PERIODS FOR KLEIN BOTTLE MAPS

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In dynamical systems, it is often the case that topological information can be used to study qualitative and quantitative properties (like the set of periods) of the system. This note deals with the problem of determining the set of periods (of the periodic points) of a continuous self-map of the Klein bottle. Our interest in this problem comes from the fact that the unique manifolds in dimensions 1 and 2, with zero Euler characteristic are the circle, torus and Klein bottle, and for the two first the structure of the set of periods of their continuous self-maps has been determined.

To fix terminology, suppose f is a continuous self-map on the manifold M. A *fixed point* of f is a point x in M such that $f(x) = x$. We shall call x a *periodic point of period n* if x is a fixed point of f^n but is not fixed by any f^k, for $1 \leq k < n$. We denote by $\text{Per}(f)$ the set of natural numbers corresponding to periods of periodic orbits of f.

Even for continuous self-maps f on the circle the relation between the degree of f and the set $\text{Per}(f)$ is interesting and nontrivial (see [5,3], and for more details [2]). Let $\mathbb{N}$ denote the set of natural numbers. Suppose f has degree d.

(1) For $d \notin \{-2, -1, 0, 1\}$, $\text{Per}(f) = \mathbb{N}$.
(2) For $d = -2$, $\text{Per}(f)$ is either $\mathbb{N}$ or $\mathbb{N} \setminus \{2\}$.
(3) For $d = -1, 0$, $\text{Per}(f) \supset \{1\}$.
(4) For $d = 1$, the set $\text{Per}(f)$ can be empty.

Recently, in [1] these results have been extended to continuous self-maps on the 2-dimensional torus, and many of them to the m-dimensional torus with $m > 2$.

The goal of this note is to provide a similar description of the set of periods for continuous self-maps on the Klein bottle, or simply *Klein bottle maps*.

Let K be the Klein bottle and lef f be a Klein bottle map. If p is a base point for K, then f is homotopic to a continuous map h, $f \sim h$, such that $h(p) = p$ and a presentation of the group $\Pi(K, p)$ is given by two generators a and b and the relation $abab^{-1}$. Halpern in [6] shows that $h_p(a) = a^u$ and $h_p(b) = a^w b^v$ for some integers u, v and w, where h_p is the induced endomorphism on $\Pi(K, p)$. In what follows u and v will be called the *integers associated to f*.

Now our main result can be stated as follows. Here, $2\mathbb{N}$ denotes the set of all even natural numbers.

THEOREM. *Let f be a Klein bottle map and let u and v its integers associated.*

(1) *Suppose* $|u| > 1$. *Then*

$$\begin{aligned} \text{Per}(f) &= \mathbb{N} && \text{if } |v| \neq 1, \text{ and} \\ \text{Per}(f) &\supset \mathbb{N} \setminus 2\mathbb{N} && \text{if } v = -1. \end{aligned}$$

Furthermore, the last result cannot be improved without additional hypotheses on f.