Periodic orbits of periodic differential systems

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Abstract. We study the periodic solutions of a class of periodic differential systems on a compact boundaryless C^1 differentiable manifold. We essentially consider the following two questions: (1) How to know if our periodic differential systems have periodic solutions of a given period? (2) If our periodic differential systems have infinitely many periodic solutions, what can be said on their periods?

1. Introduction.

Consider a differential system

$$\frac{dx}{dt} = X(t,x) \tag{1}$$

on the compact boundaryless C^1 differentiable manifold M. We always assume that

(i) *$X : R \times M \to M$ is a C^1 map,*

(ii) *$X(t+1,x) = X(t,x)$ for all $(t,x) \in R \times M$* (i.e., the differential system (1) is 1-periodic in t).

For a given $(t_0, x_0) \in R \times M$ we denote by $\varphi(t) = \varphi(t; t_0, x_0)$ the solution of the differential system (1) satisfying the initial condition $\varphi(t_0) = x_0$. Notice that since M is closed the solution $\varphi(t)$ is defined for all $t \in R$. We say that $\varphi(t)$ is an *n-periodic solution* or a *periodic solution of period n* if $\varphi(t_0+n) = x_0$ and $\varphi(t) \neq x_0$ for any $t \in (t_0, t_0+n)$.

We also assume that

(iii) *if $\varphi(t) = \varphi(t; t_0, x_0)$ is an n-periodic solution of the differential system* (1), *then the eigenvalues of $d\varphi(t)(x_0)|_{t=t_0+n}$ are no roots of unity.*

Notice that assumption (iii) on the periodic solutions is weaker than hyperbolicity assumption. Recall that an n-periodic solution $\varphi(t) = \varphi(t; t_0, x_0)$ of (1) is *hyperbolic* if the eigenvalues of $d\varphi(t)(x_0)|_{t=t_0+n}$ have modulus different from 1.

Our purpose is to study the periodic solutions of the periodic differential system (1) under assumptions (i)-(iii).

The map $T : M \to M$ defined by $T(x) = \varphi(1; 0, x)$ is usually called the *1-time map* or the *Poincaré map* of the 1-periodic differential system (1). A point $x \in M$ is *n-periodic* or *periodic of period n* if $T^n(x) = x$ and $T^k(x) \neq x$ for $k = 1, \dots, n-1$.

As usual we denote by N the set of natural numbers, for us zero is *not* a natural number.