International Journal of Bifurcation and Chaos

Global phase portraits of the quadratic systems having a singular and irreducible invariant curve of degree 3

J. Llibre
Departament de Matemàtiques, Universitat Autònoma de Barcelona, Edifici C, 08193 Bellaterra, Barcelona, Catalonia, Spain
jllibre@mat.uab.cat

C. Pantazi
Departament de Matemàtiques, Universitat Politècnica de Catalunya, (EPSEB), Av. Doctor Marañón, 44–50, 08028 Barcelona, Spain
chara.pantazi@upc.edu

Any singular irreducible cubic curve (or simply, cubic) after an affine transformation can be written as either $y^2 = x^3$, or $y^2 = x^2(x+1)$, or $y^2 = x^2(x-1)$. We classify the phase portraits of all quadratic polynomial differential systems having the invariant cubic $y^2 = x^2(x+1)$. We prove that there are 63 different topological phase portraits for such quadratic polynomial differential systems. We control all the bifurcations among these distinct topological phase portraits. These systems have no limit cycles. Only 3 phase portraits have a center, 19 of these phase portraits have one polycycle, 3 of these phase portraits have 2 polycycles. The maximum number of separartices that have these phase portraits is 26 and the minimum number is 9, the maximum number of canonical regions of these phase portraits is 7 and the minimum is 3.

Keywords: Poincaré disk, global phase portrait, singular curve, cubic curve, polycycles, canonical regions, separatrices.

1. Introduction and statement of the main result

Quadratic polynomial differential systems (or simply *quadratic systems*) are systems that can be written into the form

$$\dot{x} = P(x, y) = P_0 + P_1 + P_2, \qquad \dot{y} = Q(x, y) = Q_0 + Q_1 + Q_2, \tag{1}$$

where P_i and Q_i are real polynomials of degree i in the variables (x, y) and $P_2^2 + Q_2^2 \neq 0$.

An extensive literature is dedicated to the study of the quadratic systems these last years. For a good survey see the book of Reyn [Reyn, 1994] or the book of Artés et.al [Artés *et al.*, 2021], and references therein. For example, the following families of quadratic systems have been studied: homogeneous [Coll *et al.*, 1987], semi-homogeneous [Cairó & Llibre, 1997], bounded [Dickson *et al.*, 1970], reversible [Gavrilov & Iliev, 2000; Coll *et al.*, 2009], Hamiltonian [Artés & Llibre, 1994a; Chow *et al.*, 2002], Lienard [Dumortier & Li, 1997], integrable using Carleman and Painlevé tools [Hua *et al.*, 1996], rational integrable [Artés *et al.*, 2007, 2010, 2009], the ones having a star nodal point [Berlinski, 1966], a center [Vulpe, 1983; Lunkevich & Sibirskii, 1982; Coll *et al.*, 2009; Vulpe, 1983], one focus and one antisaddle [Artés & Llibre, 1994b], with a semi–elementary triple node [Artés *et al.*, 2013], chordal [Gasull *et al.*, 1986; Gasull & Llibre, 1988], with