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HORSESHOES, ENTROPY AND PERIODS FOR GRAPH MAPS

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1. INTRODUCTION

IN RECENT years, many papers (and even some books) have appeared, where topological entropy and cycles (periodic orbits) are studied for maps of compact one-dimensional spaces into themselves. However, usually those one-dimensional spaces are an interval or a circle. Maps of other graphs are studied much less (see e.g. [11, 12, 1, 3, 4, 16, 15, 6, 7]).

Here we would like to contribute to filling up this gap by applying some of the machinery developed to study interval maps, to the case of graphs (see Section 2 for a precise definition). Our main goal is to understand the properties of topological entropy of the graph maps and its relation to horseshoes and periods of cycles.

For continuous maps of an interval and a circle into itself it is well known that the magnitude of the topological entropy $h(f)$ of such a map f depends on the horseshoes (see [18, 17, 2]). The same turns out to be true in our case. We get also the usual corollary about lower semi-continuity of topological entropy. We define an s-horseshoe for f in much the same way as for interval maps: there is an interval I and s its subintervals with pairwise disjoint interiors, each of them mapped by f onto the whole I. Here we require that an "interval" is contained in an edge of the graph. For precise definitions, see Section 2.

THEOREM A. *If a continuous map f of a graph into itself has an s-horseshoe then* $h(f) \geqslant \log s$.

THEOREM B. *If a continuous map f of a graph into itself has positive topological entropy then there exist sequences $(k_n)_{n=1}^{\infty}$ and $(s_n)_{n=1}^{\infty}$ of positive integers such that for each n the map f^{k_n} has an s_n-horseshoe and*

$$\limsup_{n\to\infty} \frac{1}{k_n} \log s_n = h(f).$$

THEOREM C. *Topological entropy, as a function of a continuous map of a graph into itself, is lower semi-continuous.*

By the same reasons as for interval maps, Theorem C cannot be improved, that is the entropy is not upper semi-continuous (see e.g. [2]).

As for other spaces, also for graphs the existence of horseshoes implies the existence of cycles of many periods. Thus, using Theorem B we are able to give various characterizations of maps with positive entropy or maps with zero entropy via the set $\mathrm{Per}(f)$ of periods of cycles of f. Those results generalize the existing theorems for interval and circle maps (see [8, 18, 17, 2]).