Chaos ARTICLE pubs.aip.org/aip/cha

Existence of a cylinder foliated by periodic orbits in the generalized Chazy differential equation

Cite as: Chaos 33, 073104 (2023); doi: 10.1063/5.0138309
Submitted: 10 December 2022 · Accepted: 13 June 2023 ·
Published Online: 5 July 2023

View Online Export Citation CrossMark

Jaume Llibre,[1,a)] Douglas D. Novaes,[2,b)] and Claudia Valls[3,c)]

AFFILIATIONS

[1] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain
[2] Departamento de Matemática, Instituto de Matemática, Estatística e Computação Científica (IMECC), Universidade Estadual de Campinas (UNICAMP), Rua Sérgio Buarque de Holanda, 651, Cidade Universitária Zeferino Vaz, 13083-859 Campinas, SP, Brazil
[3] Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais, 1049–001 Lisboa, Portugal

[a)] **Email address:** jllibre@mat.uab.cat
[b)] **Author to whom correspondence should be addressed:** ddnovaes@unicamp.br
[c)] **Email address:** cvalls@math.tecnico.ulisboa.pt

ABSTRACT

The generalized Chazy differential equation corresponds to the following two-parameter family of differential equations $\dddot{x} + |x|^q\ddot{x} + \frac{k|x|^q}{x}\dot{x}^2 = 0$, which has its regularity varying with q, a positive integer. Indeed, for $q = 1$, it is discontinuous on the straight line $x = 0$, whereas for q a positive even integer it is polynomial, and for $q > 1$ a positive odd integer it is continuous but not differentiable on the straight line $x = 0$. In 1999, the existence of periodic solutions in the generalized Chazy differential equation was numerically observed for $q = 2$ and $k = 3$. In this paper, we prove analytically the existence of such periodic solutions. Our strategy allows to establish sufficient conditions ensuring that the generalized Chazy differential equation, for $k = q + 1$ and any positive integer q, has actually an invariant topological cylinder foliated by periodic solutions in the $(x, \dot{x}, \ddot{x})$-space. In order to set forth the bases of our approach, we start by considering $q = 1, 2, 3$, which are representatives of the different classes of regularity. For an arbitrary positive integer q, an algorithm is provided for checking the sufficient conditions for the existence of such an invariant cylinder, which we conjecture that always exists. The algorithm was successfully applied up to $q = 100$.

In 1999, Géronimi *et al.*[1] introduced a two-parameter family of third order differential equations called generalized Chazy differential equation. An interesting feature of the generalized Chazy differential equation is that its regularity changes with a positive integer parameter q. Indeed, for $q = 1$, the generalized Chazy differential equation is discontinuous at $x = 0$, whereas for q, a positive even integer it is polynomial, and for $q > 1$, a positive odd integer it is continuous but not differentiable at $x = 0$. They performed numerical computations and, under some constraints, they observed (only numerically) the existence of periodic solutions. Usually, to prove analytically the existence of periodic solutions in a given differential equation is not an easy problem. The difficulty increases significantly when the dimension or the order is greater than two. In this paper, we develop a strategy to detect analytically the existence of periodic solutions in the generalized Chazy differential equation. Our strategy allows to establish sufficient conditions ensuring that the generalized Chazy differential equation, for any positive integer q, has actually an invariant topological cylinder foliated by periodic solutions in the $(x, \dot{x}, \ddot{x})$-space. First, we will focus our analysis in the cases $q = 1, 2, 3$ (representatives of the above different classes of regularity), for which we will prove the existence of such an invariant cylinder. These initial cases will set forth the bases for an algorithmical approach when q is an arbitrary given positive integer, which will be successfully applied up to $q = 100$.

I. INTRODUCTION AND STATEMENTS OF THE MAIN RESULTS

In 1997, Feix *et al.*[2] introduced the following general third order ordinary differential equation:

$$\dddot{x} + x^{3q+1}f(a, b) = 0, \text{ where } a = \frac{\dot{x}}{x^{q+1}} \text{ and } b = \frac{\ddot{x}}{x^{2q+1}}, \quad (1)$$