LIMIT CYCLES OF PLANAR CONTINUOUS PIECEWISE DIFFERENTIAL SYSTEMS SEPARATED BY A PARABOLA AND FORMED BY AN ARBITRARY LINEAR AND QUADRATIC CENTERS

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Abstract. Due to their applications to many physical phenomena during these last decades the interest for studying the continuous or discontinuous piecewise differential systems has increased strongly. The limit cycles play a main role in the study of any planar differential system. Up to now the major part of papers which study the limit cycles of the planar piecewise differential systems have considered systems formed by two pieces separated by one straight line. Here we consider planar continuous piecewise differential systems separated by a parabola.

We prove that the planar continuous piecewise differential systems separated by a parabola and formed by a linear center and a quadratic center have at most one limit cycle. Moreover there are systems in this class exhibiting one limit cycle. So in particular we have solved the extension of the 16th Hilbert problem to this class of differential systems.

1. Introduction and results

Andronov, Vitt and Khaikin [1] started in a serious way the study of the piecewise differential systems mainly motivated for their applications to some mechanical systems, and now these systems still continue to receive the attention of many researchers. Recently these differential systems are widely used to model processes appearing in mechanics, electronics, economy, etc., see for instance the books [3] and [19], and the survey [17], as well as the hundreds of references cited there.

While the more studied piecewise differential systems are the discontinuous ones, here we will deal with a class of the continuous piecewise differential systems in the plane.

The simplest possible continuous but nonsmooth piecewise differential systems are the ones having only two pieces formed by two linear differential systems separated by a straight line in the plane $\mathbb{R}^2$. Thus in 1990 Lum and Chua conjectured in [15, 16] that such continuous piecewise linear differential systems have at most one limit cycle. In 1998 this conjecture was proved by Freire et al. [6]. In 2013 a new and shorter proof was done by Llibre, Ordóñez and E. Ponce [12], and recently another proof has been provided by Carmona, Fernández-Sánchez and Novaes [4]. But in all these papers the authors forgot to analyze the case when the two linear differential systems which form the piecewise linear differential system have no equilibrium points, this case was studied in Llibre and Teixeira [13] in 2016 where it is proved that such continuous piecewise systems have no limit cycles.

Let $p(x, y) = 0$ be a parabola. A *continuous piecewise differential system in the plane with two pieces separated by a parabola* is a differential system of the form

$$\dot{x} = f_+(x, y), \quad \dot{y} = g_+(x, y), \quad \text{in the region } p(x, y) \geq 0;$$

and

$$\dot{x} = f_-(x, y), \quad \dot{y} = g_-(x, y), \quad \text{in the region } p(x, y) \leq 0;$$

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such that $f_+(x,y) = f_-(x,y)$ and $g_+(x,y) = g_-(x,y)$ on the points of the parabola $p(x,y) = 0$, being the functions f_+, f_-, g_+ and g_- at least $C^!$ functions. As usual the dot on the variables x and y denotes derivative with respect to the time t.

If we have a continuous piecewise differential system in the plane with two pieces separated by a parabola doing an affine change of variables it is not restrictive to assume that the parabola is $y = x^2$.

We recall that a *center* is an equilibrium point p of a planar differential system having a neighborhood U such that all the orbits of the system in $U \setminus \{p\}$ are periodic.

A differential system

$$\dot{x} = P(x,y), \qquad \dot{y} = Q(x,y), \tag{1}$$

with P and Q real polynomials whose maximum degrees is n is called a *polynomial differential system of degree n*. When $n = 1$ system (1) is usually called *linear differential system* or *linear system*, and when $n = 2$ system (1) is simply called *quadratic differential system* or *quadratic system*.

When a linear system has a center such system is called *linear center*, and when a quadratic system has one or two centers we called it in this paper a *quadratic center*. It is known that the quadratic systems can have at most two centers, see for instance Vulpe [20] or Schlomiuk [18].

A *limit cycle* of a planar differential system is a periodic orbit of the system isolated in the set of all periodic orbits of the system. In general it is a very difficult problem to know the non-existence or the existence of the maximum number of limit cycles that a given class of planar differential systems can have, see for instance the famous 16th Hilbert problem [7, 8, 11].

The main objective of this paper is to study the problem of Lum and Chua extended to the class of continuous piecewise differential systems in the plane with two pieces separated by a parabola and when in each piece we have either two linear centers, or a linear center and a quadratic center. That is, *what is the maximum number of limit cycles that such classes of continuous piecewise differential systems can exhibit*?

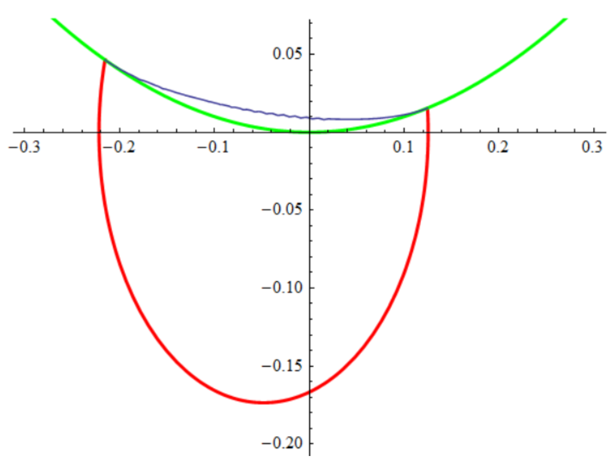


FIGURE 1. The limit cycle of the planar continuous piecewise differential system separated by the parabola $y = x^2$ and formed by the linear center (26) and the quadratic center (27). This limit cycle is traveled in counterclockwise.

Our main results are the following.

Proposition 1. *The continuous piecewise differential systems in the plane with two pieces separated by a parabola and having in each piece two arbitrary linear centers have no limit cycles.*

Theorem 2. *The continuous piecewise differential systems in the plane with two pieces separated by a parabola and having in one piece an arbitrary linear center and in the other piece an arbitrary quadratic center have at most one limit cycle. Moreover there are systems in this class exhibiting one limit cycle, see for instance Figure* 2.

Proposition 1 and Theorem 2 are proved in section 3. In section 2 we recall some well known results that we shall need for proving Proposition 1 and Theorem 2 .

2. Preliminaries

Let U be an open subset of $\mathbb{R}^2$ and $H : U \to \mathbb{R}$ be a C^1 function. We recall that H is a *first integral* of a differential system (1) if

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}P + \frac{\partial H}{\partial y}Q = 0 \quad \text{in all the points of } U.$$

In other words if the function H is constant on the orbits of system (1) contained in U.

In the next lemma we give a normal form for an arbitrary linear center, for a proof see Lemma 1 of [14].

Lemma 3. *A linear differential system having a center can be written as*

$$\dot{x} = -ax - \frac{a^2 + w^2}{d}y + b, \qquad \dot{y} = dx + ay + c, \tag{2}$$

with $d > 0$ *and* $w > 0$.

The quadratic centers where characterized in the following theorem, for a proof see Kapteyn [9, 10] and Bautin [2].

Theorem 4 (Kapteyn–Bautin Theorem)**.** *Any quadratic system candidate to have a center can be written in the form*

$$\dot{x} = -y - Bx^2 - Cxy - Dy^2, \qquad \dot{y} = x + Ax^2 + Exy - Ay^2. \tag{3}$$

This system has a center at the origin if and only if one of the following conditions holds
$A - 2B = C + 2A = 0$,
$C = A = 0$,
$B + D = 0$,
$C + 2A = E + 3B + 5D = A^2 + BD + 2D^2 = 0$.

3. The proofs

Proof of Proposition 1. Under the assumptions of this proposition let (2) be one of the two linear centers of a continuous piecewise differential system in the plane with two pieces separated by the parabola $y = x^2$. The other arbitrary linear center can be

$$\dot{x} = -\alpha x - \frac{\alpha^2 + \omega^2}{\delta}y + \beta, \qquad \dot{y} = \delta x + \alpha y + \gamma, \tag{4}$$

with $\delta > 0$ and $\omega > 0$.

Now when we impose that on the points of the parabola systems (2) and (4) coincide, i.e.

$$-ax - \frac{a^2 + w^2}{d}x^2 + b = -\alpha x - \frac{\alpha^2 + \omega^2}{\delta}x^2 + \beta,$$

and

$$dx + ax^2 + c = \delta x + \alpha x^2 + \gamma,$$

for all $x \in \mathbb{R}$. Then we obtain that both differential systems coincide. So we have a unique linear center in the whole plane $\mathbb{R}^2$, and it is well known that the linear centers have no limit cycles. This completes the proof of the proposition. $\square$

Proof of Theorem 2. Under the hypotheses of this theorem let (2) be the linear center of a continuous piecewise differential system in the plane with two pieces separated by the parabola $y = x^2$. The other arbitrary quadratic center will be the quadratic differential system

$$\begin{aligned} \dot{x} &= c_0 + c_1 x + c_2 y + c_3 x^2 + c_4 xy + c_5 y^2, \\ \dot{y} &= d_0 + d_1 x + d_2 y + d_3 x^2 + d_4 xy + d_5 y^2, \end{aligned} \tag{5}$$

for convenient values of its parameters that we shall determine in order that the piecewise differential system formed by the differential systems (2) and (5) be continuous, and system (5) be a quadratic center.

We start imposing that the piecewise differential system formed by the differential systems (2) and (5) and separated by the parabola $y = x^2$ be continuous, i.e.

$$-ax - \frac{a^2+w^2}{d}x^2 + b = c_0 + c_1 x + (c_2 + c_3)x^2 + c_4 x^3 + c_5 x^4,$$

and

$$dx + ax^2 + c = d_0 + d_1 x + (d_2 + d_3)x^2 + d_4 x^3 + d_5 x^4,$$

for all $x \in \mathbb{R}$. Therefore system (5) becomes

$$\begin{aligned} \dot{x} &= b - ax - \frac{a^2 + c_3 d + w^2}{d} y + c_3 x^2, \\ \dot{y} &= c + dx + (a - d_3)y + d_3 x^2. \end{aligned} \tag{6}$$

We define

$$R = d^2\left(ad_3 + c_3 d + w^2\right)^2 - 4\left(a^2 d_3 + ac_3 d + d_3 w^2\right)\left(a^2 c + abd - bdd_3 + c\left(c_3 d + w^2\right)\right),$$

and assume that

$$R > 0 \quad \text{and} \quad adc_3 + d_3(a^2 + w^2) \neq 0. \tag{7}$$

Then system (6) has two finite equilibria, namely $p_\pm = (x_\pm, y_\pm)$ where

$$\begin{aligned} x_\pm &= \frac{\pm\sqrt{R} - d(ad_3 + c_3 d + w^2)}{2(adc_3 + d_3(a^2 + w^2))}, \\ y_\pm &= \frac{d(ad_3 + c_3 d)\left(ad_3(2b + d) + c_3\left(d^2 - 2ac\right) \mp \sqrt{R}\right) + dw^2\left(d_3(ad + 2bd_3) + c_3\left(d^2 - 2cd_3\right)\right)}{2\left(adc_3 + d_3(a^2 + w^2)\right)^2}. \end{aligned}$$

Note that condition $adc_3 + d_3(a^2 + w^2) \neq 0$ of (7) is necessary in order that system (6) has finite equilibria, and we will see that condition $R > 0$ of (7) will be necessary for having a quadratic center.

Since the determinant of the linear part of system (6) at the equilibria $p_\pm$ are $\pm\sqrt{R}/d$ and $d > 0$, we have that p_+ is either a node, a focus or a center, and p_- is a saddle. Note that when $R = 0$ the equilibria $p_\pm$ coincide producing a saddle-node, and consequently the quadratic system cannot have a center. So R must be positive. For more details on the results of this paragraph see Chapter 2 of [5].

Now we must determine the coefficients of system (6) in such a way that the equilibrium p_+ be a center.

The eigenvalues of the linear part of system (6) at p_+ are $(T \pm \sqrt{T^2-4D}/2$ where

$$
\begin{aligned}
T = & -\frac{c_3(add_3+c_3d^2+dw^2-\sqrt{R})}{adc_3+d_3(a^2+w^2)}-d_3,\\
T^2-4D = & \frac{c_3^2(\sqrt{R}-d(ad_3+c_3d+w^2))^2}{(adc_3+d_3(a^2+w^2))^2}+\frac{2(2d_3(a^2+w^2)+c_3d(2a+d_3))(d(ad_3+c_3d+w^2)-\sqrt{R})}{d(adc_3+d_3(a^2+w^2))}\\
& -4ad_3-4c_3d+d_3^2-4w^2.
\end{aligned}
$$

Of course, T and D are the trace and the determinant of the linear part of system (6) at p_+. In order that p_+ can be a center we must have $T=0$ and $T^2-4D<0$, see again Chapter 2 of [5]. Isolating $\sqrt{R}$ from $T=0$ and substituing it into T^2-4D we get that

$$
T^2-4D=\frac{-4(ad_3+dc_3)^2-4w^2(dc_3+d_3^2)}{dc_3}. \tag{8}
$$

We note that $c_3 \neq 0$, otherwise the trace of system (6) is constant and equal to $-d_3$, and then system (6) would have a center if and only if $d_3=0$, but then the system becomes linear instead of quadratic.

In order to simplify the computations we change the parameter $w>0$ by the new parameter $k>0$ defined from $T^2-4D=-k^2$, i.e.

$$
w=\frac{1}{2}\sqrt{\frac{dk^2c_3-4(ad_3+dc_3)^2}{dc_3+d_3^2}}.
$$

Note that $dc_3+d_3^2 \neq 0$, otherwise from (8) we obtain that $T^2-4D=4(ad_3+dc_3)^2/d_3^2 \geq 0$, and consequently system (6) would not have a center. Moreover $(dk^2c_3-4(ad_3+dc_3)^2)(dc_3+d_3^2)>0$, because $w>0$.

For forcing that $R>0$ we also change the parameter c by a new parameter $L>0$ taking $R=L^2$. Now from $T=0$ we obtain

$$
-\frac{\left(c_3d+d_3^2\right)\left(dk^2-4L\right)}{d\big(4(a-d_3)(ad_3+c_3d)+d_3k^2\big)}=0,
$$

and of course $d\big(4(a-d_3)(ad_3+c_3d)+d_3k^2\big)$ cannot be zero, otherwise system (6) would not have a center. Since $c_3d+d_3^2$ is not zero, we obtain that

$$
L=\frac{dk^2}{4}. \tag{9}
$$

Now from $R=L^2$ we obtain that

$$
c=\frac{4(a-d_3)\left(d_3^2(ad_3+c_3d)-4bc_3\left(c_3d+d_3^2\right)\right)-d_3k^2\left(2c_3d+d_3^2\right)}{4c_3^2\left(4(a-d_3)^2+k^2\right)}.
$$

In the new variable k, because the variable L disappears using (9), the quadratic differential system (6) writes

$$
\begin{aligned}
\dot{x} = & \; b-ax-\frac{c_3y\left(4(a-d_3)^2+k^2\right)}{4(c_3d+d_3^2)}+c_3x^2,\\
\dot{y} = & \; dx+(a-d_3)y+d_3x^2+\frac{4(a-d_3)\left(d_3^2(ad_3+c_3d)-4bc_3\left(c_3d+d_3^2\right)\right)-d_3k^2\left(2c_3d+d_3^2\right)}{4c_3^2\left(4(a-d_3)^2+k^2\right)},
\end{aligned} \tag{10}
$$

and the equilibrium point p_+, which now is a weak focus or a center because the eigevalues of the linear part of system (10) at p_+ are $\pm ki/2$ with $k>0$, writes

$$
p_+=\left(\frac{d_3}{2c_3},\frac{\left(c_3d+d_3^2\right)\left(d_3(d_3-2a)+4bc_3\right)}{c_3^2\left(4(a-d_3)^2+k^2\right)}\right).
$$

In order to decide when the equilibrium p_+ is a quadratic center we shall use Theorem 4. Therefore we need first to translate the equilibrium p_+ to the origin of coordinates and after to write the differential system (10) in the normal form (3).

For doing the translation of p_+ to the origin of coordinates we pass from the coordinates (x, y) to the coordinates (X, Y) through

$$(x,y) = \left(X + \frac{d_3}{2c_3}, Y + \frac{\left(c_3 d + d_3^2\right)\left(d_3(d_3 - 2a) + 4bc_3\right)}{c_3^2\left(4(a-d_3)^2 + k^2\right)} \right),$$

and system (10) becomes

$$\begin{aligned} \dot{X} = & \quad (d_3 - a)X - \frac{c_3\left(4(a-d_3)^2 + k^2\right)}{4\left(c_3 d + d_3^2\right)} Y + c_3 X^2, \\ \dot{Y} = & \quad \left(\frac{d_3^2}{c_3} + d\right) X + (a - d_3)Y + d_3 X^2. \end{aligned} \tag{11}$$

The more general linear change of variables from (X, Y) to (u, v) wich writes the linear part of system (11) into its real Jordan normal form is

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \\ \dfrac{2c_3(a-d_3)y_1 - 2(d_3^2 + c_3 d)y_2}{c_3 k} & \dfrac{1}{2k}\left(\dfrac{c_3(4(a-d_3)^2 + k^2)y_1}{d_3^2 + c_3 d} + 4(d_3 - a)y_2\right) \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}, \tag{12}$$

of course having the determinant of the matrix nonzero. Doing this change of variables we get system (11) in the new variables (u, v) being the linear part at the origin of system $(\dot{u}, \dot{v})$ given by the matrix

$$\begin{pmatrix} 0 & -\dfrac{k}{2} \\ \dfrac{k}{2} & 0 \end{pmatrix}.$$

Doing the scaling of the time $s = kt/2$, the differential system $(\dot{u}, \dot{v})$ in the new time becomes of the form

$$\begin{aligned} u' = & \quad -u + a_0 u^2 + a_1 uv + a_2 v^2, \\ v' = & \quad v + b_0 u^2 + b_1 uv + b_2 v^2, \end{aligned} \tag{13}$$

where the prime denotes derivative with respect to the new time s, and the coefficients a_i and b_i for $i = 1, 2, 3$ depend on the parameters $(a, c_3, d, d_3, k, y_1, y_2)$.

In order that system (13) be into the normal form (3) we need that $b_0 + b_2 = 0$, from this equation we get

$$y_2 = -\frac{c_3(4(a-d_3)(ad_3 + c_3 d) + d_3 k^2)}{4\left(c_3 d + d_3^2\right)\left(ad_3 + c_3 d\right)} y_1.$$

Hence taking $y_1 = ad_3 + c_3 d$ the determinant of the matrix of the change of variables (12) is

$$\frac{c_3\left(4k(ad_3 + c_3 d)^2 + d_3^2 k^3\right)}{8\left(c_3 d + d_3^2\right)} \neq 0.$$

Now system (13) writes in the normal form (3):

$$u' = -u - Bu^2 - Cuv - Dv^2, \qquad v' = u + Au^2 + Euv - Av^2, \tag{14}$$

where

$$\begin{aligned} & A = E = 0, \\ & B = -\frac{8c_3\left(c_3 d + d_3^2\right)}{4k(ad_3 + c_3 d)^2 + d_3^2 k^3}, \\ & C = \frac{8c_3(4(a-d_3)(ad_3 + c_3 d) + d_3 k^2)}{4k^2(ad_3 + c_3 d)^2 + d_3^2 k^4}, \\ & D = -\frac{2c_3\left(4(a-d_3)(ad_3 + c_3 d) + d_3 k^2\right)^2}{k^3\left(c_3 d + d_3^2\right)\left(4(ad_3 + c_3 d)^2 + d_3^2 k^2\right)}. \end{aligned}$$

We know that the denominators of B, C and D do not vanish.

From Theorem 4 the origin of system (14) is a center if either $B + D = 0$, or $C = 0$. Since

$$B + D = -\frac{2c_3\left(4(a-d_3)^2+k^2\right)}{k^3\left(c_3 d + d_3^2\right)} \neq 0,$$

the unique possibility to have a center is that $C = 0$, or equivalently

$$4(a-d_3)(ad_3+c_3 d)+d_3 k^2 = 0.$$

If $a - d_3 \neq 0$ then isolating c_3 from this last equation we get either

$$c_3 = -\frac{d_3\left(4a(a-d_3)+k^2\right)}{4d(a-d_3)}. \tag{15}$$

Later on, in (19), we will see that $a - d_3$ cannot be zero, otherwise the linear system (2) would not have a center. Hence, from (15) when the quadratic system (14) has a center it becomes

$$u' = -v + \frac{2u^2\left(4a(a-d_3)+k^2\right)}{dk^3}, \qquad v' = u. \tag{16}$$

If we write the quadratic center (16) in the original coordinates (x, y) going back through the changes of coordinates we get

$$\begin{aligned} \dot{x} &= b - ax - \frac{(4a(a-d_3)+k^2)}{4d}y - \frac{d_3(4a(a-d_3)+k^2)}{4d(a-d_3)}x^2, \\ \dot{y} &= \frac{2d(a-d_3)(-8a^2b+8abd_3+k^2(d-2b))}{(4a(a-d_3)+k^2)^2} + dx + (a-d_3)y + d_3x^2. \end{aligned} \tag{17}$$

This system has the center at the point

$$\left(\frac{2d(d_3-a)}{4a(a-d_3)+k^2}, \frac{4d\left((a-d_3)(2a(2b+d)-dd_3)+bk^2\right)}{\left(4a(a-d_3)+k^2\right)^2}\right),$$

and the eigenvalues of the linear part of the system at this point are $\pm ki/2$.

The linear system (2), taking into account all the values of the parameters for arriving to have the quadratic center (16), is

$$\begin{aligned} \dot{x} &= b - ax - \frac{a\left(4a(a-d_3)+k^2\right)}{4d(a-d_3)}y, \\ \dot{y} &= \frac{2d(a-d_3)\left(-8a^2b+8abd_3+k^2(d-2b)\right)}{\left(4a(a-d_3)+k^2\right)^2} + dx + ay. \end{aligned} \tag{18}$$

This system has the equilibrium point

$$\left(\frac{2d(d_3-a)}{4a(a-d_3)+k^2}, \frac{4d(a-d_3)\left(2a(a-d_3)(2b+d)+bk^2\right)}{a\left(4a(a-d_3)+k^2\right)^2}\right),$$

and the eigenvalues of the linear part of the system at this point are $\pm k\sqrt{a/(d_3-a)}/2$. So in order this equilibrium be a linear center we need that

$$a(d_3 - a) < 0. \tag{19}$$

Notice that in order that systems (17) and (18) will be well defined and have centers we need (19) and that

$$4a(a-d_3)+k^2 \neq 0. \tag{20}$$

In summary all the planar continuous piecewise differential systems separated by the parabola $y = x^2$ formed by a linear and a quadratic center are such that the linear center is given in (18) and the quadratic center is given in (17).

It is well known that the linear centers have a first integral and that also the quadratic centers have a first integral, see for instance [18]. Then it is easy to check that

$$H_1(x,y)=\frac{4d^2x(a-d_3)\left(-8a^2b+8abd_3+k^2(d-2b)\right)}{\left(4a(a-d_3)+k^2\right)^2}+(ay+dx)^2+\frac{ak^2y^2}{4a-4d_3}-2bdy,$$

and

$$H_2(x,y)=\left({d_3}^2k^4\left(x^2-y\right)+k_3+k_4x+k_5y+k_6x^2\right)e^{\dfrac{d_3(k_0+k_1x+k_2y)}{2d^2k^2(a-d_3)}},$$

are first integrals of the systems (18) and (17) respectively, where

$$\begin{aligned}
k_0&=-4d(a-d_3)(4ab+dd_3)-4bdk^2,\\
k_1&=4d(a-d_3)(4a(a-d_3)+k^2),\\
k_2&=(4a(a-d_3)+k^2)^2,\\
k_3&=2d(a-d_3)(8abd_3(-a+d_3)+(-ad+(-2b+d)d_3)k^2),\\
k_4&=4ad(a-d_3)d_3(4a(a-d_3)+k^2),\\
k_5&=ad_3(16a(a-d_3)^3+8(a-d_3)^2k^2+k^4),\\
k_6&=8a(a-d_3)d_3^2(2a(a-d_3)+k^2).
\end{aligned}$$

If a planar continuous piecewise differential system separated by the parabola $y=x^2$ formed by the linear center (18) and the quadratic center (17) has a limit cycle intersecting the parabola in the points (x_1,x_1^2) and (x_2,x_2^2) with $x_1\neq x_2$, the coordinates x_1 and x_2 must be an isolated solution of the equations

$$e_1=H_1(x_1,x_1^2)-H_1(x_2,x_2^2)=0,\qquad e_2=H_2(x_1,x_1^2)-H_2(x_2,x_2^2)=0, \tag{21}$$

or equivalently

$$\begin{aligned}
e_1=&\ \frac{x_1-x_2}{4(a-d_3)\left(4a^2-4ad_3+k^2\right)^2}e_{11}e_{12}=0,\\
e_2=&\ \left(k_3+k_4x_1+x_1^2(k_5+k_6)\right)e^{\dfrac{d_3\left(k_0+k_1x_1+k_2x_1^2\right)}{2d^2k^2(a-d_3)}}\\
&-\left(k_3+k_4x_2+x_2^2(k_5+k_6)\right)e^{\dfrac{d_3\left(k_0+k_1x_2+k_2x_2^2\right)}{2d^2k^2(a-d_3)}}=0,
\end{aligned}$$

where

$$\begin{aligned}
e_{11}=&\ x_1\left(4a^2-4ad_3+k^2\right)+x_2\left(4a^2-4ad_3+k^2\right)+4d(a-d_3),\\
e_{12}=&\ 4d(a-d_3)\left(-8a^2b+8abd_3+k^2(d-2b)\right)+ax_1^2\left(4a^2-4ad_3+k^2\right)^2+ax_2^2\left(4a^2-4ad_3+k^2\right)^2\\
&+4adx_1\left(4a^3-8a^2d_3+a\left(4d_3^2+k^2\right)-d_3k^2\right)+4adx_2\left(4a^3-8a^2d_3+a\left(4d_3^2+k^2\right)-d_3k^2\right).
\end{aligned}$$

Remark 5. *From the definitions of e_1 and e_2 it is clear that if (x_1,x_2) is a solution of system (21), then (x_2,x_1) is also a solution of system (21).*

For solving system (21) with $x_1\neq x_2$ it is sufficient to solve the two following subsystems

$$e_{11}=0,\qquad e_2=0. \tag{22}$$

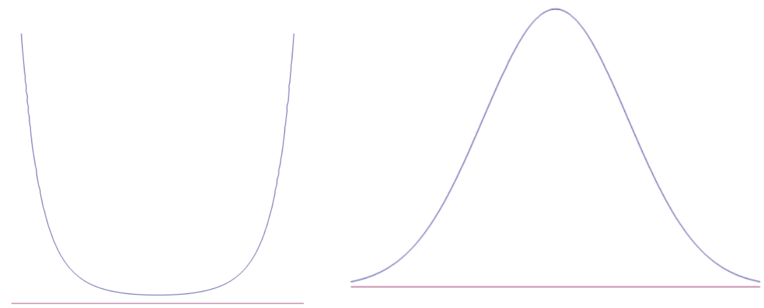
and

$$e_{12}=0,\qquad e_2=0. \tag{23}$$

First we solve subsystem (22). Solving $e_{11}=0$ we get that

$$x_1=\frac{4d(d_3-a)}{4a(a-d_3)+k^2}-x_2.$$

Substituting x_1 into $e_2 = 0$ we obtain that e_2 becomes identically zero, so there is a continuum of solutions, and consequently these solutions cannot produce limit cycles, only periodic solutions.



(a) $f(x_2)$ has a minimum (b) $f(x_2)$ has a maximum

FIGURE 2. The graphic of the function $f(x_2)$. The horitzontal straigh line is the x_2-axis.

Now we shall study the solutions of subsystem (23). First we write the equation $e_2 = 0$ as

$$E_2 = \left(k_3 + k_4 x_1 + x_1^2(k_5 + k_6)\right) - \left(k_3 + k_4 x_2 + x_2^2(k_5 + k_6)\right) e^{\dfrac{d_3\left(k_1(x_2 - x_1) + k_2(x_2^2 - x_1^2)\right)}{2d^2k^2(a - d_3)}} = 0,$$

Solving $e_{12} = 0$ we obtain the two solutions

$$x_1^{\pm} = \frac{-2ad(a - d_3)\left(4a(a - d_3) + k^2\right) \pm \sqrt{U}}{a\left(4a(a - d_3) + k^2\right)^2}, \tag{24}$$

where

$$\begin{aligned} U = \; & -a\left(4a(a - d_3) + k^2\right)^2 \left(4d(a - d_3)\left(k^2(d - 2b) - a(a - d_3)(8b + d)\right)\right. \\ & \left. + 4adx_2(a - d_3)\left(4a(a - d_3) + k^2\right) + ax_2^2\left(4a(a - d_3) + k^2\right)^2\right). \end{aligned}$$

Remark 6. *It is easy to check that* $H_1(x_1^+, (x_1^+)^2) = H_1(x_1^-, (x_1^-)^2)$, *therefore the ellipses* $H_1(x, y) = H_1(x_1^+, (x_1^+)^2)$ *and* $H_1(x, y) = H_1(x_1^-, (x_1^-)^2)$ *are the same.*

Substituting every one of the two solutions $x_1^{\pm}$ in $E_2 = 0$ we obtain for both that

$$E_2 = -\left(ad_3k_2x_2^2 + k_3 + k_4x_2\right) e^{\dfrac{ad_3k_2x_2^2 + k_4x_2 + k_7}{ad^2k^2(a - d_3)}} + 4ad^2k^2(d_3 - a) - k_3 - k_4x_2 - ad_3k_2x_2^2 = 0,$$

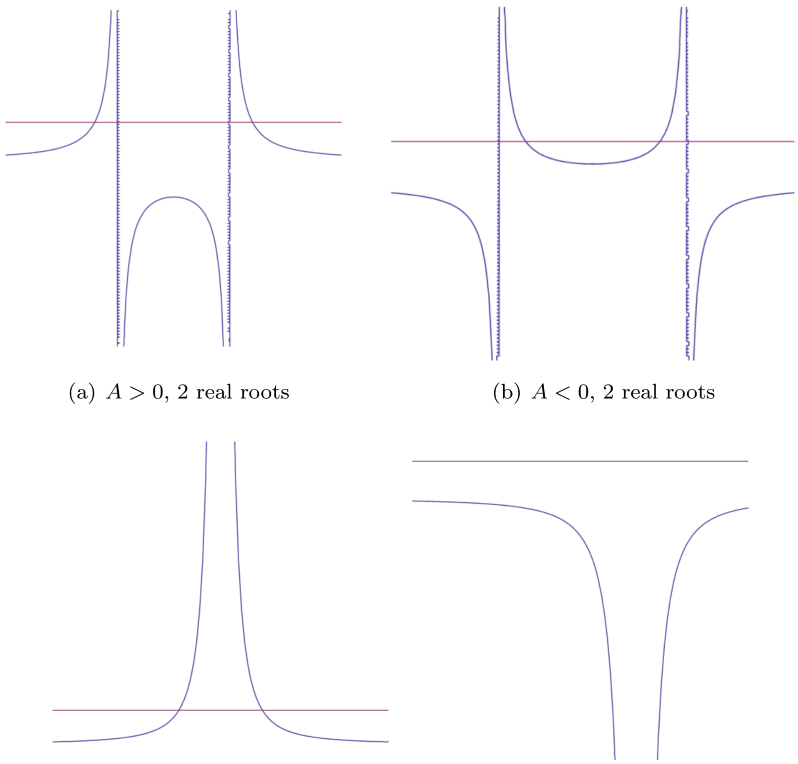
where $k_7 = 2d(a - d_3)d_3(-8a^2b + 8abd_3 + (-2b + d)k^2)$. To study the solutions of x_2 satisfying $E_2 = 0$ is equivalent to study the solutions of x_2 of the equation

$$e^{\dfrac{ad_3k_2x_2^2 + k_4x_2 + k_7}{ad^2k^2(a - d_3)}} = \frac{4ad^2k^2(d_3 - a)}{ad_3k_2x_2^2 + k_4x_2 + k_3} - 1. \tag{25}$$

Since $ad_3k_2 \neq 0$ the function $f(x_2) = e^{\dfrac{ad_3k_2x_2^2 + k_4x_2 + k_7}{ad^2k^2(a - d_3)}}$ is positive and has a unique extremum which can be a minimun or a maximum, and then its possible graphics $(x_2, f(x_2))$ is shown in Figure 2. The horitzontal straight line which appears in Figure 2 is the x_2-axis.

We define $A = ad_3k_2$ and the function $g(x_2) = \dfrac{4ad^2k^2(d_3 - a)}{ad_3k_2x_2^2 + k_4x_2 + k_3} - 1$. Then in Figure 3 we show the graphic of the function $g(x_2)$ according if A is positive or negative, and according with the different kind of roots of the quadratic polynomial $ad_3k_2x_2^2 + k_4x_2 + k_3$ in the variable x_2.

Clearly that the graphics of Figure 2 with the graphics of Figure 3 can intersect in at most two points. Hence equaton (25) can have at most two solutions for x_2, denote them by x_{21} and x_{22}. From (24) we obtain



(a) $A > 0$, 2 real roots

(b) $A < 0$, 2 real roots

(c) $A > 0$, 1 doble or 2 complex roots

(d) $A < 0$, 1 doble or 2 complex roots

FIGURE 3. The graphic of the function $g(x_2)$. The horitzontal straigh line is the x_2-axis. In the case $A < 0$, 2 real roots eventually the minimum can be positive.

for each one of these values of x_2 produce two values of x_1, denoted by x_{11} and x_{12} the ones produced by x_{21}, and by x_{13} and x_{14} the ones produced by x_{22}. So if a planar continuous piecewise differential system separated by the parabola $y = x^2$ formed by the linear center (18) and the quadratic center (17) has a limit cycle this must intersect the parabola in the two points given for one of the following four pairs of points:

$$(x_{11}, x_{11}^2), (x_{21}, x_{21}^2); \qquad (x_{12}, x_{12}^2), (x_{21}, x_{21}^2); \qquad (x_{13}, x_{13}^2), (x_{22}, x_{22}^2); \qquad (x_{14}, x_{14}^2), (x_{22}, x_{22}^2).$$

Since the points (x_{21}, x_{21}^2) and (x_{22}, x_{22}^2) only can appear in a unique limit cycle, at most two pairs of these four pairs can produce two limit cycles, assume that they are the pairs

$$(x_{11}, x_{11}^2), (x_{21}, x_{21}^2); \qquad (x_{13}, x_{13}^2), (x_{22}, x_{22}^2).$$

This agrees with the remark 6.

In short, at most we have the two solutions (x_{11}, x_{21}) and (x_{13}, x_{22}) of system (21) which can provide limit cycles of our continuous piecewise differential system formed by a linear center (18) and a quadratic center (17). But from remark 5 if (x_{11}, x_{21}) is a solution also (x_{21}, x_{11}) is another solution, hence $(x_{21}, x_{11}) = (x_{13}, x_{22})$. Consequently at most we have one limit cycle because the solutions (x_{11}, x_{21}) and (x_{21}, x_{11}) define the same limit cycle.

In order to complete the proof of the theorem we shall prove that the planar continuous piecewise differential system separated by the parabola $y = x^2$ and formed by the linear center

$$\dot{x} = -y, \qquad \dot{y} = \frac{257}{5324} + x, \tag{26}$$

and the quadrtic center

$$\dot{x} = -1 + 2x + 11y + 11x^2, \qquad \dot{y} = \frac{13}{242} - x - y - x^2, \tag{27}$$

has one limit cycle. Indeed, it is easy to see that equation (25), for the piecewise differential system formed by systems (26) and (27) separated by the parabola $y = x^2$, has for the variable x_2 two solutions $x_{21} = \dfrac{-176 - 176\sqrt{14}}{3872}$ and $x_{22} = \dfrac{-176 + 176\sqrt{14}}{3872}$. Then we obtain the limit cycle of Figure 1 which intersects the parabola $y = x^2$ in the two points (x_{21}, x_{21}^2) and (x_{22}, x_{22}^2) according with the previous paragraph. □

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