s.$$

We recall the Vieta's formulae:

$$\begin{aligned} a_s &= \lambda_1 \cdots \lambda_s, \\ a_{s-1} &= \sum_{i_1 < \cdots < i_{s-1}} \lambda_{i_1} \cdots \lambda_{i_{s-1}}, \\ &\vdots \\ a_2 &= \sum_{i<j} \lambda_i \lambda_j, \\ a_1 &= \lambda_1 + \cdots + \lambda_s. \end{aligned} \tag{3}$$

If we suppose that $h(f) = 0$, by statement (a) of Theorem 2 we have that $|\lambda_i| \leq 1$ for all $1 \leq i \leq s$. From (3) it follows

$$|a_1| \leq \sum_{i=1}^{s} |\lambda_i| \leq s.$$

For the general term:

$$|a_k| \leq \sum_{1 \leq i_1 < \cdots < i_k \leq s} |\lambda_{i_1} \cdots \lambda_{i_k}| \leq \sum_{1 \leq i_1 < \cdots < i_k \leq s} 1 = \binom{s}{k}.$$

This proves statement (a).

The Lefschetz numbers are $L(f^m) = 1 - \text{trace}(f_{*1}^m)$. If $h(f) = 0$, then $|\lambda_i| \leq 1$ for all $1 \leq i \leq s$. Therefore $|\text{trace}(f_{*1}^m)| \leq s$ for all $m \geq 1$. This proves statement (b).

The Lefschetz zeta function can be written as a rational function, the expression (2) in our context is

$$\zeta_f(t) = \frac{\det(Id_1 - t f_{*1})}{\det(Id_0 - t f_{*0})},$$

where Id_i is the identity map on $H_i(G, \mathbb{Q})$ for $i = 0, 1$. Therefore $\zeta_f(t) = q(t)(1-t)^{-1}$, where $q(t) = \det(Id_1 - t f_{*1})$.

If $p(t)$ is the characteristic polynomial of f_{*1}, i.e. $p(t) = \det(f_{*1} - tId_1)$, then $q(t) = (-1)^s t^s p(t^{-1})$. If r is the multiplicity of 0 as eigenvalue of f_{*1}, we set $r = 0$ if 0 is not an eigenvalue of f_{*1}, then

$$\begin{aligned} (-1)^s p(t) &= t^s - a_1 t^{s-1} + \cdots + (-1)^r a_{s-r} t^r \\ &= t^r (t^{s-r} - a_1 t^{s-r-1} + \cdots + (-1)^r a_{s-r}). \end{aligned}$$

So, if $q(t) = b_0 t^{s-r} + \cdots + b_{s-r}$ then $b_k = (-1)^k a_{s-r-k}$ for $0 \leq k \leq s-r$.

If we suppose that $h(f) = 0$, then all the roots of $p(t)$ have modulus smaller than or equal to one. We apply the argument based on the Vieta's formulae used in statement (a), to the polynomial $u(t) = t^{s-r} - a_1 t^{s-r-1} + \cdots + (-1)^r a_{s-r}$, if all its roots have norm smaller than or equal to 1, then $|b_k| \leq \binom{s-r}{k}$, because the degree of $u(t)$ is $s - r$. Therefore statement (c) follows from statement (a). $\square$

Proof of Proposition 5. We shall prove the proposition for $s = 2$, but this proof can be extended immediately to arbitrary s.

Let $g : \mathbb{S}^1 \to \mathbb{S}^1$ be a continuous circle map of degree 1 with positive topological entropy, infinite $\mathrm{Per}(g)$ and a fixed point p, see [2] for the existence of this kind of circle maps, and in particular, the map shown in Figure 1. Let $G_2 = \mathbb{S}^1 \vee \mathbb{S}^1$ the wedge sum of two circles, by identifying a point, we choose p for a such point. We denote the two circles of G_2 by S_1 and S_2. Let $f : G_2 \to G_2$ be the map defined as follows: $f(p) = p$, $f(x) = g(x)$ if $x \in S_1$ in such way that $f(S_1) = S_1$ and $f(S_2) = p$ for $x \in S_2$, see Figure 1. Then clearly

$$f_{*1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

Consequently $L(f^k) = 0$ for all $k \in \mathbb{N}$ because $\mathrm{trace}(f_{*1}^k) = 1$, and $\mathrm{Per}(f) = \mathbb{N}$ because it is easy to prove that such map restricted to the interval when we separate the circle S_1 by the point p has a periodic orbit of period 3 and by the Sharkovskii's theorem (see [2]) it follows that $\mathrm{Per}(f) = \mathbb{N}$. $\square$

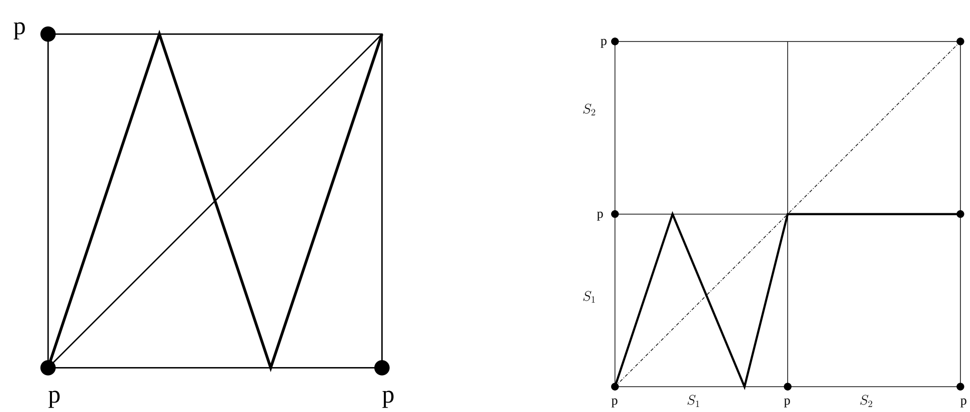

FIGURE 1. On the right the graphic of the circle-map g, and on the left the graphic of the G_2–map f.

Proof of Corollary 9. According to Theorem 8, if $L(f^k) = 0$ for $0 \leq k \leq n$ and f_{*1} invertible then $n < s$. Therefore if $L(f^k) = 0$ for all k then f_{*1} is not invertible. $\square$

Proof of Theorem 10. Let $\alpha_k := \mathrm{trace}(f_{*1}^k) = \sum_{i=1}^{s} \lambda_i^k$. Then the Newton's formulae for symmetric polynomials (*cf.* [19]) state

$$\begin{aligned}
\alpha_1 - a_1 &= 0, && (4)\\
\alpha_2 - a_1\alpha_1 + 2a_2 &= 0, && (5)\\
&\vdots \\
\alpha_s + \sum_{i=1}^{s-1}(-1)^i a_i \alpha_{s-i} + (-1)^s s a_s &= 0.
\end{aligned}$$

If $L(f) = 0$ then, from (1), (3) and (4) we have $a_1 = \alpha_1 = 1$. From equation (5) we get that $a_2 = 0$ if and only if $L(f^2) = 0$, provided that $L(f) = 0$. By induction we get $a_2 = \cdots = a_j = 0$ if and only if $L(f) = \cdots = L(f^j) = 0$ with $j \in \{2, \ldots, s\}$. $\square$

Proof of Corollary 11. By Theorem 10 and its proof, if $L(f) = \cdots = L(f^s) = 0$ then $a_s = \cdots = a_2 = 0$ and $a_1 = 1$, i.e. the characteristic polynomial of f_{*1} is $t^{s-1}(t-1)$. So all the eigenvalues of f_{*1} are zero except one eigenvalue which is equal to 1. Hence $\mathrm{trace}(f_{*1}^m) = 1$, i.e. $L(f^m) = 0$ for all m. $\square$

3. Some graph maps on G_2

Using the notation introduced in the proof of Proposition 5 let $f : G_2 \to G_2$ be the map defined as follows: $f(p) = p$, $f(x) = g(x)$ if $x \in S_1$ or $x \in S_2$ in such way that $f(S_1) = S_2$ preserving orientation and $f(S_2) = S_1$ in reversing orientation, see Figure 2. The map defined in this way, has positive topological entropy, infinite $\mathrm{Per}(f)$, moreover

$$f_{*1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{6}$$

and consequently its characteristic polynomial is $\Phi_4(t) = t^2 + 1$, the 4-th cyclotomic polynomial, see [12, 20].

Similarly we can get a map $f : G_2 \to G_2$ with positive topological entropy, infinite $\mathrm{Per}(f)$ with the characteristic polynomial of f_{*1} equal to the 3-th cyclotonic polynomial $\Phi_3(t) = t^2 + t + 1$ (see Figure 2) because

$$f_{*1} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}. \tag{7}$$

Using this method we can get $f : G_s \to G_s$ with positive topological entropy, infinite $\mathrm{Per}(f)$ and the characteristic polynomial of f_{*1} being any product of cyclotomic polynomials of total degree s.

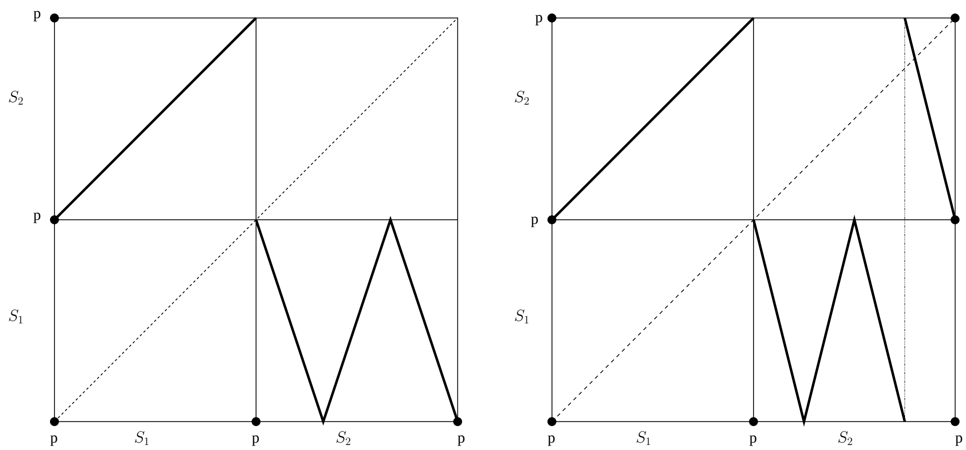

FIGURE 2. Maps $f : G_2 \to G_2$ such that f_{*1} is given by matrix (6) and (7) on the right and left respectively.

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REFERENCES

[1] R.L. ADLER, A.G. KONHEIM AND M.H. MCANDREW, Topological Entropy. Trans. Amer. Math. Soc. **114** (1965), 309–319.

[2] LL. ALSEDÀ, J. LLIBRE AND M. MISIUREWICZ, *Combinatorial Dynamics and Entropy in dimension one* (Second Edition), Advanced Series in Nonlinear Dynamics, vol **5**, World Scientific, Singapore 2000.

[3] A.M. BLOKH, *The spectral decomposition, periods of cycles, and Misiurewicz conjecture for graph maps*, Lecture Notes in Mathematics , Springer, Berlin, (1992), 24–31

[4] R.F. Brown, *The Lefschetz fixed point theorem*, Scott, Foresman and Company, Glenview, IL, 1971.
[5] J. Franks, *Homology and dynamical systems*, CBSM Regional Conf. Ser. in Math. **49**, Amer. Math. Soc., Providence, R.I. 1982.
[6] J.L. García Guirao. and J. Llibre, On the periods of a continuous selfmap on a graph, *Comput. Appl. Math.* **38**(78) (2019), pp 6.
[7] J.L. García Guirao, J. Llibre and W. Gao, Topological entropy of continuous self-maps on a graph, *Comput. Appl. Math.* **38**(154) (2019), pp. 10.
[8] J. Guaschi and J. Llibre, Periodic points of C^1 maps and the asymptotic Lefschetz number, *Inter. J. of Bifurcation and Chaos*, **5** (1995), 1369–1373.
[9] A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2002.
[10] B. Jiang, Nielsen theory for periodic orbits and applications to dynamical systems, *Contemp. Math.*, **152** (1993), 183–202.
[11] B. Jiang, Estimation of the number of periodic orbits, *Pacific J. of Math.*, **172** (1996), 151–185.
[12] S. Lang, *Algebra*, Third Edition, Graduate Texts in Mathematics **211**, Springer, 2005.
[13] S. Lefschetz, Intersections and transformations of complexes and manifolds, *Trans. Amer. Math. Soc.* **28** (1926), 1–49.
[14] J. Llibre, Brief survey on the topological entropy, *Discrete Contin. Dyn. Syst. Ser. B* **20** (2015), 3363–3374.
[15] J. Llibre and M. Misiurewicz, Horseshoes, entropy and periods for graphs maps, *Topology*, **52**(1993), 649–664.
[16] J. Llibre and A. Sá, Periods for continuous self-maps of a bouquet of circles, *C. R. Acad. Sci. Paris. Sér I Maths.* **318** (1994), 1035–1040.
[17] J. Llibre and V.F. Sirvent, *Partially periodic point free self-maps on surfaces, graphs, wedge sums and products of spheres*, J. Difference Equ. Appl. **19** (2013), 1654–1662.
[18] A. Manning, Topological entropy and the first homology group, in *Dynamical systems – Warwick 1974*, Lecture Notes in Math. 468, Springer–Verlag, Berlin, 1975, 185–190.
[19] J.P. Tignol, *Galois' theory of algebraic equations*, Singapore, World Scientific, 2001.
[20] L.C. Washington, *Introduction to cyclotomic fields*, Springer, Berlin, 1982.

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