PERIODIC POINTS OF ONE DIMENSIONAL MAPS

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Abstract. The goal of this paper is to give information on the periodic points of a continuous self-map of a connected finite graph by using its induced homology endomorphisms.

1. Introduction and statement of the results

A *connected finite graph* G is a connected topological space formed by a finite set of points $v_1, \dots, v_N$ (the *vertices*) and a finite set of open arcs $e_1, \dots, e_M$ (the *edges*), in such a way that each open arc is attached by its endpoints to vertices (see Figure 1). That is a connected finite graph is a connected finite complex of dimension 1. A graph may be embedded into the 3-dimensional Euclidean space $\mathbb{R}^3$ where any set of points can be joined in pairs by non-intersecting arcs. We consider in the finite graph the relative topology given by the standard topology of $\mathbb{R}^3$, which coincides with the weak topology of the graph because it is finite. For more details on topological graphs see [S].

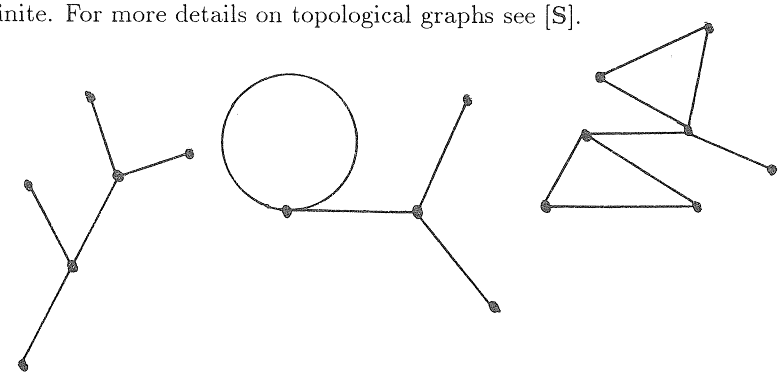

Figure 1. Three connected finite graphs.

If G is a connected finite graph the integral homology groups of G are $H_0(G) = H_0(G;\mathbb{Z}) \approx \mathbb{Z}$ and $H_1(G) = H_1(G;\mathbb{Z}) \approx \mathbb{Z} \oplus \overset{1-N+M}{\cdots} \oplus \mathbb{Z}$, i.e. $H_1(G)$ is free abelian of rank $1 - N + M$.

The following result about fixed points for a continuous self-map of a connected finite graph G follows immediately from the Lefschetz fixed point theorem, see [B] or Section 2.

THEOREM 1. *Let G be a connected finite graph and $f : G \to G$ a continuous map. Assume that either $H_1(G) = 0$, or $H_1(G) \neq 0$ and trace$(f_{*1}) \neq 1$ where $f_{*1} : H_1(G) \to H_1(G)$ is the induced endomorphism by f on the first homology group. Then f has a fixed point.*

There are well-known corollaries of Theorem 1. Thus, let f be a continuous self-map of a connected finite graph G. Then, since $H_1(G) = 0$ if and only if G is contractible (i.e. G is a *tree*), from Theorem 1 it follows that every continuous self-map of a tree has a fixed point. Also, if G is the circle and the degree of f is different from 1, then $H_1(G) \neq 0$ and trace$(f_{*1}) \neq 1$. Hence, from Theorem 1 every continuous self-map of the circle of degree different from 1 has a fixed point.

In this paper we study the periodic points for a continuous self-map f of a connected finite graph G when $H_1(G) \neq 0$ and trace$(f_{*1}) = 1$. To state our results we need some notation.

Let A be an $n \times n$ complex matrix. A $k \times k$ *principal submatrix* of A is a submatrix lying in the same set of k rows and columns, and a $k \times k$ *principal minor* is the determinant of such a principal submatrix. There are $\binom{n}{k}$ different $k \times k$ principal minors of A, and the sum of these is denoted by $E_k(A)$. In particular, $E_1(A)$ is the trace of A and $E_n(A)$ is the determinat of A, denoted by $\det(A)$.

If $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A, then it is well-known that the characteristic polynomial of A satisfies

$$\begin{aligned}\det(tI - A) &= t^n - E_1(A)t^{n-1} + E_2(A)t^{n-2} - \cdots + (-1)^n E_n(A) \\ &= t^n - S_1(\lambda_1, \dots, \lambda_n)t^{n-1} + S_2(\lambda_1, \dots, \lambda_n)t^{n-2} - \cdots \\ &\quad + (-1)^n S_n(\lambda_1, \dots, \lambda_n),\end{aligned}$$

where $S_k(\lambda_1, \dots, \lambda_n)$ is the kth *elementary symmetric function* of the n numbers $\lambda_1, \dots, \lambda_n$ for $k \leq n$, defined as follows:

$$S_k(\lambda_1, \dots, \lambda_n) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \prod_{j=1}^{k} \lambda_{i_j},$$

i.e. the sum of all $\binom{n}{k}$ k-fold products of distinct items from $\lambda_1, \dots, \lambda_n$. Thus $S_1(\lambda_1, \dots, \lambda_n) = \lambda_1 + \cdots + \lambda_n$ and $S_n(\lambda_1, \dots, \lambda_n) = \lambda_1 \cdot \ldots \cdot \lambda_n$.

Our main result is the following.

THEOREM 2. *Let G be a connected finite graph and $f : G \to G$ a continuous map. Suppose that $H_1(G) \approx \mathbb{Z} \oplus \overset{n}{\cdots} \oplus \mathbb{Z}$ with $n > 1$ and that A is the $n \times n$ integral matrix of the endomorphism $f_{*1} : H_1(G) \to H_1(G)$ induced by f. If $E_1(A) = 1$ and k is the smallest integer of the set*