

# Some families of quadratic systems with at most one limit cycle

Jaume Llibre

**Abstract.** The work of Chicone and Shafer published in 1982 together with the work of Bamon published in 1986 proved that any polynomial differential system of degree two has finitely many limit cycles. But the problem remains open of providing a uniform upper bound for the maximum number of limit cycles that a polynomial differential system of degree two can have, i.e. the second part of the 16th Hilbert problem restricted to the polynomial differential systems of degree two remains open. Here we present six subclasses of polynomial differential systems of degree two for which we can prove that an upper bound for their maximum number of limit cycles is one.

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## 1 Introduction and statement of the main results

We deal with polynomial differential systems in  $\mathbb{R}^2$  of the form

$$\frac{dx}{dt} = \dot{x} = P(x, y), \quad \frac{dy}{dt} = \dot{y} = Q(x, y).$$

The degree of such a polynomial system is the maximum of the degrees of the polynomials  $P$  and  $Q$ . In what follows the polynomial differential systems of degree 2 are simply called *quadratic systems*.

We recall that a *limit cycle* of a differential system is a periodic orbit of this system isolated in the set of all periodic orbits of the system. As far as we know the notion of limit cycle appeared in the work of Poincaré [14] in the year 1885.

At the Second International Congress of Mathematicians, held in Paris in 1900, Hilbert [8] proposed his famous 16th problem, whose second part essentially says: *Find an upper bound for the maximum number of limit cycles that the polynomial differential systems in  $\mathbb{R}^2$  of a given degree can have.*

The works of Chicone and Shafer [5] and of Bamon [1] proved that any polynomial differential system of degree 2 has finitely many limit cycles. This result uses previous results of Ilyashenko [9]. Up to now the second part of the 16th Hilbert problem remains unsolved, also for the quadratic systems.

In 1957 Petrovskii and Landis [12] claimed that the polynomial differential systems of degree  $n = 2$  have at most 3 limit cycles. Soon (in 1959) a gap was found