



Research article

Time-reversibility and integrability of $p : -q$ resonant vector fields

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Abstract: We study the local analytical integrability in a neighborhood of $p : -q$ resonant singular point of a two-dimensional vector field and its connection to time-reversibility with respect to the non-smooth involution $\varphi(x, y) = (y^{p/q}, x^{q/p})$. Some generalizations of the theory developed by Sibirsky for the $1 : -1$ resonant case to the $p : -q$ resonant case are presented.

Keywords: planar systems of ODEs; time-reversibility; integrability; resonant singularity

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1. Introduction

Consider an n -dimensional system of ordinary differential equations

$$\dot{x} = F(x), \tag{1.1}$$

where $F(x)$ is an n -dimensional vector-function defined on some domain Ω of \mathbb{R}^n or \mathbb{C}^n . It is said (see e.g., [2, 15]) that system (1.1) is *time-reversible* on Ω if there exists an involution ψ defined on Ω such

that

$$D_{\psi}^{-1} \cdot F \circ \psi = -F. \quad (1.2)$$

We say that system (1.1) is *completely analytically integrable* on Ω if it admits $n - 1$ functionally independent analytic first integrals on Ω .

A time-reversal symmetry is one of the fundamental symmetries that appears in nature, in particular, important both for classical and quantum mechanics. Various properties of systems exhibiting such symmetries have been studied by many authors, see e.g., [1, 2, 12, 13, 15, 27, 29] and the references therein.

Our paper is devoted to the investigation of the interconnection of time-reversibility and local integrability in a neighborhood of a singular point of systems of the form

$$\dot{\mathbf{x}} = A\mathbf{x} + X(\mathbf{x}), \quad (1.3)$$

where A is an $n \times n$ matrix with entries in \mathbb{R} or \mathbb{C} being $\mathbf{x} = (x_1, \dots, x_n)$, and $X(\mathbf{x})$ is a vector-function without constant and linear terms defined on some domain Ω of \mathbb{R}^n or \mathbb{C}^n .

One of the first results in such studies is due to Poincaré. It follows from his results that if in the two-dimensional case the eigenvalues of A are pure imaginary and the system has an axis of symmetry passing through the origin, then it admits an analytic first integral in a neighborhood of the origin.

A generalization of this result is presented in [3], where it is shown that if system (1.3) is time-reversible with respect to a certain linear involution and two eigenvalues of the matrix A are pure imaginary, then under some assumptions, the system has at least one analytic first integral in a neighborhood of the origin.

A detailed study of the interconnection of time-reversibility and local integrability for systems (1.3) was presented in [17]. In [17, 28] the notion of time-reversibility was generalized to the case when on the right-hand side of (1.2) “ -1 ” is replaced by a primitive root of unity.

In this paper, we limit our consideration to the two-dimensional systems (1.3) with non-degenerate matrix A . In the case when a two-dimensional system (1.3) is real and the eigenvalues of A are pure imaginary, and the vector field is symmetric with respect to a curve passing through the origin, the origin of (1.3) is a center, and, therefore, has an analytic local integral in a neighborhood of the origin. This geometric argument was used in [31] in order to find some integrable systems in the family of real cubic systems (see also [2] for recent developments in this direction). The symmetry axis is, in the general case, an analytic curve passing through the origin. However, from the work of Montgomery and Zippin [18], any analytic involution ψ associated with a time-reversal symmetry can be linearized in such a way that the symmetry axis becomes a straight line. From this result in [1] the normal form theory is used to establish an algorithm to determine if a two-dimensional system (1.3) is orbitally reversible.

A detailed study of real polynomial systems which are time-reversible under reflection with respect to a line was performed by Sibirsky [24, 25]. In particular, he showed that in the polynomial case the set of such systems in the space of parameters is the variety of a binomial ideal defined by invariants of the rotation group of the system (some similar results were also obtained in [5, 16]). Later on, the results obtained by Sibirsky were generalized to the case of complex systems (1.3) with a $1 : -1$ resonant singular point at the origin in [14, 19, 20].

In this paper, we consider system (1.3) having a $p : -q$ resonant singular point at the origin, which we write in the form

$$\begin{aligned}\dot{x} &= px - \sum_{j+k \geq 1, j \geq -1}^{\infty} a_{jk} x^{j+1} y^k = px \left(1 - \sum_{j+k \geq 1, j \geq -1}^{\infty} \frac{1}{p} a_{jk} x^j y^k\right), \\ \dot{y} &= -qy + \sum_{j+k \geq 1, j \geq -1}^{\infty} b_{kj} x^k y^{j+1} = -qy \left(1 - \sum_{j+k \geq 1, j \geq -1}^{\infty} \frac{1}{q} b_{kj} x^k y^j\right),\end{aligned}\tag{1.4}$$

where $p, q \in \mathbb{N}$, $\gcd(p, q) = 1$. Both the vector field (1.4) and the associated differential operator are denoted by \mathcal{X} .

Let \mathbb{N}_0 be the set of non-negative integers, and for a given positive integer n we denote by \mathbb{N}_{-n} the set $\{-n, -n+1, \dots, -1\} \cup \mathbb{N}_0$. For system (1.4), it is always possible to find a series of the form

$$\Psi(x, y) = x^q y^p + \sum_{\substack{j+k > p+q \\ j, k \in \mathbb{N}_0}} v_{j-q, k-p} x^j y^k\tag{1.5}$$

for which

$$\mathcal{X}\Psi = g_{q,p}(x^q y^p)^2 + g_{2q,2p}(x^q y^p)^3 + g_{3q,3p}(x^q y^p)^4 + \dots,\tag{1.6}$$

where $g_{kq, kp}$, for $k = 1, 2, \dots$, are polynomials in the parameters a_{jk}, b_{kj} of system (1.4). Polynomials $g_{kq, kp}$ are called the saddle quantities of system (1.4) (sometimes also the focus quantities). System (1.4) corresponding to some fixed values a_{jk}^*, b_{kj}^* of the parameters has a local analytical first integral in a neighborhood of the origin if and only if $g_{kq, kp}(a^*, b^*) = 0$ for all $k \in \mathbb{N}$ (see e.g., [20, 22]).

Unless $p = q = 1$, this system is not time-reversible under a linear transformation. Our study deals with the time-reversibility of system (1.4) with respect to the involution

$$\varphi(x, y) = (y^{p/q}, x^{q/p}).\tag{1.7}$$

We first prove that if system (1.4) is time-reversible with respect to (1.7), then it admits an analytic first integral on a neighborhood of the origin. It can be done by reducing the system to the $1 : -1$ resonant case and applying known results (e.g., [3, 17, 28]), but we will give a different proof. In Section 2, properties of functions (1.5) satisfying (1.6) are studied. In Section 3 we present our main result, Theorem 9, which describes the set of time-reversible systems and their Zariski closure as an algebraic variety in the space of parameters (it extends the results of [24, 25] and their generalizations obtained in [14, 19, 20]) and give an algorithm for computing this variety. Our study shows that, in fact, the theory developed in [23–25] for the $1 : -1$ resonant case can be extended to system (1.4), however not to the whole family, but only to a certain subfamily of (1.4).

2. Integrability of time-reversible systems

We first show that if system (1.4) is time-reversible with respect to (1.7), then it has an analytic first integral of the form (1.5). Observe that if $p = q = 1$ then map (1.7) is just a permutation of the variables, so the statement presents a generalization of known results of [3, 17, 19] to the case of $p : -q$ resonant systems.

Theorem 1. Assume that system (1.4) is time-reversible with respect to the involution (1.7), that is,

$$D_\varphi^{-1} \cdot \mathcal{X} \circ \varphi = -\mathcal{X}. \quad (2.1)$$

Then it admits an analytic first integral of the form (1.5) in a neighborhood of the origin.

To prove the above theorem we will need the following results.

Lemma 2. System (1.4) with p or q different from 1 is time-reversible with respect to (2.1) if and only if

$$b_{qv,up} = \frac{q}{p} a_{qu,pv}, \quad (2.2)$$

where $u, v = 0, 1, 2, \dots$ and the other coefficients in (1.4) are equal to zero.

Proof. Using involution (1.7), that is, performing the substitution

$$x_1 = y^{p/q}, \quad y_1 = x^{q/p}, \quad (2.3)$$

after straightforward calculations we obtain

$$\begin{aligned} \dot{x}_1 &= -px_1 \left(1 - \sum_{j_1+k_1 \geq 1, j_1 \geq -1}^{\infty} \frac{1}{q} b_{k_1 j_1} x_1^{\frac{qj_1}{p}} y_1^{\frac{pk_1}{q}} \right), \\ \dot{y}_1 &= qy_1 \left(1 - \sum_{j_1+k_1 \geq 1, j_1 \geq -1}^{\infty} \frac{1}{p} a_{j_1 k_1} x_1^{\frac{qk_1}{p}} y_1^{\frac{pj_1}{q}} \right). \end{aligned}$$

In view of (2.1) it should hold that

$$\frac{1}{p} a_{jk} x^j y^k = \frac{1}{q} b_{k_1 j_1} x^{\frac{qj_1}{p}} y^{\frac{pk_1}{q}}, \quad (2.4)$$

where the exponents on the right-hand side should be non-negative integers or

$$\frac{qj_1}{p} = \frac{pk_1}{q} = -1.$$

However, the latter equality is impossible unless $p = q = 1$. Thus, (2.4) can take place if we set $j_1 = pu$, $k_1 = qv$, where $u, v = 0, 1, 2, \dots$. This yields formula (2.2). \square

One possibility to prove Theorem 1 is to use the substitution $x \rightarrow x^q$, $y \rightarrow y^p$. Then, in view of (2.2), we obtain a $1 : -1$ resonant vector field which is time-reversible with respect to the involution $x \rightarrow y, y \rightarrow x$, and then by the results of [3, 17, 28] it has a first integral of the form (1.5). The proofs of [3, 17, 28] are obtained using the normal form theory. The main step in these proofs is to show that time-reversibility is preserved by certain transformations into a normal form.

We will present another proof, which does not rely on the normal form theory. For our proof we will use some properties of series (1.5) satisfying (1.6).

Formula (2.2) and the results obtained below also remain valid in the case of $1 : -1$ resonant singular points. But, since the results in the $1 : -1$ resonant case are known, below we work under

the assumption $p/q \neq 1$, taking advantage of the fact that in such case the subscripts of the parameters a_{jk} , b_{kj} of (1.4) are non-negative.

Observe that we can write function (1.5) in the form

$$\Psi(x, y) = x^q y^p \left(\sum_{s=0}^{p+q} v_{-q+s, q-s} + \sum_{\substack{j+k > p+q \\ j, k \in \mathbb{N}_0}} v_{j-q, k-p} x^{j-q} y^{k-p} \right), \quad (2.5)$$

where

$$v_{00} = 1, \quad v_{-q+s, q-s} = 0 \text{ for } s = 0, \dots, p+q, s \neq q. \quad (2.6)$$

By [23, p. 117], the coefficients v_{k_1, k_2} of the series (1.5) can be computed recursively using the formula

$$v_{k_1, k_2} = \begin{cases} \frac{1}{pk_1 - qk_2} \sum_{\substack{s_1 + s_2 = 0 \\ s_1 \geq -q, s_2 \geq -p}}^{k_1 + k_2 - 1} [(s_1 + q)a_{k_1 - s_1, k_2 - s_2} - (s_2 + p)b_{k_1 - s_1, k_2 - s_2}] v_{s_1, s_2} & \text{if } pk_1 \neq qk_2, \\ 0 & \text{if } pk_1 = qk_2, \end{cases} \quad (2.7)$$

(in [23] formula (2.7) was obtained for the case of polynomial system (1.4), but, obviously, it also remains valid in the case when the right-hand sides of (1.4) are series). Using (2.7), the computation of coefficients v_{k_1, k_2} is performed recursively with the recursion on $k_1 + k_2$. For the initial step $k_1 + k_2 = 0$, the values of v_{k_1, k_2} are set accordingly to (2.6).

We order the index set of parameters a_{jk} in the first equation of (1.4) in some manner, say by degree lexicographic order from least to greatest, and write the ordered set as

$$S = \{(1, 0), (0, 1), (-1, 2), (2, 0), \dots\}.$$

Consistent with this we then order the parameters as $(a_{10}, a_{01}, a_{-1,2}, a_{20}, \dots, b_{02}, b_{2,-1}, b_{10}, b_{01})$ so that any monomial appearing in v_{ij} has the form $a_{10}^{\nu_1} a_{01}^{\nu_2} \cdots a_{s,t}^{\nu_\ell} b_{t,s}^{\nu_{\ell+1}} \cdots b_{10}^{\nu_{2\ell-1}} b_{01}^{\nu_{2\ell}}$ for some $\nu = (\nu_1, \dots, \nu_{2\ell})$ and $\ell = 1, 2, \dots$. To simplify the notation, for $\nu \in \mathbb{N}_0^{2\ell}$ we write

$$[\nu] \stackrel{\text{def}}{=} a_{10}^{\nu_1} a_{01}^{\nu_2} \cdots a_{s,t}^{\nu_\ell} b_{t,s}^{\nu_{\ell+1}} \cdots b_{10}^{\nu_{2\ell-1}} b_{01}^{\nu_{2\ell}}, \quad (2.8)$$

so if the k -th variable in the product is a_{pq} , then the $2\ell - k + 1$ -st variable is b_{qp} .

For each $m \in \mathbb{N}$ we consider the finite subset S_m of the set S which corresponds to the case when system (1.4) is a polynomial system of degree m , so

$$S_m = \{(1, 0), (0, 1), (-1, 2), (2, 0), \dots, (-1, m)\}.$$

Denote by $\ell(m)$ the number of elements in S_m and let $L^m : \mathbb{N}_0^{2\ell(m)} \rightarrow \mathbb{Z}^2$ be the linear map defined by

$$\begin{aligned} L^m(\nu) &= (L_1^m(\nu), L_2^m(\nu)) \\ &= \nu_1(1, 0) + \nu_2(0, 1) + \cdots + \nu_{\ell(m)}(-1, m) + \nu_{\ell(m)+1}(m, -1) + \cdots + \nu_{2\ell(m)-1}(1, 0) + \nu_{2\ell(m)}(0, 1). \end{aligned} \quad (2.9)$$

Let $\mathbb{K}[a, b]$ be the ring of polynomials in parameters a_{jk}, b_{jk} of system (1.4) over the field \mathbb{K} and for $f \in \mathbb{K}[a, b]$ we write $f = \sum_{\nu \in \text{Supp}(f)} f^{(\nu)}[\nu]$, where $\text{Supp}(f)$ denotes those $\nu \in \mathbb{N}_0^{2\ell(m)}$, $m = 1, 2, \dots$, for which the coefficient $f^{(\nu)} \in \mathbb{K}$ of $[\nu]$ in the polynomial f is nonzero.

Definition 3. For $(j, k) \in \mathbb{N}_{-q} \times \mathbb{N}_{-p}$, a polynomial

$$f = \sum_{v \in \text{Supp}(f)} f^{(v)}[v]$$

in the polynomial ring $\mathbb{C}[a, b]$ is a (j, k) -polynomial if, for every $v \in \text{Supp}(f)$, $L^\ell(v) = (j, k)$ for all sufficiently large ℓ .

The reader can consult [23, Section 3.4] for more details about (j, k) -polynomials in the case of polynomial system (1.4).

From now on, we will limit our consideration to systems of the form

$$\begin{aligned} \dot{x} &= x(p - \sum_{\tau_1 + \tau_2 = 1}^{\infty} a_{q\tau_1, p\tau_2} x^{q\tau_1} y^{p\tau_2}), \\ \dot{y} &= -y(q - \sum_{\tau_1 + \tau_2 = 1}^{\infty} b_{q\tau_2, p\tau_1} x^{q\tau_2} y^{p\tau_1}). \end{aligned} \quad (2.10)$$

By Lemma 2, in the case when $p/q \neq 1$, systems (1.4), which are time-reversible with respect to involution (1.7), form a subfamily of systems (2.10), so we do not lose generality working with family (2.10) if we are interested in time-reversibility with respect to (1.7).

Lemma 4. For system (2.10), if v_{k_1, k_2} is a nonzero coefficient of series (1.5) computed by (2.7), then

$$k_1 = \tau_1 q, \quad k_2 = \tau_2 p, \quad (2.11)$$

for some non-negative integers τ_1, τ_2 . Moreover, under involution (1.7) the term

$$v_{q\tau_1, p\tau_2} x^{q\tau_1 + q} y^{p\tau_2 + p} \quad (2.12)$$

of (1.5) is changed to the term

$$v_{q\tau_2, p\tau_1} x^{q\tau_2 + q} y^{p\tau_1 + p}, \quad (2.13)$$

and vice versa.

Proof. For system (2.10) the linear map (2.9) can be written in the form

$$L^{qm}(v) = (qL_1^m(v), pL_2^m(v)),$$

where $(L_1^m(v), L_2^m(v))$ is defined by (2.9) and $L^{qm-q+1}(v) = \dots = L^{q^{m-1}}(v) = L^{qm}(v)$. By Theorem 4 of [21], v_{k_1, k_2} is a (k_1, k_2) -polynomial. Therefore for each monomial $[v]$ of v_{k_1, k_2} it holds that

$$(qL_1^m(v), pL_2^m(v)) = (k_1, k_2),$$

for all sufficiently large m . It means that q divides k_1 and p divides k_2 , that is, (2.11) holds.

Performing in (2.12) substitution (2.3) we see that (2.13) holds. \square

Remark 5. Theorem 4 of [21] mentioned above was formulated in [21] for the case of polynomial systems (1.4), but it also remains correct in the case when the right-hand sides of (1.4) are series.

Theorem 6. The formal series (1.5) computed according to (2.7) is unchanged under involution (1.7), that is, in view of Lemma 4,

$$v_{q\tau_1, p\tau_2} = v_{q\tau_2, p\tau_1}. \quad (2.14)$$

Proof. We prove the claim using induction on $\tau_1 + \tau_2$. When $\tau_1 = \tau_2 = 0$, we have by definition $v_{00} = 1$, so the claim holds.

Using Lemma 4 we can write formula (2.7) as

$$v_{q\tau_1, p\tau_2} = \begin{cases} \frac{1}{pq(\tau_1 - \tau_2)} \sum_{\substack{s_1 + s_2 = 0 \\ s_1 \geq -q, s_2 \geq -p}}^{q\tau_1 + p\tau_2 - 1} [(s_1 + q)a_{q\tau_1 - s_1, p\tau_2 - s_2} - (s_2 + p)b_{q\tau_1 - s_1, p\tau_2 - s_2}] v_{s_1, s_2}, & \text{if } \tau_1 \neq \tau_2, \\ 0, & \text{if } \tau_1 = \tau_2. \end{cases} \quad (2.15)$$

In view of (2.2) and taking into account that $v_{j,k}$ are (j, k) -polynomials for $\tau_1 \neq \tau_2$, we can change the rule of summation obtaining from (2.15)

$$\begin{aligned} v_{q\tau_1, p\tau_2} &= \frac{1}{pq(\tau_1 - \tau_2)} \sum_{s_1=0}^{q\tau_1} \sum_{s_2=0}^{p\tau_2} [(s_1 + q)a_{q\tau_1 - s_1, p\tau_2 - s_2} - (s_2 + p)b_{q\tau_1 - s_1, p\tau_2 - s_2}] v_{s_1, s_2} \\ &= \frac{1}{pq(\tau_1 - \tau_2)} \sum_{\tilde{s}_1=0}^{\tau_1} \sum_{\tilde{s}_2=0}^{\tau_2} [(\tilde{s}_1 + 1)qa_{q(\tau_1 - \tilde{s}_1), p(\tau_2 - \tilde{s}_2)} - (\tilde{s}_2 + 1)pb_{q(\tau_1 - \tilde{s}_1), p(\tau_2 - \tilde{s}_2)}] v_{q\tilde{s}_1, p\tilde{s}_2}, \end{aligned} \quad (2.16)$$

where $s_1 = q\tilde{s}_1$, $s_2 = p\tilde{s}_2$.

Performing similar computations we have

$$\begin{aligned} v_{q\tau_2, p\tau_1} &= \frac{1}{pq(\tau_2 - \tau_1)} \sum_{s_1=0}^{q\tau_2} \sum_{s_2=0}^{p\tau_1} [(s_1 + q)a_{q\tau_2 - s_1, p\tau_1 - s_2} - (s_2 + p)b_{q\tau_2 - s_1, p\tau_1 - s_2}] v_{s_1, s_2} \\ &= \frac{1}{pq(\tau_2 - \tau_1)} \sum_{\tilde{s}_2=0}^{\tau_2} \sum_{\tilde{s}_1=0}^{\tau_1} [(\tilde{s}_2 + 1)qa_{q(\tau_2 - \tilde{s}_2), p(\tau_1 - \tilde{s}_1)} - (\tilde{s}_1 + 1)pb_{q(\tau_2 - \tilde{s}_2), p(\tau_1 - \tilde{s}_1)}] v_{q\tilde{s}_2, p\tilde{s}_1}, \end{aligned} \quad (2.17)$$

where $s_1 = q\tilde{s}_2$, $s_2 = p\tilde{s}_1$.

Using (2.2) we further obtain from (2.17)

$$\begin{aligned} v_{q\tau_2, p\tau_1} &= \frac{1}{pq(\tau_2 - \tau_1)} \sum_{\tilde{s}_2=0}^{\tau_2} \sum_{\tilde{s}_1=0}^{\tau_1} [(\tilde{s}_2 + 1)pb_{q(\tau_1 - \tilde{s}_1), p(\tau_2 - \tilde{s}_2)} - (\tilde{s}_1 + 1)qa_{q(\tau_1 - \tilde{s}_1), p(\tau_2 - \tilde{s}_2)}] v_{q\tilde{s}_2, p\tilde{s}_1} \\ &= \frac{1}{pq(\tau_1 - \tau_2)} \sum_{\tilde{s}_1=0}^{\tau_1} \sum_{\tilde{s}_2=0}^{\tau_2} [(\tilde{s}_1 + 1)qa_{q(\tau_1 - \tilde{s}_1), p(\tau_2 - \tilde{s}_2)} - (\tilde{s}_2 + 1)pb_{q(\tau_1 - \tilde{s}_1), p(\tau_2 - \tilde{s}_2)}] v_{q\tilde{s}_1, p\tilde{s}_2}, \end{aligned} \quad (2.18)$$

where we have changed $v_{q\tilde{s}_2, p\tilde{s}_1}$ to $v_{q\tilde{s}_1, p\tilde{s}_2}$ using the induction hypothesis. Comparing the expressions for (2.16) and (2.18), we conclude that (2.14) holds, that is, the series $\Psi(x, y)$ computed by (2.7) is unchanged under the involution (1.7). \square

Using the obtained results we prove Theorem 1 as follows.

Proof of Theorem 1. Denote by \mathfrak{X} the vector field of system (2.10). By Theorem 6 the series $\Psi(x, y)$ computed by (2.7) is unchanged under involution (1.7).

Assume that

$$\mathfrak{X}\Psi = \alpha(x, y).$$

Since, by our assumption, the system is time-reversible, it also holds that

$$\mathfrak{X}\Psi = -\alpha(x, y),$$

yielding $\alpha(x, y) \equiv 0$. That means, $\Psi(x, y)$ is a formal first integral of (1.4). Then, there also exists an analytic first integral of the form (1.5) (see e.g., [21, 30]). \square

Remark 7. *Our proof of Theorem 1 is based on the fact that series (1.5) has property (2.14). We emphasize that this property is true for general systems (1.4), but for time-reversible systems it yields that, in the reasoning above, $\alpha(x, y) \equiv 0$.*

Corollary 8. *If system (1.4) is time-reversible with respect to involution (1.7), then the system in the distinguished Poincaré-Dulac normal form is also time-reversible with respect to the same involution.*

Proof. Since by Theorem 1 any time-reversible system (1.4) is locally analytically integrable, its distinguished normal form can be written as

$$\dot{x} = px(1 + \sum_{k=1}^{\infty} g_k(x^q y^p)^k), \quad \dot{y} = -qy(1 + \sum_{k=1}^{\infty} g_k(x^q y^p)^k), \quad (2.19)$$

where g_k are numbers (see e.g., [21, 30]). Clearly, the latter system is time-reversible with respect to (1.7). \square

The normal form of any locally analytically integrable system (1.4) is given by (2.19). System (2.19) is time-reversible with respect to the involution (1.7). Therefore, any locally analytically integrable system (1.4) is conjugate to a time-reversible system, in the sense that there exists a change of variables ϕ that transforms the original system to the normal form (2.19), and consequently the original system is time-reversible with respect to the involution $\tilde{\psi} = \phi^{-1} \circ \psi \circ \phi$, where ψ is the involution (1.7). Hence, the analytical integrability of system (1.4) is always associated with a time-reversal symmetry. In fact, all nondegenerate centers are conjugate to a time-reversible system, and all the nilpotent centers are orbitally time-reversible. This does not happen for systems with the null linear part, see [12].

The problem with the map $\tilde{\psi}$ is that we have no idea about the form of $\tilde{\psi}$, not even the leading terms of such involution. Therefore, from the found results, we cannot deduce an algorithm based on the computation of the involution of the original system. However, several methods to compute the saddle or focus quantities are known, see for instance [8, 10, 11, 23] and references therein.

Nevertheless, there always exists a change ϕ such that any differential system (1.4) is transformed to its normal form

$$\dot{y}_1 = py_1(1 + Y_1(y_1^q y_2^p)), \quad \dot{y}_2 = -qy_2(1 + Y_2(y_1^q y_2^p)),$$

(where $Y_1(w) = \sum_{k=1}^{\infty} Y_1^{(k)} w^k$, $Y_2(w) = \sum_{k=1}^{\infty} Y_2^{(k)} w^k$ are formal or convergent series of variable w) and the results known for such resonance can be applied to the $p : -q$ resonance but only in the normal form. The change $z_1 = x^q$, $z_2 = y^p$ does not work for the original system (1.4), but it is possible to apply it to system (2.10), reducing the study to the $1 : -1$ resonant case. However, we have chosen to work with system (2.10), directly obtaining the important property of series (1.5) given in Theorem 6, which is related to the results of the next section.

3. Conditions of time-reversibility and the Sibirsky ideal

In this section we propose an algorithmic approach which allows for a given polynomial family (2.10) to find the set of systems which are time-reversible with respect to (1.7). We also give a description of the set using the so-called Sibirsky ideal, obtaining some generalizations of the results of [14, 19].

We will limit our consideration to polynomial systems of the form (2.10), that is, systems of the form

$$\begin{aligned}\dot{x} &= x(p - \sum_{u+v=1}^n a_{qu,pv} x^{qu} y^{pv}), \\ \dot{y} &= -y(q - \sum_{u+v=1}^n b_{qv,pu} x^{qu} y^{pv}),\end{aligned}\tag{3.1}$$

assuming that $p/q \neq 1$.

Denote by ℓ the number of parameters in the first equation of (3.1). For $k = 1, \dots, \ell$, let

$$\zeta_k = u_k - v_k\tag{3.2}$$

and consider the ideal

$$H = \langle 1 - w\gamma, a_{qu_k, pv_k} - t_k, b_{qv_k, pu_k} - \frac{q}{p} \gamma^{\zeta_k} t_k : k = 1, \dots, \ell \rangle,\tag{3.3}$$

where w is a new variable, so H is an ideal in the ring $\mathbb{C}[w, a, b]$.

Theorem 9. *The following statements hold:*

(a) *The Zariski closure of the set of systems in family (3.1), which are time-reversible with respect to involution (1.7) after the transformation*

$$x \rightarrow \alpha x, \quad y \rightarrow \alpha^{-1} y\tag{3.4}$$

with $\alpha \in \mathbb{C} \setminus \{0\}$, is the variety $\mathbf{V}(\mathcal{I})$ of the ideal

$$\mathcal{I} = H \cap \mathbb{C}[a, b].\tag{3.5}$$

(b) *If the parameters $a_{qu,pv}$, $b_{qv,pu}$ of system (3.1) belong to the variety $\mathbf{V}(\mathcal{I})$, then the system admits a local analytic first integral of the form (1.5).*

Proof. (a) Performing in system (3.1) transformation (3.4) we obtain the system of the same shape with the parameters $a_{qu,pv}$, $b_{qv,pu}$ changed according to the rule

$$a_{qu,pv} \mapsto \alpha^{pv-qu} a_{qu,pv}, \quad b_{qv,pu} \mapsto \alpha^{pu-qv} b_{qv,pu},$$

where $u + v = 1, \dots, n$.

By Lemma 2, the system obtained after transformation (3.4) is time-reversible with respect to involution (1.7) if and only if for some $\alpha \neq 0$,

$$\alpha^{pv-qu} a_{qu,pv} = \frac{p}{q} \alpha^{pu-qv} b_{qv,pu},\tag{3.6}$$

where $u + v = 1, \dots, n$. Equivalently, we can rewrite (3.6) as

$$a_{qu_k, pv_k} = t_k, \quad b_{qv_k, pu_k} = \frac{q}{p} \gamma^{\zeta_k} t_k, \quad (3.7)$$

where $\gamma = \alpha^{-(p+q)}$, $k = 1, \dots, \ell$ and ζ_k are defined by (3.2).

From (3.7), using the Implicitization Theorem (see e.g., [7]), we conclude that the first statement holds.

(b) By construction, $\mathbf{V}(\mathcal{I})$ is the Zariski closure of systems which are time-reversible with respect to (1.7) after a linear transformation (3.4), so, in view of Theorem 1, it is the Zariski closure of systems which admit a first integral of the form (1.5). However, the set of systems in the space of parameters of (3.1) having an analytic first integral of the form (1.5) is an algebraic set (see e.g., Theorem 3.2.5 of [23]). Therefore, all systems from $\mathbf{V}(\mathcal{I})$ admit an analytic first integral of the form (1.5). \square

Remark 10. Obviously, generically the set of time-reversible systems is a proper subset of $\mathbf{V}(\mathcal{I})$.

As an example we consider the $1 : -2$ resonant system of the form (3.1) of degree five:

$$\begin{aligned} \dot{x} &= x \left(1 - \sum_{u+v=1}^2 a_{2u,v} x^{2u} y^v \right) = x - a_{01}xy - a_{20}x^3 - a_{21}x^3y - a_{02}xy^2 - a_{40}x^5, \\ \dot{y} &= -y \left(2 - \sum_{u+v=1}^2 b_{2v,u} x^{2v} y^u \right) = -2y + b_{01}y^2 + b_{20}x^2y + b_{21}x^2y^2 + b_{02}y^3 + b_{40}x^4y. \end{aligned} \quad (3.8)$$

Proposition 11. System (3.8) admits an analytic first integral of the form (1.5) if the 10-tuple $(a_{01}, \dots, a_{40}, b_{40}, \dots, b_{01})$ of its coefficients belong to the variety of the ideal

$$\begin{aligned} \tilde{\mathcal{I}} &= \langle 2a_{21} - b_{21}, -a_{40}b_{01}^2 + 2a_{20}^2b_{02}, 4a_{02}a_{40} - b_{02}b_{40}, 8a_{02}a_{20}^2 - b_{01}^2b_{40}, -2a_{02}a_{20}b_{20} \\ &\quad + a_{01}b_{01}b_{40}, 2a_{01}a_{40}b_{01} - a_{20}b_{02}b_{20}, 4a_{01}a_{20} - b_{01}b_{20}, -a_{02}b_{20}^2 + 2a_{01}^2b_{40}, 8a_{01}^2a_{40} - b_{02}b_{20}^2 \rangle. \end{aligned}$$

Proof. In the case of system (3.8) the ideal H used in Theorem 9 is

$$\begin{aligned} &\langle a_{01} - t_1, a_{20} - t_2, a_{02} - t_3, a_{40} - t_4, a_{21} - t_5, \\ &\quad b_{20} - 2t_1\gamma^{-1}, b_{01} - 2t_2\gamma, b_{40} - 2t_3\gamma^{-2}, b_{02} - 2t_4\gamma^2, b_{21} - 2t_5, 1 - w\gamma \rangle. \end{aligned}$$

Computing the reduced Gröbner basis of this ideal with respect to the lexicographic ordering with $\gamma > w > t_1 > t_2 > t_3 > t_4 > a_{01} > a_{02} > a_{20} > a_{21} > a_{40} > b_{01} > b_{02} > b_{20} > b_{21} > b_{40}$ we obtain the set of polynomials

$$\begin{aligned} &\{2a_{21} - b_{21}, 4a_{01}a_{20} - b_{01}b_{20}, a_{02}b_{20}^2 - 2a_{01}^2b_{40}, 2a_{02}a_{20}b_{20} - a_{01}b_{01}b_{40}, 8a_{02}a_{20}^2 - b_{01}^2b_{40}, \\ &\quad a_{40}b_{01}^2 - 2a_{20}^2b_{02}, 2a_{01}a_{40}b_{01} - a_{20}b_{02}b_{20}, 8a_{01}^2a_{40} - b_{02}b_{20}^2, 4a_{02}a_{40} - b_{02}b_{40}, \dots\}, \end{aligned}$$

where the dots stand for the polynomials which depend on $\gamma, w, t_1, t_2, t_3, t_4, t_5$.

The polynomials of the Gröbner basis which do not depend on $\gamma, w, t_1, t_2, t_3, t_4$ form a basis of the ideal \mathcal{I} of Theorem 9 and they are exactly the polynomials defining the ideal in the statement of the present proposition. \square

Remark 12. By (a) of Theorem 9, the variety $\mathbf{V}(\tilde{I})$ is the Zariski closure of the set of time-reversible systems in family (3.8).

We denote by S the ordered set of subscripts of the coefficients of the nonlinear terms of the first equation in (3.1). Letting ℓ denote the number of elements of S , S can be written as

$$S = \{(qu_1, pv_1), \dots, (qu_\ell, pv_\ell)\} = \{\bar{i}_1, \dots, \bar{i}_\ell\}.$$

For $\bar{i}_s = (qu_s, pv_s)$, let $\bar{j}_s = (qv_s, pu_s)$. We call \bar{i}_s and \bar{j}_s *conjugate* vectors (or conjugate indices). Any monomial appearing in the coefficient $v_{k_1 k_2}$ of (1.5) has the form $a_{\bar{i}_1}^{v_1} \cdots a_{\bar{i}_\ell}^{v_\ell} b_{\bar{j}_\ell}^{v_{\ell+1}} \cdots b_{\bar{j}_1}^{v_{2\ell}}$ for some $v = (v_1, \dots, v_{2\ell})$. We use notation (2.8) adapted to the case of system (3.1), so now

$$[v] \stackrel{\text{def}}{=} a_{\bar{i}_1}^{v_1} \cdots a_{\bar{i}_\ell}^{v_\ell} b_{\bar{j}_\ell}^{v_{\ell+1}} \cdots b_{\bar{j}_1}^{v_{2\ell}}.$$

For a given field \mathbb{K} we will write just $\mathbb{K}[a, b]$ in place of $\mathbb{K}[a_{\bar{i}_1}, \dots, a_{\bar{i}_\ell}, b_{\bar{j}_\ell}, \dots, b_{\bar{j}_1}]$, and for $f \in \mathbb{K}[a, b]$ write $f = \sum_{v \in \text{Supp}(f)} f^{(v)} [v]$, where $\text{Supp}(f)$ denotes those $v \in \mathbb{N}_0^{2\ell}$ such that the coefficient of $[v]$ in the polynomial f is nonzero.

Definition 13. Let

$$f = \sum_{v \in \text{Supp}(f)} f^{(v)} a_{qu_1, pv_1}^{v_1} \cdots a_{qu_\ell, pv_\ell}^{v_\ell} b_{qv_\ell, pu_\ell}^{v_{\ell+1}} \cdots b_{qv_1, pu_1}^{v_{2\ell}} \in \mathbb{C}[a, b].$$

The conjugate \widehat{f} of f is the polynomial obtained from f by the involution

$$f^{(v)} \rightarrow \widehat{f}^{(v)} \quad a_{qi, pj} \rightarrow b_{qj, pi} \quad b_{qj, pi} \rightarrow a_{qi, pj};$$

that is,

$$\widehat{f} = \sum_{v \in \text{Supp}(f)} \widehat{f}^{(v)} a_{qu_1, pv_1}^{v_{2\ell}} \cdots a_{qu_\ell, pv_\ell}^{v_{\ell+1}} b_{qv_\ell, pu_\ell}^{v_\ell} \cdots b_{qv_1, pu_1}^{v_1} \in \mathbb{C}[a, b].$$

Since $[v] = a_{qu_1, pv_1}^{v_1} \cdots a_{qu_\ell, pv_\ell}^{v_\ell} b_{qv_\ell, pu_\ell}^{v_{\ell+1}} \cdots b_{qv_1, pu_1}^{v_{2\ell}}$, we have

$$[\widehat{v}] = a_{qu_1, pv_1}^{v_{2\ell}} \cdots a_{qu_\ell, pv_\ell}^{v_{\ell+1}} b_{qv_\ell, pu_\ell}^{v_\ell} \cdots b_{qv_1, pu_1}^{v_1},$$

so that

$$[(v_1, \dots, v_{2\ell})] = [(v_{2\ell}, \dots, v_1)].$$

For this reason we will also write, for $v = (v_1, \dots, v_{2\ell})$, $\widehat{v} = (v_{2\ell}, \dots, v_1)$.

Once the ℓ -element set S has been specified and ordered we let $L : \mathbb{N}_0^{2\ell} \rightarrow \mathbb{N}_0^{2\ell}$ be the map defined by

$$L(v) = (L_1(v), L_2(v)) = v_1 \bar{i}_1 + \cdots + v_\ell \bar{i}_\ell + v_{\ell+1} \bar{j}_\ell + \cdots + v_{2\ell} \bar{j}_1, \quad (3.9)$$

which is similar to the map (2.9).

Let

$$\mathcal{M} = \{v \in \mathbb{N}_0^{2\ell} : L(v) = (qk, pk), k = 0, 1, 2, \dots\}.$$

Clearly, \mathcal{M} is an Abelian monoid.

Let \mathfrak{X} be the vector field of system (3.1). The following result was obtained in [21] (where speaking about (s, t) -polynomials we mean (s, t) -polynomials with respect to map (3.9)).

Theorem 14. Let family (3.1) be given. There exists a formal series $\Psi(x, y)$ of the form (1.5) and polynomials $g_{q,p}, g_{2q,2p}, \dots$ in $\mathbb{Q}[a, b]$ such that

(a)

$$\mathfrak{X}\Psi = \sum_{k=1}^{\infty} g_{qk,pk} x^{qk} y^{pk}; \quad (3.10)$$

(b) for every pair $(i, j) \in \mathbb{N}_{-q} \times \mathbb{N}_{-p}$, $i + j \geq 0$, $v_{ij} \in \mathbb{Q}[a, b]$, and v_{ij} is an (i, j) -polynomial;

(c) for every $k \geq 1$, $v_{qk,pk} = 0$; and

(d) for every $k \geq 1$, $g_{qk,pk} \in \mathbb{Q}[a, b]$, and $g_{qk,pk}$ is a (qk, pk) -polynomial.

For a given family (3.1) and ordered set S of indices for any $\nu \in \mathbb{N}_0^{2\ell}$, define $V(\nu) \in \mathbb{Q}$ recursively, with respect to $|\nu| = \nu_1 + \dots + \nu_{2\ell}$, as follows:

$$V(0, \dots, 0) = 1;$$

for $\nu \neq (0, \dots, 0)$

$$V(\nu) = 0 \quad \text{if} \quad pL_1(\nu) = qL_2(\nu);$$

and when $pL_1(\nu) \neq qL_2(\nu)$,

$$\begin{aligned} V(\nu) = & \frac{1}{pL_1(\nu) - qL_2(\nu)} \\ & \times \left[\sum_{j=1}^{\ell} V(\nu_1, \dots, \nu_j - 1, \dots, \nu_{2\ell})(L_1(\nu_1, \dots, \nu_j - 1, \dots, \nu_{2\ell}) + q) \right. \\ & \left. - \sum_{j=\ell+1}^{2\ell} V(\nu_1, \dots, \nu_j - 1, \dots, \nu_{2\ell})(L_2(\nu_1, \dots, \nu_j - 1, \dots, \nu_{2\ell}) + p) \right], \end{aligned}$$

where $L(\nu)$ is defined by (3.9).

Theorem 15. For a family of systems of the form (3.1) let Ψ be the formal series of the form (1.5) computed by (2.7), $\{g_{qk,pk} : k \in \mathbb{N}\}$ be the polynomials in $\mathbb{C}[a, b]$ satisfying (3.10). Then,

(a) for $\nu \in \text{Supp}(v_{k_1,k_2})$, the coefficient $v_{k_1,k_2}^{(\nu)}$ of $[\nu]$ in v_{k_1,k_2} is $V(\nu)$,

(b) for $\nu \in \text{Supp}(g_{qk,pk})$, the coefficient $g_{qk,pk}^{(\nu)}$ of $[\nu]$ in $g_{qk,pk}$ is

$$\begin{aligned} g_{qk,pk}^{(\nu)} = & - \left[\sum_{j=1}^{\ell} V(\nu_1, \dots, \nu_j - 1, \dots, \nu_{2\ell})(L_1(\nu_1, \dots, \nu_j - 1, \dots, \nu_{2\ell}) + q) \right. \\ & \left. - \sum_{j=\ell+1}^{2\ell} V(\nu_1, \dots, \nu_j - 1, \dots, \nu_{2\ell})(L_2(\nu_1, \dots, \nu_j - 1, \dots, \nu_{2\ell}) + p) \right], \end{aligned}$$

and

(c) the following identities hold:

$$\kappa V(\widehat{\nu}) = V(\nu) \quad \text{and} \quad \kappa g_{qk,pk}^{(\widehat{\nu})} = -g_{qk,pk}^{(\nu)}, \quad \text{for all} \quad \nu \in \mathbb{N}_0^{2\ell},$$

$$V(\nu) = g_{qk,pk}^{(\nu)} = 0 \quad \text{if} \quad \widehat{\nu} = \nu \neq (0, \dots, 0), \quad \text{where} \quad \kappa = \left(\frac{q}{p}\right)^{(\nu_1 + \dots + \nu_{\ell}) - (\nu_{\ell+1} + \dots + \nu_{2\ell})}.$$

Statements (a) and (b) of the theorem are proved in [21], statement (c) can be proved similarly as statement 3) of Theorem 3.4.5 of [23].

The next statements follows immediately from c) of Theorem 15.

Corollary 16. *The saddle quantities $g_{qk,pk}$ of system (3.1) have the form*

$$g_{qk,pk} = \frac{1}{2} \sum_{v:L(v)=(qk,pk)} g_{qk,pk}^{(v)} (\kappa[v] - [\hat{v}]). \quad (3.12)$$

By the analogy with the $1 : -1$ resonant case, we call the ideal

$$I_{Sib} = \langle \kappa[v] - [\hat{v}] : v \in \mathcal{M} \rangle \quad (3.13)$$

the *Sibirsky ideal* of system (3.1). Obviously, transformations (3.4) form a group. It is easy to see that any $v \in \mathcal{M}$ $[v]$ is an invariant of group (3.4). Sibirsky studied such invariants for the case of the $1 : -1$ resonant system (1.4) and used the ideal (3.13) to describe the basis of the invariants and the number of symmetry axis of the corresponding real systems [24, 25].

Theorem 17. *If the 2ℓ -tuple of parameters of (3.1) belong to $\mathbf{V}(I_{Sib})$, then the corresponding system admits an analytic first integral of the form (1.5).*

Proof. The conclusion follows from formula (3.12). \square

The following statement is similar to the one of [14] and shows that the variety of the Sibirsky ideal is the Zariski closure of the set of systems, which are time-reversible with respect to (1.7). The proof is based on an adaption of the ideas of [6, 26].

Theorem 18. *Let \mathcal{I} be the ideal defined by (3.5). Then*

$$I_{Sib} = \mathcal{I}. \quad (3.14)$$

Proof. For $k = 1, \dots, \ell$ let, as above, $\zeta_k = u_k - v_k$ and consider the ring homomorphism

$$\theta : \mathbb{Q}(a, b, t_1, \dots, t_\ell, \gamma, w) \longrightarrow \mathbb{Q}(\gamma, t_1, \dots, t_\ell)$$

defined by

$$a_{qu_k,pv_k} \mapsto t_k, \quad b_{qv_k,pu_k} \mapsto \frac{q}{p} \gamma^{\zeta_k} t_k, \quad w \mapsto 1/\gamma, \quad (k = 1, \dots, \ell). \quad (3.15)$$

Let H be the ideal (3.3). Clearly,

$$H = \ker(\theta).$$

A reduced Gröbner basis G of $\mathbb{Q}[a, b] \cap H$ can be found computing a reduced Gröbner basis of H using an elimination ordering with $\{a_{qu_j,pv_j}, b_{qv_j,pu_j}\} < \{w, \gamma, t_j\}$ for all $j = 1, \dots, \ell$, and then intersecting it with $\mathbb{Q}[a, b]$. Since H is binomial, any reduced Gröbner basis G of H also consists of binomials. This means that $\mathcal{I} = H \cap \mathbb{Q}[a, b]$ is a binomial ideal.

We show that $I_{Sib} \subset \mathcal{I}$. Taking into account that $\zeta_k = -\zeta_{2\ell-k}$ by (3.15) for any $\alpha = (\alpha_1, \dots, \alpha_{2\ell}) \in \mathcal{M}$ we have

$$\begin{aligned} \theta([\alpha]) &= t_1^{\alpha_1} \dots t_\ell^{\alpha_\ell} t_\ell^{\alpha_{\ell+1}} \left(\frac{q}{p}\right)^{\alpha_{\ell+1}} \gamma^{\zeta_\ell \alpha_{\ell+1}} \dots t_1^{\alpha_{2\ell}} \left(\frac{q}{p}\right)^{\alpha_{2\ell}} \gamma^{\zeta_1 \alpha_{2\ell}} \\ &= \left(\frac{q}{p}\right)^{\alpha_{\ell+1} + \dots + \alpha_{2\ell}} t_1^{\alpha_1} \dots t_\ell^{\alpha_\ell} t_\ell^{\alpha_{\ell+1}} \dots t_1^{\alpha_{2\ell}} \gamma^{-(\zeta_{\ell+1} \alpha_{\ell+1} + \dots + \zeta_{2\ell} \alpha_{2\ell})} \end{aligned}$$

and

$$\begin{aligned}\theta([\hat{\alpha}]) &= t_1^{\alpha_{2\ell}} \cdots t_\ell^{\alpha_{\ell+1}} t_\ell^{\alpha_\ell} \left(\frac{q}{p}\right)^{\alpha_\ell} \gamma^{\zeta_\ell \alpha_\ell} \cdots t_1^{\alpha_1} \left(\frac{q}{p}\right)^{\alpha_1} \gamma^{\zeta_1 \alpha_1} \\ &= \left(\frac{q}{p}\right)^{\alpha_1 + \cdots + \alpha_\ell} t_1^{\alpha_1} \cdots t_\ell^{\alpha_\ell} t_\ell^{\alpha_{\ell+1}} \cdots t_1^{\alpha_{2\ell}} \gamma^{\zeta_1 \alpha_1 + \cdots + \zeta_\ell \alpha_\ell}.\end{aligned}$$

Since

$$-(\zeta_{\ell+1} \alpha_{\ell+1} + \cdots + \zeta_{2\ell} \alpha_{2\ell}) = \zeta_1 \alpha_1 + \cdots + \zeta_\ell \alpha_\ell,$$

we obtain

$$\theta(\kappa[\alpha] - [\hat{\alpha}]) = 0.$$

Thus, $\kappa[\alpha] - [\hat{\alpha}] \in \ker(\theta)$ yielding $\kappa[\alpha] - [\hat{\alpha}] \in \mathcal{I}$.

The proof of the inclusion $\mathcal{I} \subset I_{Sib}$ is similar as the proof in Theorem 5.2.2 of [23]. \square

As an immediate consequence of Theorem 3.14 and Theorem 9, we have the next result.

Theorem 19. *The variety of the Sibirsky ideal I_{Sib} is the Zariski closure of the set \mathcal{R} of all time-reversible systems in family (3.1).*

4. Conclusions

The above studies show that the theory regarding the computation and the structure of the saddle quantities for $1 : -1$ resonant systems also has a counterpart in the family of $p : -q$ resonant systems, however not in the whole family, but just in subfamilies of the form (2.10) and (3.1). It is in agreement with the known fact that the study of local integrability of $p : -q$ resonant systems is much more difficult than the studies in the $1 : -1$ case, which can be observed already in the quadratic and the cubic case [4, 9].

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

Prof. Jaume Giné and Prof. Valery G. Romanovski are the Guest Editors of special issue “Advances in Qualitative Theory of Differential Equations” for AIMS Mathematics. Prof. Jaume Giné and Prof. Valery G. Romanovski were not involved in the editorial review and the decision to publish this article. The authors declare no conflicts of interest.

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