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Sufficient conditions for a continuous map of R^n into itself to have m-periodic points for all m>0

INTRODUCTION AND RESULTS

In the last few years there has been considerable interest in the study of continuous maps of $\mathbb{R}^n$ into itself from a dynamic point of view. That is, let $T:\mathbb{R}^n \to \mathbb{R}^n$ be a continuous map and let p be an initial point; we are interested in the sequence $p, Tp, T^2p, \ldots$ which reflects some properties of the map T. Periodic points are of particular interest. We shall say p is an m-periodic point if $p = T^m p$ and $p \neq T^k p$ for $1 \leq k \leq m-1$. We say p is a periodic point of p is an m-periodic point for some $m > 0$. We are interested in the answers to the following two questions: (i) when does a continuous map T of R^n into itself have m-periodic points for all $m > 0$? (ii) when does T have the shift on two elements as a subsystem?

Let $T:\mathbb{R}^n \to \mathbb{R}^n$ be a continuous map with $n = 1$. Then the three following statements are equivalent, and give a complete and simple answer to question (i) in dimension one.

(a) The map T has m-periodic points for all $m > 0$.

(b) The map T has a 3-periodic point.

(c) There exist two closed intervals L and R such that $R \subset TL$, $R \cup L \subset TR$ and $T^2(R \cap L) \cap R = \emptyset$.

Sarkovskii [4] proves (a) $\Longleftrightarrow$ (b), and Li and Yorke [1] that (b) $\Longleftrightarrow$ (c).

The following result is a generalization of the above theorem of Li and Yorke to dimensions greater than one.

Theorem 1

Let $T:\mathbb{R}^n \to \mathbb{R}^n$ *be a continuous map with* $n > 1$. *Assume that there exist two subsets* L *and* R *of* $\mathbb{R}^n$ *homeomorphic to the closed unit ball of* $\mathbb{R}^n$ *such that:* (1) $R \subset TL$, (2) $L \cup R \subset TR$, (3) $T^2(L \cap R) \cap R = \emptyset$ *and* (4) *the map* T *restricted to* R (resp. L) *is a homeomorphism between* R (resp. L) *and* TR (resp. TL). *Then for every* $m = 1,2,\ldots$ *there exists an m-periodic point in* R.

It is not difficult to construct examples which prove that if the map