Periodic orbits and equilibria for a seventh-order generalized Hénon-Heiles Hamiltonian system

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Abstract

In this paper we study analytically the existence of two families of periodic orbits using the averaging theory of second order, and the finite and infinite equilibria of a generalized Hénon-Heiles Hamiltonian system which includes the classical Hénon-Heiles Hamiltonian. Moreover we show that this generalized Hénon-Heiles Hamiltonian system is not C^1 integrable in the sense of Liouville–Arnol'd, i.e. it has not a second C^1 first integral independent with the Hamiltonian. The techniques that we use for obtaining analytically the periodic orbits and the non C^1 Liouville-Arnol'd integrability, can be applied to Hamiltonian systems with an arbitrary number of degrees of freedom.

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1. Introduction and statement of results

The classical Hénon-Heiles Hamiltonian consist of a two dimensional harmonic potential plus two cubic terms,i.e.

$$H = \frac{1}{2}(p_x^2 + p_y^2 + x^2 + y^2) + x^2 y - \frac{y^3}{3}.$$

This Hamiltonian was introduced in 1964, it is a model for studying the existence of a third integral of motion of a star in an rotating meridian plane of a galaxy in the neighborhood of a circular orbit [1].

The generalized Hénon-Heiles Hamiltonian system here studied is

$$\begin{aligned} H_\varepsilon = \; & \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + x^2 y - \frac{y^3}{3} + \varepsilon\Big(x^6 y + x^4 y^3 + x^4 y \\ & + x^2 y^5 + x^2 y^3 - \frac{y^7}{7} - \frac{y^5}{5} + \frac{1}{4}(x^2+y^2)^2 + \frac{1}{6}(x^2+y^2)^3\Big), \end{aligned} \tag{1}$$

where $\varepsilon \geq 0$ is a small parameter. Of course, when $\varepsilon = 0$ the Hamiltonian H_0 is the classical Hénon-Heiles Hamiltonian. The Hamiltonian (1) was introduced in [2].

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The Hamiltonian system associated to the Hamiltonian (1) is

$$
\begin{aligned}
\dot{x} = &\quad p_x,\\
\dot{y} = &\quad p_y,\\
\dot{p}_x = &\quad -x-2xy-\varepsilon\Big(6x^5y+4x^3y^3+4x^3y+2xy^5+2xy^3\\
&\qquad\qquad +x(x^2+y^2)+x(x^2+y^2)^2\Big),\\
\dot{p}_y = &\quad -y-x^2+y^2-\varepsilon\Big(x^6+3x^4y^2+x^4+5x^2y^4+3x^2y^2-y^6-y^4\\
&\qquad\qquad +y(x^2+y^2)+y(x^2+y^2)^2\Big),
\end{aligned}
\tag{2}
$$

here the dot denotes derivative with respect to the time t.

The equilibrium points of the generalized Hénon-Heiles Hamiltonian system are analyzed in section 2. As we shall see are the four equilibrium points of the classical Hénon-Heiles Hamiltonian system slightly perturbed.

In general to study the orbits which go or come from the infinity of a differential system defined in $\mathbb{R}^4$ is a difficult task, but when the differential system is polynomial as system (2), Poincaré in [3] and its generalization in [4], provided a tool for doing this study. This tool is now called the Poincaré compactification. Here we will describe the Poincaré compactification for a polynomial differential system in $\mathbb{R}^4$, because the domain of definition of system (2) is $\mathbb{R}^4$.

Roughly speaking the Poincaré compactification consists in identifying $\mathbb{R}^4$ with the interior of the unit closed ball $\mathbb{B}^6$ of $\mathbb{R}^4$ centered at the origin of coordinates. Then the boundary of this ball, the 5–dimensional sphere $\mathbb{S}^3$, is identified with the infinity of $\mathbb{R}^4$, because in $\mathbb{R}^4$ we can go to infinity in as many as directions as points has the sphere $\mathbb{S}^3$. There is a technique which extend a polynomial differential system from the interior of the ball $\mathbb{B}^4$ to its boundary in an analytic way, in such a way that the boundary $\mathbb{S}^3$ is invariant by the extended flow, i.e. if an orbit of the extended differential system has a point in the sphere $\mathbb{S}^3$ the whole orbit remains on $\mathbb{S}^3$. Then studying the dynamics of the extended differential system on $\mathbb{S}^3$ we understand the dynamics in a neighborhood of the infinity of the initial polynomial differential system defined in $\mathbb{R}^4$. For instance, if the extended differential system has an equilibrium point on $\mathbb{S}^3$ which is a local attractor, then we know that there is a set of dimension 4 of orbits of the polynomial differential system which escapes to infinity in the direction defined by this equilibrium point.

In section 3 we provide the explicit equations of the Poincaré compactification in $\mathbb{R}^4$. Using this compactification we shall prove that the classical and the generalized Hénon-Heiles Hamiltonian systems has no infinite equilibrium points, i.e. equilibrium point in the sphere $\mathbb{S}^3$.

After the equilibrium points the periodic orbits are the most simple interesting orbits of a differential system. This is due mainly to the following two facts. First the periodic orbits provide information on the motion in their neighborhoods studying their type of stability. Moreover, if there are isolated periodic orbits having some multiplier distinct from 1 in the energy levels of the Hamiltonian system (2) this orbit prevents the existence of a second $\mathcal{C}^1$ first integral independent with the Hamiltonian, see details in section 6.

In section 4 we present a brief introduction to the averaging theory of second order. Using this theory we shall compute two families of periodic orbits of the classical and generalized Hénon–Heiles Hamiltonian system (2), and we obtain the following result.

Theorem 1. *The generalized Hamiltonian system for ε sufficiently small in each Hamiltonian level $H = \varepsilon^2 h > 0$ has two periodic solutions of the form*

$$
\begin{aligned}
&(x(t,\varepsilon),y(t,\varepsilon),p_x(t,\varepsilon),p_y(t,\varepsilon)) =\\
&(\varepsilon\sqrt{h}\cos t+O(\varepsilon^2),\pm\varepsilon\sqrt{h}\sin t+O(\varepsilon^2),-\varepsilon\sqrt{h}\sin t+O(\varepsilon^2),\pm\varepsilon\sqrt{h}\cos t+O(\varepsilon^2)).
\end{aligned}
$$

Theorem 1 is proved in section 5.

Theorem 2. *The generalized Hamiltonian system for ε sufficiently small in each Hamiltonian level $H = \varepsilon^2 h > 0$ satisfies*

(a) either it is Liouville-Arnol'd integrable and the gradients of the two constants of motion are linearly dependent on some points of the two periodic orbits found in Theorem 1,

(b) or it is not Liouville–Arnol'd integrable with any second C^1 first integral.

Theorem 2 is proved in section 6. At the beginning of this section we recall the notion of Liouville-Arnol'd integrability.

2. The finite equilibria

The classical Hénon-Heiles Hamiltonian system has four finite equilibria (x, y, p_x, p_y), namely

$$p_1 = (0,0,0,0), p_2 = (0,1,0,0), p_3 = \left(\frac{-\sqrt{3}}{2}, -\frac{1}{2}, 0, 0\right), p_4 = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}, 0, 0\right),$$

The eigenvalues of the linear part of the Hamiltonian system at p_1 are $\pm i$ with multiplicity two, and the corresponding eigenvalues at p_j for $j = 2, 3, 4$ are ± 1 and $\pm\sqrt{3}\, i$.

For $\varepsilon > 0$ sufficiently small the equilibria of the Hamiltonian system (2) are

$$\begin{aligned} &p_1(\varepsilon) = (0,0,0,0), && p_2(\varepsilon) = (0,1,0,0),\\ &p_3(\varepsilon) = (x_3(\varepsilon), y(\varepsilon), 0, 0), && p_4(\varepsilon) = (x_4(\varepsilon), y(\varepsilon), 0, 0), \end{aligned} \tag{3}$$

where

$$\begin{aligned} x_3(\varepsilon) &= -\frac{\sqrt{3}}{2} + \sqrt{3}\varepsilon - \frac{2585\sqrt{3}\varepsilon^2}{256} + \frac{144335\sqrt{3}\varepsilon^3}{1024} - \frac{149836699\sqrt{3}\varepsilon^4}{65536} + O\left(\varepsilon^5\right),\\ x_4(\varepsilon) &= \frac{\sqrt{3}}{2} - \sqrt{3}\varepsilon + \frac{2585\sqrt{3}\varepsilon^2}{256} - \frac{144335\sqrt{3}\varepsilon^3}{1024} + \frac{149836699\sqrt{3}\varepsilon^4}{65536} + O\left(\varepsilon^5\right),\\ y(\varepsilon) &= -\frac{1}{2} + \frac{15\varepsilon}{16} - \frac{1233\varepsilon^2}{128} + \frac{138483\varepsilon^3}{1024} - \frac{18018531\varepsilon^4}{8192} + O\left(\varepsilon^5\right). \end{aligned}$$

The eigenvalues of the linear part of the Hamiltonian system at $p_1(\varepsilon)$ are again $\pm i$ with multiplicity two. The corresponding eigenvalues at $p_2(\varepsilon)$ are $\pm\sqrt{1+2\varepsilon}$ and $\pm\sqrt{3(1+2\varepsilon)}i$. While the eigenvalues of $p_j(\varepsilon)$ for $j = 3, 4$ are

$$\pm\frac{65536 + 268288\varepsilon - 3097664\varepsilon^2 + 48298085\varepsilon^3}{65536} + O\left(\varepsilon^4\right),$$

and

$$\pm\frac{\sqrt{3}(65536 - 14336\varepsilon + 273280\varepsilon^2 - 4085287\varepsilon^3)}{65536}\, i + O\left(\varepsilon^4\right).$$

In short we have proved the following proposition.

Proposition 3. *For $\varepsilon \geq 0$ sufficiently small the generalized Hénon-Heiles Hamiltonian system has four finite singular points, namely $p_j(\varepsilon)$ for $j = 1, 2, 3, 4$ given in (3).*

3. The infinite equilibria:

3.1. The Poincaré compactification in $\mathbb{R}^4$

In $\mathbb{R}^4$ we consider a polynomial differential system

$$\dot{x}_k = P_k(x_1, x_2, x_3, x_4), \quad \text{for } k = 1, 2, 3, 4,$$

or equivalently its associated polynomial vector field $\mathcal{X} = (P_1, P_2, P_3, P_4)$. The degree n of X is defined as $n = \max\{\deg(P_i) : i = 1, 2, 3, 4\}$.

We have that $\mathbb{S}^4 = \{y = (y_1, y_2, y_3, y_4, y_5) \in \mathbb{R}^5 : \|y\| = 1\}$, and let

$$\mathbb{S}_+ = \{y \in \mathbb{S}^4 : y_5 > 0\} \quad \text{and} \quad \mathbb{S}_- = \{y \in \mathbb{S}^4 : y_5 < 0\}$$

be the northern and southern hemispheres of the sphere $\mathbb{S}^4$, respectively. The tangent space to $\mathbb{S}^4$ at the point y is denoted by $T_y\mathbb{S}^4$. Then the tangent hyperplane

$$T_{(0,0,0,0,1)}\mathbb{S}^4 = \{(x_1, x_2, x_3, x_4, 1) \in \mathbb{R}^5 : (x_1, , x_2, x_3, x_4) \in \mathbb{R}^4\}$$

is identified with $\mathbb{R}^4$.

We consider the central projections

$$f_+ : \mathbb{R}^4 = T_{(0,0,0,0,1)}\mathbb{S}^4 \to \mathbb{S}_+ \quad \text{and} \quad f_- : \mathbb{R}^4 = T_{(0,0,0,0,1)}\mathbb{S}^4 \to \mathbb{S}_-,$$

defined by

$$f_+(x) = \frac{1}{\Delta x}(x_1, x_2, x_3, x_4, 1) \quad \text{and} \quad f_-(x) = -\frac{1}{\Delta x}(x_1, x_2, x_3, x_4, 1)\,,$$

where $\Delta x = \left(1 + \sum_{i=1}^4 x_i^2\right)^{1/2}$. Through these central projections, $\mathbb{R}^4$ can be identified with the northern and the southern hemispheres, respectively. The equator of $\mathbb{S}^4$ is $\mathbb{S}^3 = \{y \in \mathbb{S}^4 : y_5 = 0\}$. Clearly $\mathbb{S}^3$ can be identified with the *infinity* of $\mathbb{R}^4$.

The maps f_+ and f_- define two copies of X, one $Df_+ \circ X$ in the northern hemisphere and the other $Df_- \circ X$ in the southern one. Denote by $\overline{X}$ the vector field on $\mathbb{S}^4 \setminus \mathbb{S}^3 = \mathbb{S}_+ \cup \mathbb{S}_-$ which restricted to $\mathbb{S}_+$ coincides with $Df_+ \circ X$ and restricted to sss_- coincides with $Df_- \circ X$.

In what follows we shall work with the orthogonal projection of the closed northern hemisphere to $y_5 = 0$. Note that this projection is the closed ball $\mathbb{B}^4$ of radius one centered at the origin of coordinates, whose interior is diffeomorphic to $\mathbb{R}^4$ and whose boundary $\mathbb{S}^3$ corresponds to the infinity of $\mathbb{R}^4$. We shall extend analytically the polynomial vector field $\overline{X}$ defined on $\mathbb{S}_+ \cup \mathbb{S}_-$ to its boundary $\mathbb{S}^3$, in such a way that the flow on the boundary be invariant. This new vector field on $\mathbb{B}^4$ will be called the *Poincaré compactification* of the vector field $\mathcal{X}$, and $\mathbb{B}^6$ will be called the *Poincaré ball*. Poincaré [3] introduced this compactification for polynomial vector fields in $\mathbb{R}^2$, and its extension to $\mathbb{R}^m$ can be found in [4].

Now we can extend analytically the vector field $\overline{X}(y)$ to the whole sphere $\mathbb{S}^4$ defining the new vector field

$$p(X)(y) = y_5^{n-1}\overline{X}(y).$$

This extended vector field $p(X)$ is called the *Poincaré compactification* of $\mathcal{X}$.

As $\mathbb{S}^4$ is a differentiable manifold, to compute the expression for $p(X)$ we can consider the ten local charts (U_i, F_i), (V_i, G_i) where $U_i = \{y \in \mathbb{S}^4 : y_i > 0\}$, and $V_i = \{y \in \mathbb{S}^4 : y_i < 0\}$ for $i = 1, \ldots, 5$; the diffeomorphisms $F_i : U_i \to \mathbb{R}^4$ and $G_i : V_i \to \mathbb{R}^4$ for $i = 1, \ldots, 5$ the inverses of the central projections from the origin to the tangent planes at the points $(\pm 1, 0, 0, 0, 0), (0, \pm 1, 0, 0, 0), \ldots, (0, 0, 0, 0, \pm 1)$, respectively.

We now determine the expression of the extended vector field $p(X)$ on the local chart U_1. Suppose that the origin $(0, 0, 0, 0, 0)$, the point $(y_1, \ldots, y_5) \in \mathbb{S}^4$ and the point $(1, z_1, \ldots, z_4)$ in the tangent plane to $\mathbb{S}^4$ at $(1, 0, 0, 0, 0)$ are collinear, then we have

$$\frac{1}{y_1} = \frac{z_1}{y_2} = \frac{z_2}{y_3} = \frac{z_3}{y_4} = \frac{z_4}{y_5},$$

and consequently

$$F_1(y) = \left(\frac{y_2}{y_1}, \frac{y_3}{y_1}, \frac{y_4}{y_1}\frac{y_5}{y_1}\right) = (z_1, z_2, z_3, z_4)$$

defines the coordinates on U_1.

As

$$DF_1(y) = \begin{pmatrix} -y_2/y_1^2 & 1/y_1 & 0 & 0 & 0 \\ -y_3/y_1^2 & 0 & 1/y_1 & 0 & 0 \\ -y_4/y_1^2 & 0 & 0 & 1/y_1 & 0 \\ -y_5/y_1^2 & 0 & 0 & 0 & 1/y_1 \end{pmatrix}$$

and $y_5^{n-1} = \left(\frac{z_4}{\Delta z}\right)^{n-1}$. Then the analytical vector field $p(X)$ in the local chart U_1 becomes

$$\frac{z_4^n}{(\Delta z)^{n-1}} \left(-z_1P_1 + P_2, -z_2P_1 + P_3, -z_3P_1 + P_4, -z_4P_1\right), \tag{4}$$

where $P_i = P_i\left(1/z_4, z_1/z_4, z_2/z_4, z_3/z_4\right)$.

In a similar way we can deduce the expressions of $p(X)$ in the local charts U_k for $k = 2, 3, 4, 5$. These are

$$\frac{z_4^n}{(\Delta z)^{n-1}} \left(-z_1P_2 + P_1, -z_2P_2 + P_3, -z_3P_2 + P_4, -z_4P_2\right), \tag{5}$$

where $P_i = P_i\left(z_1/z_4, 1/z_4, z_2/z_4, z_3/z_4\right)$ in U_2,

$$\frac{z_4^n}{(\Delta z)^{n-1}} \left(-z_1P_3 + P_1, -z_2P_3 + P_2, -z_3P_3 + P_4, -z_4P_3\right), \tag{6}$$

where $P^i = P^i\left(z_1/z_4, z_2/z_4, 1/z_4, z_3/z_4\right)$ in U_3,

$$\frac{z_4^n}{(\Delta z)^{n-1}} \left(-z_1P_4 + P_1, -z_2P_4 + P_2, -z_3P_4 + P_3, -z_4P_4\right), \tag{7}$$

where $P^i = P^i\left(z_1/z_4, z_2/z_4, z_3/z_4, 1/z_4\right)$ in U_4. The expression of $p(X)$ in U_5 is $z_4^{n+1}\left(P_1, P_2, P_3, P_4\right)$ where the component $P_i = P_i\left(z_1, z_2, z_3, z_4\right)$.

The expression of $p(X)$ in the local chart V_i is the same as in U_i multiplied by $(-1)^{n-1}$.

When we shall work with the expression of the compactified vector field $p(X)$ in the local charts we shall omit the factor $1/(\Delta z)^{n-1}$. We can do that through a rescaling of the time.

We remark that all the points on the sphere at infinity in the coordinates of any local chart have $z_4 = 0$.

3.2. The Poincaré compactification of system (2) in the local chart U_1

In this chart from (4) system (2) writes

$$\begin{aligned}
\dot{z}_1 =& \quad -(z_2 - z_3)z_4^5, \\
\dot{z}_2 =& \quad -z_4^4(2z_1 + z_4 + z_2^2 z_4) + \varepsilon(-6z_1 - 4z_1^3 - 2z_1^5 - z_4 - 2z_1^2 z_4 \\
& \quad -z_1^4 z_4 - 4z_1 z_4^2 - 2z_1^3 z_4^2 - z_4^3 - z_1^2 z_4^3), \\
\dot{z}_3 =& \quad -z_4^4(1 - z_1^2 + z_1 z_4 + z_2 z_3 z_4) + \varepsilon(-1 - 3z_1^2 - 5z_1^4 + z_1^6 \\
& \quad -z_1 z_4 - 2z_1^3 z_4 - z_1^5 z_4 - z_4^2 - 3z_1^2 z_4^2 + z_1^4 z_4^2 - z_1 z_4^3 - z_1^3 z_4^3), \\
\dot{z}_4 =& \quad -z_2 z_4^6.
\end{aligned}$$

This system has no infinite equilibria $(z_1, z_2, z_3, 0)$.

3.3. The Poincaré compactification of system (2) in the local chart U_2

In this chart from (5) system (2) writes

$$\begin{aligned}
\dot z_1 =&\quad (z_2 - z_1 z_3) z_4^5,\\
\dot z_2 =&\quad -z_4^4(2z_1 + z_1 z_4 + z_2 z_3 z_4) - \varepsilon z_1(2 + 4z_1^2 + 6z_1^4 + z_4 + 2z_1^2 z_4\\
&\quad + z_1^4 z_4 + 2z_4^2 + 4z_1^2 z_4^2 + z_4^3 + z_1^2 z_4^3),\\
\dot z_3 =&\quad -z_4^4(-1 + z_1^2 + z_4 + z_3^2 z_4) + \varepsilon(1 - 5z_1^2 - 3z_1^4 - z_1^6 - z_4\\
&\quad -2z_1^2 z_4 - z_1^4 z_4 + z_4^2 - 3z_1^2 z_4^2 - z_1^4 z_4^2 - z_4^3 - z_1^2 z_4^3),\\
\dot z_4 =&\quad -z_3 z_4^6.
\end{aligned} \tag{8}$$

In system (8) we must look for the infinite equilibria of the form $(0, z_2, z_3, 0)$, because if there were infinite equilibria with $z_1 \neq 0$ these would be appear in the local chart U_1. But system (8) has no equilibria of the form $(0, z_2, z_3, 0)$.

3.4. The Poincaré compactification of system (2) in the local chart U_3

In this chart from (6) system (2) writes

$$\begin{aligned}
\dot z_1 =&\quad z_4^4(2z_1^2 z_2 + z_4 + z_1^2 z_4) + \varepsilon z_1^2(6z_1^4 z_2 + 4z_1^2 z_2^3 + 2z_2^5 + z_1^4 z_4\\
&\quad +2z_1^2 z_2^2 z_4 + z_2^4 z_4 + 4z_1^2 z_2 z_4^2 + 2z_2^3 z_4^2 + z_1^2 z_4^3 + z_2^2 z_4^3),\\
\dot z_2 =&\quad z_4^4(2z_1 z_2^2 + z_1 z_2 z_4 + z_3 z_4) + \varepsilon z_1 z_2(6z_1^4 z_2 + 4z_1^2 z_2^3 + 2z_2^5\\
&\quad +z_1^4 z_4 + 2z_1^2 z_2^2 z_4 + z_2^4 z_4 + 4z_1^2 z_2 z_4^2 + 2z_2^3 z_4^2 + z_1^2 z_4^3 + z_2^2 z_4^3),\\
\dot z_3 =&\quad -z_4^4(z_1^2 - z_2^2 - 2z_1 z_2 z_3 + z_2 z_4 - z_1 z_3 z_4) + \varepsilon(-z_1^6 - 3z_1^4 z_2^2\\
&\quad -5z_1^2 z_2^4 + z_2^6 + 6z_1^5 z_2 z_3 + 4z_1^3 z_2^3 z_3 + 2z_1 z_2^5 z_3 - z_1^4 z_2 z_4\\
&\quad -2z_1^2 z_2^3 z_4 - z_2^5 z_4 + z_1^5 z_3 z_4 + 2z_1^3 z_2^2 z_3 z_4 + z_1 z_2^4 z_3 z_4 - z_1^4 z_4^2\\
&\quad -3z_1^2 z_2^2 z_4^2 + z_2^4 z_4^2 + 4z_1^3 z_2 z_3 z_4^2 + 2z_1 z_2^3 z_3 z_4^2 - z_1^2 z_2 z_4^3 - z_2^3 z_4^3\\
&\quad +z_1^3 z_3 z_4^3 + z_1 z_2^2 z_3 z_4^3),\\
\dot z_4 =&\quad z_1 z_4^5(2z_2 + z_4) + \varepsilon z_1 z_4(6z_1^4 z_2 + 4z_1^2 z_2^3 + 2z_2^5 + z_1^4 z_4 + 2z_1^2 z_2^2 z_4\\
&\quad +z_2^4 z_4 + 4z_1^2 z_2 z_4^2 + 2z_2^3 z_4^2 + z_1^2 z_4^3 + z_2^2 z_4^3).
\end{aligned} \tag{9}$$

In system (9) we must look for the infinite equilibria of the form $(0, 0, z_3, 0)$, because if there were infinite equilibria with $z_1^2 + z_2^2 \neq 0$ these would be appear in the local charts $U_1 \cup U_2$. But system (9) has no equilibria of the form $(0, 0, z_3, 0)$.

3.5. The Poincaré compactification of system (2) in the local chart U_4

In this chart from (7) system (2) writes

$$\begin{aligned}
\dot z_1 =&\quad z_4^4(z_1^3 - z_1 z_2^2 + z_1 z_2 z_4 + z_3 z_4) + \varepsilon z_1(z_1^6 + 3z_1^4 z_2^2 + 5z_1^2 z_2^4 - z_2^6\\
&\quad +z_1^4 z_2 z_4 + 2z_1^2 z_2^3 z_4 + z_2^5 z_4 + z_1^4 z_4^2 + 3z_1^2 z_2^2 z_4^2 - z_2^4 z_4^2 + z_1^2 z_2 z_4^3\\
&\quad +z_2^3 z_4^3),\\
\dot z_2 =&\quad z_4^4(z_1^2 z_2 - z_2^3 + z_4 + z_2^2 z_4) - \varepsilon z_2(-z_1^6 - 3z_1^4 z_2^2 - 5z_1^2 z_2^4 + z_2^6\\
&\quad -z_1^4 z_2 z_4 - 2z_1^2 z_2^3 z_4 - z_2^5 z_4 - z_1^4 z_4^2 - 3z_1^2 z_2^2 z_4^2 + z_2^4 z_4^2 - z_1^2 z_2 z_4^3\\
&\quad -z_2^3 z_4^3),\\
\dot z_3 =&\quad z_4^4(-2z_1 z_2 + z_1^2 z_3 - z_2^2 z_3 - z_1 z_4 + z_2 z_3 z_4) + \varepsilon(-6z_1^5 z_2 - 4z_1^3 z_2^3\\
&\quad -2z_1 z_2^5 + z_1^6 z_3 + 3z_1^4 z_2^2 z_3 + 5z_1^2 z_2^4 z_3 - z_2^6 z_3 - z_1^5 z_4 - 2z_1^3 z_2^2 z_4\\
&\quad -z_1 z_2^4 z_4 + z_1^4 z_2 z_3 z_4 + 2z_1^2 z_2^3 z_3 z_4 + z_2^5 z_3 z_4 - 4z_1^3 z_2 z_4^2 - 2z_1 z_2^3 z_4^2\\
&\quad +z_1^4 z_3 z_4^2 + 3z_1^2 z_2^2 z_3 z_4^2 - z_2^4 z_3 z_4^2 - z_1^3 z_4^3 - z_1 z_2^2 z_4^3 + z_1^2 z_2 z_3 z_4^3\\
&\quad +z_2^3 z_3 z_4^3),\\
\dot z_4 =&\quad z_4^5(z_1^2 - z_2^2 + z_2 z_4) + \varepsilon z_4(z_1^6 + 3z_1^4 z_2^2 + 5z_1^2 z_2^4 - z_2^6 + z_1^4 z_2 z_4\\
&\quad +2z_1^2 z_2^3 z_4 + z_2^5 z_4 + z_1^4 z_4^2 + 3z_1^2 z_2^2 z_4^2 - z_2^4 z_4^2 + z_1^2 z_2 z_4^3 + z_2^3 z_4^3).
\end{aligned} \tag{10}$$

In system (10) we must look for the infinite equilibria of the form $(0,0,0,0)$, because if there were infinite equilibria with $z_1^2+z_2^2+z_3^2 \neq 0$ these would be appear in the local charts $U_1 \cup U_2 \cup U_3$. But the point $(0,0,0,0)$ is not an equilibrium point for system (10).

In short we have proved the following proposition.

Proposition 4. *The generalized Hénon-Heiles Hamiltonian system has no infinite singular points.*

4. The averaging theory of first and second order

Here we summarize the averaging theory of order two for finding periodic orbits. See the paper [5] the proofs of the results presented in his section.

Theorem 5. *We assume that the non-autonomous differential system*

$$\dot{x}(t) = \varepsilon F_1(t,x) + \varepsilon^2 F_2(t,x) + \varepsilon^3 R(t,x,\varepsilon), \tag{11}$$

being F_1, $F_2 : \mathbb{R} \times D \to \mathbb{R}^n$, $R : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ continuous functions, T-periodic in the t variable, and D is an open subset of $\mathbb{R}^n$, satisfies the following assumptions.

(i) The functions $F_1(t,\cdot) \in C^1(D)$ for all $t \in \mathbb{R}$, F_1, F_2, R and $D_x F_1$ are locally Lipschitz with respect to x, and R is differentiable with respect to ε. We define f_1, $f_2 : D \to \mathbb{R}^n$ as

$$f_1(z) = \int_0^T F_1(s,z)ds,$$
$$f_2(z) = \int_0^T [D_z F_1(s,z) \int_0^s F_1(t,z)dt + F_2(s,z)]ds.$$

(ii) There exists an open and bounded set $V \subset D$ such that for each $\varepsilon \in (-\varepsilon_f, \varepsilon_f)\backslash\{0\}$, there is $a \in V$ verifying $f_1(a) + \varepsilon f_2(a) = 0$ and the Brouwer degree $d_B(f_1 + \varepsilon f_2)(a)$ of the function $f_1 + \varepsilon f_2$ at its fixed point a is not zero.

Then for $|\varepsilon| > 0$ small enough, there exists a $T-$periodic solution $x(\cdot,\varepsilon)$ of system (11) verifying that $x(0,\varepsilon) \to a$ when $\varepsilon \to 0$.

We recall that if the Jacobian of a function f at its fixed point a is non–zero, then the Brouwer degree $d_B(f_1 + \varepsilon f_2)(a)$ in non-zero, see [6].

When the function $f_1 \not\equiv 0$, then for ε small enough the zeros of $f_1 + \varepsilon f_2$ are essentially the zeros of f_1. In this case Theorem 5 provides the *averaging theory of first order*.

When the function $f_1 \equiv 0$ and $f_2 \not\equiv 0$, then the zeros of f_2 are the zeros of $f_1 + \varepsilon f_2$, and Theorem 5 provides the *averaging theory of second order*.

In the following section we will show that after adequate changes of variables, we can apply Theorem 5 to the classical and generalized Hénon–Heiles Hamiltonian systems (2). And consequently we can prove the existence of two families of periodic orbits for these systems.

5. Proof of Theorem 1

It is well known that for Hamiltonian system with more than one degree of freedom their periodic orbits generically live on cylinders filled of periodic orbits. Hence it is not possible to apply directly Theorem 5 to a Hamiltonian system, because then the Jacobian of the function $f_1 + \varepsilon f_2$ at its fixed point a will be always zero. Therefore Theorem 5 must be applied to every fixed Hamiltonian level where generically the periodic orbits appear isolated. Remember that in the sense of Liouville–Arnol'd Theorem the integrable Hamiltonian systems are non–generic, see [7].

From the statement of Theorem 5 it follows that in order to apply it the differential system needs to be periodic in the independent variable. Therefore in the Hamiltonian system (2) we change the variables

(x, y, p_x, p_y) to $(r, \theta, \rho, \alpha)$ given by $(r\cos\theta, \rho\cos(\theta+\alpha), r\sin\theta, \rho\sin(\theta+\alpha))$, and later on we will take as the new independent variable the θ. In these new variables system (2) becomes

$$
\begin{aligned}
\dot{r} = {} & -\rho r\sin(2\theta)\cos(\alpha+\theta) - \varepsilon r\sin\theta\cos\theta\Big(\rho^2\cos^2(\alpha+\theta) \\
& \big(2\rho^3\cos^3(\alpha+\theta) + \rho^2\cos^2(\alpha+\theta) + 2\rho\cos(\alpha+\theta) + 1\big) \\
& + r^4\cos^4\theta(6\rho\cos(\alpha+\theta)+1) + r^2\cos^2\theta\big(4\rho^3\cos^3(\alpha+\theta) \\
& + 2\rho^2\cos^2(\alpha+\theta) + 4\rho\cos(\alpha+\theta) + 1\big)\Big), \\
\dot{\theta} = {} & -1 - 2\rho\cos^2\theta\cos(\alpha+\theta) - \varepsilon\cos^2\theta\Big(\rho^2\cos^2(\alpha+\theta) \\
& \big(1 + 2\rho^3\cos^3(\alpha+\theta) + \rho^2\cos^2(\alpha+\theta) + 2\rho\cos(\alpha+\theta)\big) \\
& + r^4\cos^4\theta(6\rho\cos(\alpha+\theta)+1) + r^2\cos^2\theta\big(1 + 4\rho^3\cos^3(\alpha+\theta) \\
& + 2\rho^2\cos^2(\alpha+\theta) + 4\rho\cos(\alpha+\theta)\big)\Big), \\
\dot{\rho} = {} & \sin(\alpha+\theta)\left(\rho^2\cos^2(\alpha+\theta) - r^2\cos^2\theta\right) - \varepsilon\sin(\alpha+\theta) \\
& \Big(\rho^3\cos^3(\alpha+\theta)\big(1 - \rho^3\cos^3(\alpha+\theta) + \rho^2\cos^2(\alpha+\theta) \\
& - \rho\cos(\alpha+\theta)\big) + r^6\cos^6\theta + r^4\cos^4\theta\big(3\rho^2\cos^2(\alpha+\theta) \\
& + \rho\cos(\alpha+\theta) + 1\big) + \rho r^2\cos^2\theta\cos(\alpha+\theta)\big(5\rho^3\cos^3(\alpha+\theta) \\
& + 2\rho^2\cos^2(\alpha+\theta) + 3\rho\cos(\alpha+\theta) + 1\big)\Big), \\
\dot{\alpha} = {} & \frac{\cos(\alpha+\theta)\big(\rho^2\cos(2(\alpha+\theta)) + 3\rho^2 - \cos(2\theta)(r^2 - 2\rho^2) - r^2\big)}{2\rho} \\
& + \varepsilon\frac{1}{\rho}\Big(\rho^3\cos^4(\alpha+\theta)\big(\rho^3\cos^3(\alpha+\theta) - \rho^2\cos^2(\alpha+\theta) \\
& + \rho\cos(\alpha+\theta) - 1\big) + r^2\cos^4\theta\big(\rho + \rho(2\rho^2 - r^2)\cos^2(\alpha+\theta) \\
& - (r^2 - 4\rho^2)\cos(\alpha+\theta) + (4\rho^4 - 3\rho^2 r^2)\cos^3(\alpha+\theta)\big) \\
& + \rho\cos^2\theta\cos^2(\alpha+\theta)\big(\rho^2 + (2\rho^3 - 3\rho r^2)\cos(\alpha+\theta) \\
& + (2\rho^5 - 5\rho^3 r^2)\cos^3(\alpha+\theta) + (\rho^4 - 2\rho^2 r^2)\cos^2(\alpha+\theta) - r^2\big) \\
& - r^4\cos^6\theta\big((r^2 - 6\rho^2)\cos(\alpha+\theta) - \rho\big)\Big).
\end{aligned}
\tag{12}
$$

This system is not Hamiltonian but it has the first integral

$$
\begin{aligned}
& \frac{1}{6}\big(\rho^2(3 - 2\rho\cos^3(\alpha+\theta)) + 3r^2\cos^2\theta(2\rho\cos(\alpha+\theta)+1) + 3r^2\sin^2\theta\big) \\
& + \varepsilon\Big(\frac{1}{420}\rho^4\cos^4(\alpha+\theta)\big(105 - 2\rho\cos(\alpha+\theta)(5\rho\cos(\alpha+\theta)(6\rho\cos(\alpha+\theta) \\
& - 7) + 42)\big) + \frac{1}{6}r^6\cos^6\theta(6\rho\cos(\alpha+\theta)+1) + \frac{1}{4}r^4\cos^4\theta\big(2\rho\cos(\alpha+\theta) \\
& (\rho\cos(\alpha+\theta)(2\rho\cos(\alpha+\theta)+1)+2)+1\big) + \frac{1}{2}\rho^2 r^2\cos^2\theta\cos^2(\alpha+\theta) \\
& (2\rho\cos(\alpha+\theta)+1)(\rho^2\cos^2(\alpha+\theta)+1)\Big).
\end{aligned}
\tag{13}
$$

In order to apply Theorem 5 we also need a small parameter in front of the vector field associated to the differential system, so we will do the rescaling $(r, \rho) = \varepsilon(R, \sigma)$ using the parameter ε, and we take θ as the new independent variable. Hence the differential system (12) becomes

$$
\begin{aligned}
R' = \;& \varepsilon R\sigma \sin(2\theta)\cos(\alpha+\theta) - \varepsilon^2 2R\sigma^2 \sin(2\theta)\cos^2\theta\cos^2(\alpha+\theta) \\
& + O(\varepsilon^3), \\
\sigma' = \;& \varepsilon\Big(R^2\cos^2\theta\sin(\alpha+\theta) - \sigma^2\sin(\alpha+\theta)\cos^2(\alpha+\theta)\Big) \\
& + \varepsilon^2\Big(2\sigma^3\cos^2\theta\sin(\alpha+\theta)\cos^3(\alpha+\theta) - 2R^2\sigma\cos^4\theta\sin(\alpha+\theta) \\
& \cos(\alpha+\theta)\Big) + O(\varepsilon^3), \\
\alpha' = \;& \frac{-\varepsilon}{2\sigma}\cos(\alpha+\theta)\Big(\sigma^2\cos(2(\alpha+\theta)) - \cos(2\theta)\left(R^2-2\sigma^2\right) - R^2 + 3\sigma^2\Big) \\
& + \varepsilon^2\cos^2\theta\cos^2(\alpha+\theta)\Big(\sigma^2\cos(2(\alpha+\theta)) - \cos(2\theta)\left(R^2-2\sigma^2\right) - R^2 \\
& + 3\sigma^2\Big) + O(\varepsilon^3),
\end{aligned}
\tag{14}
$$

here the prime denotes derivative with respect to the variable θ. Of course, system (14) is 2π-periodic in the variable θ.

System (14) has the first integral (13) which in the variables (R,ρ,α) writes

$$
H = \varepsilon^2\frac{1}{2}(R^2+\sigma^2) + \varepsilon^3\left(R^2\sigma\cos^2\theta\cos(\alpha+\theta) - \frac{1}{3}\sigma^3\cos^3(\alpha+\theta)\right) + O(\varepsilon^4). \tag{15}
$$

We fix the value of the first integral H at $\varepsilon^2 h > 0$ in order that the averaging theory can provide information about the periodic orbits of system (14). Computing σ from equation (15) we obtain

$$
\sigma = \sqrt{2h-R^2} + \frac{\varepsilon}{3}\left(2h\cos^3(\alpha+\theta) - R^2\cos^3(\alpha+\theta) - 3R^2\cos^2\theta\cos(\alpha+\theta)\right) + O(\varepsilon^2).
$$

Now substituting σ in system (14), this differential system reduces to

$$
\begin{aligned}
R' = \;& \varepsilon R\sqrt{2h-R^2}\sin(2\theta)\cos(\alpha+\theta) + \frac{\varepsilon^2}{6}R\sin(2\theta)\cos^2(\alpha+\theta) \\
& \Big((2h-R^2)\cos(2(\alpha+\theta)) + 3\left(R^2-4h\right)\cos(2\theta) + 2(R^2-5h)\Big) \\
& + O(\varepsilon^3), \\
\alpha' = \;& \frac{\varepsilon\cos(\alpha+\theta)}{2\sqrt{2h-R^2}}\Big((R^2-2h)\cos(2(\alpha+\theta)) + (3R^2-4h)\cos(2\theta) \\
& -6h+4R^2\Big) + \frac{\varepsilon^2\cos^2(\alpha+\theta)}{12h-6R^2}\Big(3\cos^2\theta\big((8h^2-6hR^2+R^4) \\
& \cos(2(\alpha+\theta)) + (16h^2-16hR^2+5R^4)\cos(2\theta) + 24h^2 - 22hR^2 \\
& + 6R^4\big) + (2h-R^2)\cos^2(\alpha+\theta)\big((R^2-2h)\cos(2(\alpha+\theta)) \\
& + (R^2-4h)\cos(2\theta) - 6h + 2R^2\big)\Big) + O(\varepsilon^3).
\end{aligned}
\tag{16}
$$

Now system (16) satisfies all the assumptions for applying Theorem 5, i.e. it has the form (11) with $x = (R,\sigma)$, $t = \theta$, $T = 2\pi$, $F_1 = (F_{11}, F_{12})$ and $F_2 = (F_{21}, F_{22})$, where

$$
\begin{aligned}
F_{11} &= R\sqrt{2h-R^2}\sin(2\theta)\cos(\alpha+\theta), \\
F_{12} &= \frac{\cos(\alpha+\theta)}{2\sqrt{2h-R^2}}\Big((R^2-2h)\cos(2(\alpha+\theta)) + (3R^2-4h)\cos(2\theta) - 6h + 4R^2\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
F_{21} = &\ \frac{1}{6} R\sin(2\theta)\cos^2(\alpha+\theta)\Big((2h-R^2)\cos(2(\alpha+\theta)) + 3\left(R^2-4h\right)\cos(2\theta) \\
&+2(R^2-5h)\Big), \\
F_{22} = &\ \frac{\varepsilon^2\cos^2(\alpha+\theta)}{12h-6R^2}\Big(3\cos^2\theta\big((8h^2-6hR^2+R^4)\cos(2(\alpha+\theta)) + (16h^2 \\
&-16hR^2+5R^4)\cos(2\theta) + 24h^2-22hR^2+6R^4\big) + (2h-R^2) \\
&\cos^2(\alpha+\theta)\big((R^2-2h)\cos(2(\alpha+\theta)) + (R^2-4h)\cos(2\theta) - 6h + 2R^2\big)\Big).
\end{aligned}
$$

The averaging theory of first order does not provide any information about the periodic solutions of system (16), because the average functions of F_{11} and F_{12} in the period becomes zero, i.e.

$$
f_1(R,\alpha) = \int_0^{2\pi} (F_{11}, F_{12})\, d\theta = (0,0).
$$

As the averaged function f_1 of Theorem 5 is identically zero, we compute the function f_2 by applying the second order averaging theory. We have that

$$
f_2(R,\alpha) = \int_0^{2\pi} \left(D_{R\alpha}F_1(\theta,R,\alpha).y_1(\theta,R,\alpha) + F_2(\theta,R,\alpha)\right)\, d\theta,
$$

where

$$
y_1(\theta,R,\alpha) = \int_0^{\theta} F_1(t,R,\alpha)\, dt.
$$

and the Jacobian matrix is

$$
D_{r\alpha}F_1(\theta,r,\alpha) = \begin{pmatrix} \dfrac{\partial F_{11}}{\partial r} & \dfrac{\partial F_{11}}{\partial \alpha} \\ \dfrac{\partial F_{12}}{\partial r} & \dfrac{\partial F_{12}}{\partial \alpha} \end{pmatrix}.
$$

The two components of the vector y_1 are

$$
\begin{aligned}
y_{11} = &\ \int_0^{\theta} F_{11}(t,R,\alpha)\, dt \\
= &\ \frac{1}{6}R\sqrt{2h-R^2}(-3\cos(\alpha-\theta) - \cos(\alpha+3\theta) + 4\cos(\alpha)), \\
y_{12} = &\ \int_0^{\theta} F_{12}(t,R,\alpha)\, dt \\
= &\ \frac{1}{12\sqrt{2h-R^2}}\Big(12h\sin(\alpha-\theta) - 42h\sin(\alpha+\theta) - 2h\sin(3(\alpha+\theta)) \\
&+(3R^2-4h)\sin(\alpha+3\theta) + \sin(\alpha)(34h-21R^2) + \sin(3\alpha)(2h-R^2) \\
&-9R^2\sin(\alpha-\theta) + 27R^2\sin(\alpha+\theta) + R^2\sin(3(\alpha+\theta))\Big).
\end{aligned}
$$

Now we calculate the averaged function of second order $f_2 = (f_{21}, f_{22})$ defined in Theorem 5 and we have

$$
\begin{aligned}
f_{21} = &\ -\frac{7}{6}\pi R\sin(2\alpha)\left(R^2-2h\right), \\
f_{22} = &\ -\frac{14}{3}\pi\sin^2\alpha\left(h-R^2\right).
\end{aligned}
$$

We must comput the zeros (R^*, α^*) of $f_2(R,\alpha)$, and to verify that the Jacobian determinant

$$
|D_{R,\alpha}f_2(R^*,\alpha^*)| \neq 0. \tag{17}
$$

Solving the system $f_2(R,\alpha)=0$ of two equations and two unknowns R and α we get six solutions (R^*,α^*) with $R^*\geq 0$, namely

$$(0,0),\ (0,\pi),\ (\sqrt{2h},0),\ (\sqrt{2h},\pi),\ (\sqrt{h},\pi/2),(\sqrt{h},-\pi/2). \tag{18}$$

The first four solutions are not good, because for them the Jacobian (18) vanishes, but for the last two solution the Jacobian (18) is $196h^2\pi^2/9\neq 0$. So, by Theorem 5 these two solutions provide two periodic solutions $(R_\pm(\theta,\varepsilon),\alpha_\pm(\theta,\varepsilon))$ of the differential system (16) with ε sufficiently small such that $(R_\pm(0,\varepsilon),\alpha_\pm(0,\varepsilon))\to(\sqrt{h},\pm\pi/2)$ when $\varepsilon\to 0$.

Going back to the differential system (14) we get for this system with ε sufficiently small two periodic solutions $(R_\pm(\theta,\varepsilon),\sigma_\pm(\theta,\varepsilon),\alpha_\pm(\theta,\varepsilon))$ such that $(R_\pm(0,\varepsilon),\sigma_\pm(0,\varepsilon),\ \alpha_\pm(0,\varepsilon))\to(\sqrt{h},\sqrt{h},\ \pm\pi/2)$ when $\varepsilon\to 0$.

Again going back to the differential system (12) we obtain for this system with ε sufficiently small two periodic solutions $(r_\pm(t,\varepsilon),\theta(t,\varepsilon),\rho_\pm(t,\varepsilon),\alpha_\pm(t,\varepsilon))=(\varepsilon\sqrt{h}+O(\varepsilon^2),-t+O(\varepsilon),\varepsilon\sqrt{h}+O(\varepsilon^2),\pm\pi/2+O(\varepsilon))$.

Finally going back to the initial Hamiltonian system (2) we have for this system with ε sufficiently small two periodic solutions

$$\begin{array}{l}(x(t,\varepsilon),y(t,\varepsilon),p_x(t,\varepsilon),p_y(t,\varepsilon))=\\(\varepsilon\sqrt{h}\cos t+O(\varepsilon^2),\pm\varepsilon\sqrt{h}\sin t+O(\varepsilon^2),-\varepsilon\sqrt{h}\sin t+O(\varepsilon^2),\pm\varepsilon\sqrt{h}\cos t+O(\varepsilon^2)),\end{array}$$

in each positive Hamiltonian level $H=\varepsilon^2 h$. This completes the proof of Theorem 1.

In the next section we will use the existence of these two periodic orbits with multipliers different from 1 to study the non–integrability of the Hamiltonian system (2).

6. Periodic orbits and the Liouville–Arnol'd integrability

First we present some results on the Liouville–Arnol'd integrability of the Hamiltonian systems, and also on the periodic orbits of the differential equations, see more details in [8, 9] and the subsection 7.1.2 of [9], respectively. We restrict our attention to the Hamiltonian systems with two degrees of freedom like our generalized Hénon-Heiles Hamiltonian system, but we remark that these results work in Hamiltonian systems with an arbitrary number of degrees of freedom.

It is well known that a Hamiltonian system with Hamiltonian H of two degrees of freedom is *integrable in the sense of Liouville–Arnol'd* if it has a second first integral C independent with H (i.e. the gradient vectors of H and C are *independent* in all the points of the phase space except perhaps in a set of zero Lebesgue measure), and in *involution* with H (i.e. the parenthesis of Poisson of H and C is zero). For Hamiltonian systems with two degrees of freedom the involution condition is redundant, because the fact that C is a first integral of the Hamiltonian system, implies that the mentioned Poisson parenthesis is always zero. A flow defined on a subspace of the phase space is *complete* if its solutions are defined for all time $t\in\mathbb{R}$.

The Liouville–Arnol'd Theorem restricted to Hamiltonian systems of two degrees of freedom is:

Theorem 6. *Consider a Hamiltonian system with two degrees of freedom defined on the phase space M with Hamiltonian H and having a second first integral C independent with H. Let $I_{hc}=\{p\in M: H(p)=h \text{ and } C(p)=c\}\neq\emptyset$ be. If (h,c) is a regular value of the map (H,C), then the following statements hold.*

- *(a) I_{hc} is a two dimensional submanifold of M invariant under the flow of the Hamiltonian system.*
- *(b) If the flow on a connected component I^*_{hc} of I_{hc} is complete, then I^*_{hc} is diffeomorphic either to the torus $\mathbb{S}^1\times\mathbb{S}^1$, or to the cylinder $\mathbb{S}^1\times\mathbb{R}$, or to the plane $\mathbb{R}^2$. If I^*_{hc} is compact, then the flow on it is always complete and $I^*_{hc}\approx\mathbb{S}^1\times\mathbb{S}^1$.*
- *(c) Under the assumptions of statement* (b) *the flow on I^*_{hc} is conjugated to a linear flow on either $\mathbb{S}^1\times\mathbb{S}^1$, or on $\mathbb{S}^1\times\mathbb{R}$, or on $\mathbb{R}^2$.*

The main result of this theorem states that the connected components of the invariant sets associated with the two independent first integrals in involution are generically submanifolds of the phase space, and if the flow on them is complete then they are diffeomorphic to a torus, a cylinder or a plane, where the flow is conjugated to a linear one.

Using the notation of Theorem 6 when a connected component I_{hc}^* is diffeomorphic to a torus, either all orbits on this torus are periodic if the rotation number associated to this torus is rational, or they are quasi-periodic (i.e. every orbit is dense in the torus) if the rotation number associated to this torus is not rational.

Consider the autonomous differential system

$$\dot{x} = f(x),$$

where $f\colon U \to \mathbb{R}^n$ is C^2, and U is an open subset of $\mathbb{R}^n$. We write its general solution as $\phi(t, x_0)$ with $\phi(0, x_0) = x_0 \in U$ and t belonging to its maximal interval of definition.

We say that $\phi(t, x_0)$ is T-*periodic* with $T > 0$ if and only if $\phi(T, x_0) = x_0$ and $\phi(t, x_0) \neq x_0$ for $t \in (0, T)$. The periodic orbit associated to the periodic solution $\phi(t, x_0)$ is $\gamma = \{\phi(t, x_0), t \in [0, T]\}$. The variational equation associated to the T-periodic solution $\phi(t, x_0)$ is

$$\dot{M} = \left(\frac{\partial f(x)}{\partial x}\Big|_{x=\phi(t,x_0)} \right) M, \tag{19}$$

where M is an $n \times n$ matrix. The *monodromy matrix* associated to the T-periodic solution $\phi(t, x_0)$ is the solution $M(T, x_0)$ of (19) satisfying that $M(0, x_0)$ is the identity matrix. The eigenvalues λ of the monodromy matrix associated to the periodic solution $\phi(t, x_0)$ are called the *multipliers* of the periodic orbit.

For an autonomous differential system, one of the multipliers is always 1, and its corresponding eigenvector is tangent to the periodic orbit.

A periodic solution of an autonomous Hamiltonian system always has two multipliers equal to one. One multiplier is 1 because the Hamiltonian system is autonomous, and the other 1 is due to the existence of the first integral given by the Hamiltonian.

Theorem 7. *If a Hamiltonian system with two degrees of freedom and Hamiltonian H is Liouville–Arnol'd integrable, and C is a second first integral such that the gradients of H and C are linearly independent at each point of a periodic orbit of the system, then all the multipliers of this periodic orbit are equal to* 1.

Theorem 7 is due to Poincaré [3], section 36, see also [10]. It provides a tool for studying the non Liouville–Arnol'd integrability, independently of the class of differentiability of the second first integral. The main problem for applying this theorem is to find periodic orbits having multipliers different from 1.

Proof of Theorem 2. Consider the two periodic solutions stated in Theorem 1. Their corresponding Jacobian $196h^2\pi^2/9 \neq 1$ playing with the energy level h. Since this Jacobian is the product of the four multipliers of these periodic solutions with two of them always equal to 1, the remainder two multipliers cannot be equal to 1. Hence, by Theorem 7, either the Hénon–Heiles systems cannot be Liouville–Arnol'd integrable with any second first integral C, or the system is Liouville-Arnol'd integrable and the differentials of H and C are linearly dependent on some points of these periodic orbits. Therefore the theorem is proved. □

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