LIMIT CYCLES IN PIECEWISE POLYNOMIAL SYSTEMS ALLOWING A NON-REGULAR SWITCHING BOUNDARY

TAO LI[1] AND JAUME LLIBRE[2]

Abstract. Continuing the investigation for the piecewise polynomial perturbations of the linear center $\dot{x} = -y, \dot{y} = x$ from [Physica D **371**(2018), 28-47] for the case where the switching boundary is a straight line, in this paper we allow that the switching boundary is non-regular, i.e. we consider a switching boundary which separates the plane into two angular sectors with angles $\alpha \in (0, \pi]$ and $2\pi - \alpha$. Moreover, unlike the aforementioned work, we allow that the polynomial differential systems in the two sectors have different degrees. Depending on α and for arbitrary given degrees we provide an upper bound for the maximum number of limit cycles that bifurcate from the periodic annulus of the linear center using the averaging method up to order N. The reachability of the upper bound is also reached for the first two orders. On the other hand, we pay attention to the perturbation of the linear center inside this class of all piecewise polynomial Liénard systems and give some better upper bounds in comparison with the one obtained in the general piecewise polynomial perturbations. Again our results imply that the non-regular switching boundary (i.e. when $\alpha \neq \pi$) the piecewise polynomial perturbations usually leads to more limit cycles than the regular case (i.e. when $\alpha = \pi$) where the switching boundary is a straight line.

1. Introduction and statement of main results

In the qualitative theory of smooth differential systems, a classical and challenging objective is to determine the maximum number of the limit cycles bifurcating from the periodic annulus of the linear center $\dot{x} = -y, \dot{y} = -x$, when it is perturbed inside the family formed by all planar polynomial differential systems of the form

$$(\dot{x}, \dot{y}) = \left(-y + \sum_{i=1}^{N} \varepsilon^i f_i(x, y), x + \sum_{i=1}^{N} \varepsilon^i g_i(x, y) \right), \tag{1}$$

where $|\varepsilon| > 0$ sufficiently small, and f_i and g_i are real polynomials of degree n. This is essentially the weak Hilbert's 16th problem, see [1, 14, 18]. It was proved in [14] that system (1) has at most $[N(n-1)/2]$ limit cycles bifurcating from the periodic annulus for $|\varepsilon| > 0$ sufficiently small, where as usual $[\cdot]$ denotes the integer part function. Since this upper bound obtained in [14] is not reached in general, up to now we still do not know what is the exact maximum number of limit cycles under the general polynomial perturbation (1) except some special families of perturbations, such as the Liénard family, i.e. $g_i(x, y) = 0$ and $f_i(x, y)$ is independent of the variable y, for which it was proved in [12] that at most $[(n-1)/2]$ limit cycles bifurcate and this number is reached due to [20].

2010 *Mathematics Subject Classification.* 34C29, 34C25, 34C05.
Key words and phrases. Averaging method, Hilbert's 16th problem, limit cycles, discontinuous piecewise polynomial systems, Liénard systems.

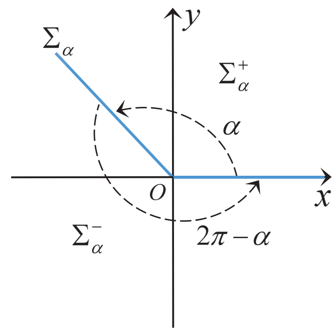

Figure 1. Switching boundary Σ_α and angular sectors $\Sigma_\alpha^\pm$.

As the discontinuity turns out to be ubiquitous in the real world (see for instance the papers in mechanical engineering [8,17], in neural sciences [11,32] and in electronic circuits [3, 26], ...), discontinuous piecewise smooth differential systems have attracted many researchers in recent years. Let Σ_α be the union of the non-negative x-axis and the ray starting at O and forming with the non-negative x-axis an angle $\alpha \in (0,\pi]$, Σ_α^+ and Σ_α^- be two angular sectors separated by Σ_α with angles α and $2\pi-\alpha$ respectively, see Figure 1. It is worth mentioning that Σ_α is just the x-axis if $\alpha=\pi$. In this paper we consider the discontinuous piecewise polynomial differential systems of the form

$$(\dot{x},\dot{y}) = \left(-y+\sum_{i=1}^{N}\varepsilon^i f_i^\pm(x,y),\, x+\sum_{i=1}^{N}\varepsilon^i g_i^\pm(x,y)\right) \qquad \text{if} \quad (x,y)\in\Sigma_\alpha^\pm, \tag{2}$$

where f_i^+ and g_i^+ (respectively f_i^- and g_i^-) are real polynomials of degree n (respectively m). In this case we say that the *degree* of system (2) is (n,m). Throughout this paper we will restrict our attention to the case of $n \ge m \ge 1$.

Following the Filippov convention [10] we know that all points in $\Sigma_\alpha \setminus \{O\}$ where the vector fields of two subsystems simultaneously point outward or inward $\Sigma_\alpha \setminus \{O\}$ form the *sliding regions*, and the complement of sliding regions in $\Sigma_\alpha \setminus \{O\}$, excluding the tangency points of the vector fields and Σ_α, defines the *crossing regions*. A periodic orbit is called *crossing periodic orbit* if it intersects Σ_α only at the crossing regions. Additionally, we call it *crossing limit cycle* if this periodic orbit is isolated. Analogous to the polynomial differential system (1), it is natural to ask *what is the maximum number, denoted by $\mathcal{M}_{\alpha,N}(n,m)$, of the crossing limit cycles of system* (2) *bifurcating from the periodic annulus of the linear center $\dot{x}=-y, \dot{y}=x$ for given α, N and degree (n,m).*

Some contributions have been made for the study of $\mathcal{M}_{\alpha,N}(n,m)$ with $n=m$. In the case of $\alpha=\pi$ where the switching boundary Σ_π is a straight line, i.e. the x-axis, Buzzi, Pessoa and Torregrosa [6] considered the perturbations inside the class of all piecewise linear differential systems and they proved $\mathcal{M}_{\pi,N}(1,1)=1,1,2,3,3,3,3$ up to order $N=1,2,3,4,5,6,7$ respectively. The piecewise quadratic and cubic perturbations were studied by Llibre and Tang in [24] where it was proved that $\mathcal{M}_{\pi,N}(2,2)=2,3,5,6,8$ and $\mathcal{M}_{\pi,N}(3,3)=3,5,8,11,13$ up to order $N=1,2,3,4,5$ respectively. For the general piecewise polynomial perturbations of degree (n,n) separated by a straight line, Buzzi, Lima and Torregrosa [5] proved that $\mathcal{M}_{\pi,1}(n,n)=n$, $\mathcal{M}_{\pi,2}(n,n)=2n-1$ and $\mathcal{M}_{\pi,N}(n,n)\le Nn-1$ for $N\ge3$. When $\alpha\in(0,\pi)$ the switching boundary Σ_α is non-regular, in this case Cardin and Torregrosa [7] studied the piecewise linear perturbations providing that $\mathcal{M}_{\alpha,N}(1,1) = 1,2,2,3,4$ up to order $N = 1,2,3,4,5$ and $\mathcal{M}_{\alpha,6}(1,1) \leq 5$.

Regarding $\mathcal{M}_{\alpha,N}(n,m)$ with $n > m$, there exist few works to be done. If $\mathcal{M}_{\alpha,N}(n,n)$ is known, it provides an upper bound of $\mathcal{M}_{\alpha,N}(n,m)$ with $n > m$, but how to know if this upper bound is exact? Therefore it is worth for us to study $\mathcal{M}_{\alpha,N}(n,m)$ allowing $n > m$, although some information could be provided by studying $\mathcal{M}_{\alpha,N}(n,n)$. On the other hand, to our knowledge an upper bound of $\mathcal{M}_{\alpha,N}(n,m)$ for $\alpha \in (0,\pi)$ is still not given, even in the case of $m = n$. The case of $\alpha = \pi$ and $n = m$ has been studied in [5] as recalled in the last paragraph. Stimulated by these two aspects, we provide some upper bounds for $\mathcal{M}_{\alpha,N}(n,m)$ with $\alpha \in (0,\pi]$ and $n \geq m$ using the averaging theory up to order N as they are stated in the following result.

Theorem 1. *For system* (2) *satisfying* $\alpha \in (0,\pi]$ *and* $n \geq m \geq 1$, *the following statements can be obtained using the averaging theory up to order* N.

(i) *If* $\alpha = \pi$, *then* $\mathcal{M}_{\alpha,1}(n,m) = n$, $\mathcal{M}_{\alpha,2}(n,m) = 2n-1$ *and* $\mathcal{M}_{\alpha,N}(n,m) \leq Nn-1$ *for* $N \geq 3$.
(ii) *If* $\alpha \in (0,\pi)$, *then* $\mathcal{M}_{\alpha,1}(n,m) = n$, $\mathcal{M}_{\alpha,2}(n,m) \leq 2n$ *and* $\mathcal{M}_{\alpha,N}(n,m) \leq Nn$ *for* $N \geq 3$. *Moreover the upper bound* $\mathcal{M}_{\alpha,2}(n,m)$ *is reached for* $\alpha \in (0,\pi/2]$.

The result in statement (i) of Theorem 1 was also obtained in [5, Theorem 1.1] in the particular case $m = n$, so the result of this statement generalizes Theorem 1.1 of [5].

From Theorem 1 we see that the upper bound of $\mathcal{M}_{\alpha,N}(n,m)$ for $\alpha \in (0,\pi)$ is the upper bound for $\alpha = \pi$ plus one, which emphasizes the importance of the shape of the switching boundary in the study of the crossing limit cycles of the piecewise differential systems. Besides we observe that the upper bound of $\mathcal{M}_{\alpha,N}(n,m)$ is usually determined by the subsystem with the higher degree, because all numbers obtained in Theorem 1 are independent of m.

Among discontinuous differential systems, one of the most studied classes is the one formed by the discontinuous piecewise Liénard systems, which is widely used to model or analyze many real problems, as for instance the mechanical engineering with dry frictions [8], the integrate-and-fire neurons [32], the discontinuous control in the buck electronic converter [3, 11], ... Of course, the study of the limit cycles for the discontinuous piecewise Liénard systems is also of fundamental importance and many researchers are devoted to the study of this subject, see the papers [5,7,19,23,27,30,31] for one switching boundary and [9,16,25,28] for multiple ones.

In this paper we also study the maximum number of crossing limit cycles bifurcating from the periodic annulus of the linear center $\dot{x} = -y, \dot{y} = x$ when the perturbations are restricted to the class of all piecewise polynomial Liénard systems of the form

$$(\dot{x},\dot{y}) = \left(-y + \sum_{i=1}^{N} \varepsilon^i f_i^{\pm}(x),\ x\right) \qquad \text{if} \quad (x,y) \in \Sigma_\alpha^{\pm}, \tag{3}$$

where f_i^+ (respectively f_i^-) are real polynomials of degree n (respectively m) with $n \geq m \geq 1$. We denote the maximum number of the crossing limit cycles of system (3) bifurcating from the periodic annulus by $\mathcal{L}_{\alpha,N}(n,m)$.

When $\alpha = \pi$, i.e. the switching boundary Σ_α is a straight line, Buzzi, Lima and Torregrosa [5] proved that $\mathcal{L}_{\pi,1}(n,n) = [(n-1)/2]$ and $\mathcal{L}_{\pi,2}(n,n) = n + [(n-1)/2]$ for any given n, and

$\mathcal{L}_{\pi,N}(n,n) = n + [(n-1)/2]$ for $n = 1,2,3,4$ up to order $N = 2,3,4,5,6$. In particular, if $f_i^{\pm}$ are even polynomials, they further proved that the origin O is a center for every ε, while if $f_i^{\pm}$ are odd polynomials, $\mathcal{L}_{\pi,N}(n,n) = (n-1)/2$ up to any order N. When $\alpha \in (0,\pi)$, i.e. the switching boundary Σ_α is non-regular, the result on the maximum number of crossing limit cycles of system (3) bifurcating from the periodic annulus is much fewer. As far as we know, only the case where $f_i^{\pm}$ are linear was considered by Cardin and Torregrosa in [7]. In that paper they proved that $\mathcal{L}_{\alpha,N}(1,1) = 1,2,2,2,2,2$ up to a study of order $N = 1,2,3,4,5,6$.

Allowing $\alpha \in (0,\pi]$ and $n \geq m \geq 1$ again, we give some upper bounds for $\mathcal{L}_{\alpha,N}(n,m)$ using the averaging theory up to order $N = 1,2$ in the next result.

Theorem 2. *For system* (3) *satisfying $\alpha \in (0,\pi]$ and $n \geq m \geq 1$, the following statements can be obtained using the averaging theory up to order $N = 1,2$.*

- (i) *If $\alpha = \pi$, then $\mathcal{L}_{\alpha,1}(n,m) = [(n-1)/2]$ and $\mathcal{L}_{\alpha,2}(n,m) = n + [(m-1)/2]$.*
- (ii) *If $\alpha \in (0,\pi)$, then $\mathcal{L}_{\alpha,1}(n,m) = n$ and $\mathcal{L}_{\alpha,2}(n,m) \leq \max\{2m-1, n\}$ (respectively $\leq \max\{2m-2, n\}$) for m odd (respectively even).*

The result in statement (i) of Theorem 2 is just Theorem 1.2 of [5] for $m = n$, while for $m < n$ we provide a better upper bound up to order 2. Again we see that the non-regular switching boundary $\alpha \in (0,\pi)$ increases the number of limit cycles in comparison with the regular one $\alpha = \pi$ as it was observed in some publications, as for instance in [7].

The paper is organized as follows. In section 2 we shortly review some main tools used in this paper, including the averaging method, Descartes Theorem and some technical results on integrals. It is worth mentioning that another method studying the limit cycle bifurcations from a periodic annulus is the Poincaré-Pontryagin-Melnikov method and it turns out to be equivalent to the averaging one for planar autonomous systems, see [4,13]. In sections 3 and 4 we are devoted to the proof of Theorems 1 and 2, respectively. Some comments and future directions are summarized in Section 5.

2. Preliminaries

The averaging theory is an important tool for studying the number of limit cycles of differential systems. The classical averaging theory [29] usually requires that the considered system is smooth. However this theory, with the efforts of many researchers, has been generalized for piecewise smooth differential systems in recent years, see [15, 21, 22, 33]. Following the work [15] in this section we introduce the averaging theory that we shall use in order to obtain information on the existence and number of crossing limit cycles for the discontinuous piecewise polynomial system (2).

To apply the averaging theory we need to write system (2) in a convenient normal form. As usual this normal form can be obtained using the polar coordinates (r,θ), where $x = r\cos\theta$ and $y = r\sin\theta$. In this case system (2) becomes

$$\begin{aligned} \frac{dr}{dt} &= \sum_{i=1}^{N} \varepsilon^i \left(\cos\theta f_i^{\pm}(r\cos\theta, r\sin\theta) + \sin\theta g_i^{\pm}(r\cos\theta, r\sin\theta)\right), \\ \frac{d\theta}{dt} &= 1 - \frac{1}{r}\sum_{i=1}^{N} \varepsilon^i \left(\sin\theta f_i^{\pm}(r\cos\theta, r\sin\theta) - \cos\theta g_i^{\pm}(r\cos\theta, r\sin\theta)\right), \end{aligned} \tag{4}$$

for $\theta \in \mathcal{I}_\alpha^\pm$, where $\mathcal{I}_\alpha^+ = [0, \alpha]$ and $\mathcal{I}_\alpha^- = [\alpha, 2\pi]$. Then taking θ as the new independent variable system (4) writes

$$\frac{dr}{d\theta} = \sum_{i=1}^{N} \varepsilon^i F_i^\pm(\theta, r) + \mathcal{O}(\varepsilon^{N+1}) \qquad \text{if} \quad \theta \in \mathcal{I}_\alpha^\pm, \tag{5}$$

where the expression of $F_i^\pm(\theta, r)$ will be determined later on.

From [15] the *averaged function* $\mathcal{F}_i(r) : (0, +\infty) \to \mathbb{R}$ *of order* i is

$$\mathcal{F}_i(r) = \frac{y_i^+(\alpha, r) - y_i^-(\alpha - 2\pi, r)}{i!}, \tag{6}$$

and the functions $y_i^\pm : [0, 2\pi] \times (0, +\infty) \to \mathbb{R}$ for $i = 1, 2, ..., N$ are defined recurrently by

$$\begin{aligned} y_1^\pm(\theta, r) &= \int_0^\theta F_1^\pm(\varphi, r) d\varphi, \\ y_i^\pm(\theta, r) &= i! \int_0^\theta \left(F_i^\pm(\varphi, r) + \sum_{l=1}^{i} \sum_{S_l} \frac{1}{K} \partial^L F_{i-l}^\pm(\varphi, r) \prod_{j=1}^{l} y_j^\pm(\varphi, r)^{b_j} \right) d\varphi. \end{aligned} \tag{7}$$

Here ∂^L denotes the derivative of order L with respect to r, S_l is the set of all l-tuples of non-negative integers $(b_1, b_2, ..., b_l)$ satisfying $b_1 + 2b_2 + \cdots + lb_l = l$, $L = b_1 + b_2 + \cdots + b_l$ and $K = b_1! b_2! 2!^{b_2} \cdots b_l! l!^{b_l}$. Moreover we are assuming that $F_0^\pm = 0$ in (7) for convenience.

From [15] we get the following result on the averaging theory for the non-autonomous discontinuous piecewise smooth differential system (5).

Theorem 3. *Consider the non-autonomous discontinuous piecewise smooth differential system* (5)*. Suppose that* i_0 *is the first positive integer such that* $\mathcal{F}_i = 0$ *for* $1 \le i \le i_0 - 1$ *and* $\mathcal{F}_{i_0} \neq 0$. *If* $\mathcal{F}_{i_0}(\rho) = 0$ *and* $\mathcal{F}'_{i_0}(\rho) \neq 0$ *for some* $\rho \in (0, +\infty)$*, then for* $|\varepsilon| > 0$ *sufficiently small there exists a* 2π*-periodic solution* $r(\theta, \varepsilon)$ *of system* (5) *such that* $r(0, \varepsilon) \to \rho$ *as* $\varepsilon \to 0$.

Theorem 3 states that a simple positive zero of the first non-vanishing averaged function provides a crossing limit cycle of system (2) bifurcating from the periodic annulus of the linear center $\dot{x} = -y, \dot{y} = x$. In other words we can obtain some information on the existence and number on the crossing limit cycles of system (2) for $|\varepsilon| > 0$ sufficiently small via studying the number of simple positive zeros of the first non-vanishing averaged function.

In order to determine the maximum number of positive roots of a real polynomial in one variable we shall use the following theorem, namely the *Descartes Theorem*, a proof of it can be found in [2].

Theorem 4. *Consider the real polynomial* $p(x) = a_{i_1 1} x^{i_1} + a_{i_2} x^{i_2} + ... + a_{i_r} x^{i_r}$ *with* $0 = i_1 < i_2 < ... < i_r$ *and* $r > 1$. *If* $a_{i_j} a_{i_{j+1}} < 0$*, we say that* a_{i_j} *and* $a_{i_{j+1}}$ *have a variation of sign. If the number of variations of signs is* $r_0 \in \{0, 1, 2, ..., r-1\}$*, then the polynomial* $p(x)$ *has at most* r_0 *positive roots. Furthermore, we can choose the coefficients of the polynomial* $p(x)$ *in such a way that* $p(x)$ *has exactly* $r - 1$ *positive roots.*

Let

$$\begin{aligned} \mathcal{C}_k(\theta) &:= \int_0^\theta \cos^k \varphi d\varphi, & \mathcal{S}_k(\theta) &:= \int_0^\theta \sin^k \varphi d\varphi, \\ c_{k,l} &:= \int_0^\pi \cos^k \varphi \sin^l \varphi d\varphi, & \mathcal{C}_{k,l}(\theta) &:= \int_0^\theta \cos^k \varphi \int_0^\varphi \cos^l \phi d\phi d\varphi, \end{aligned}$$

for $k, l \geq 0$. We summarize some technical results on these integrals that we will need in the proofs of our results.

Lemma 5. *The following propositions hold.*

(i) $\mathcal{C}_k(\pi) = \mathcal{C}_k(2\pi)/2 = \frac{(k-1)!!}{k!!}\sigma_k$, *where* $\sigma_k = \pi$ *if* k *is even and* $\sigma_k = 0$ *if* k *is odd.*
(ii) $\mathcal{C}_k(\alpha) \neq 0$ *and* $\mathcal{C}_k(\alpha - 2\pi) = \mathcal{C}_k(\alpha) - \mathcal{C}_k(2\pi) \neq 0$ *for* $\alpha \in (0, \pi)$.
(iii) $\mathcal{S}_k(\pi) = \frac{(k-1)!!}{k!!}\varsigma_k$, *where* $\varsigma_k = \pi$ *if* k *is even and* $\varsigma_k = 2$ *if* k *is odd.*
(iv) $k\mathcal{S}_k(\theta) = -\cos\theta \sin^{k-1}\theta + (k-1)\mathcal{S}_{k-2}(\theta)$.
(v) $c_{k,k} = 0$ *if* k *is odd and* $c_{k,k} = 2^{-k}\mathcal{S}_k(\pi)$ *if* k *is even.*
(vi) $c_{k,l} = \frac{k-1}{l+1}c_{k-2,l+2} = \frac{k-1}{l+k}c_{k-2,l}$.
(vii) $\mathcal{C}_{k,l}(\pi) = \frac{1-(-1)^{k+l}}{l(k+l)} + \frac{l-1}{l}\mathcal{C}_{k,l-2}(\pi) = \frac{(-1)^{k+l}-1}{k(k+l)} + \frac{k-1}{k}\mathcal{C}_{k-2,l}(\pi)$ *and* $\mathcal{C}_{k,l}(\pi) = \frac{1-(-1)^{k+l}}{l(k+l)} + \frac{(l-1)((-1)^{k+l-2}-1)}{lk(k+l-2)} + \frac{(l-1)(k-1)}{lk}\mathcal{C}_{k-2,l-2}(\pi)$.
(viii) *If* l *is odd, then* $\mathcal{C}_{k,l}(\pi) = 0$ (*respectively* > 0) *for* k *odd* (*respectively even*) *and* $\mathcal{C}_{k,l}(2\pi) = 0$.
(ix) $\mathcal{C}_{k,l}(\alpha - 2\pi) = \mathcal{C}_{k,l}(\alpha)$ *if* l *is odd.*
(x) $\mathcal{C}_{k,l}(\theta) = \mathcal{C}_k(\theta)\mathcal{C}_l(\theta) - \mathcal{C}_{l,k}(\theta)$.

We neglect the proof of Lemma 5 because these propositions can be computed using some standard methods on integrals, such as the integration by parts method, and properties on trigonometric functions. We must mention that some of the results presented in Lemma 5 already appeared in [5].

3. Proof of Theorem 1

This section is devoted to the proof of Theorem 1. To this end we next give four propositions where the maximum number of simple positive zeros of the averaged functions associated to the considered systems is computed.

Proposition 6. *The averaged function of order* N *associated to the piecewise polynomial differential system* (2) *satisfying* $\alpha \in (0, \pi]$ *and* $n \geq m \geq 1$ *has at most*

(i) n (*respectively* $Nn - 1$) *simple positive zeros if* $\alpha = \pi$ *and* $N = 1$ (*respectively* $N \geq 2$),
(ii) Nn *simple positive zeros if* $\alpha \in (0, \pi)$ *and* $N \geq 1$.

Proof. Using the polar coordinates we transform system (2) into system (5) with

$$F_i^{\pm}(\theta, r) = p_i^{\pm}(\theta, r) + \sum_{l=0}^{i-1} p_l^{\pm}(\theta, r)\left(\frac{q_{i-l}^{\pm}(\theta, r)}{r} + \sum_{\substack{k_1+k_2=i-l \\ 1\leq k_1,k_2\leq i-l}} \frac{q_{k_1}^{\pm}(\theta, r)q_{k_2}^{\pm}(\theta, r)}{r^2} + \cdots \right.$$
$$\left. + \sum_{\substack{k_1+k_2+\cdots+k_{i-l}=i-l \\ 1\leq k_1,k_2,\ldots,k_{i-l}\leq i-l}} \frac{q_{k_1}^{\pm}(\theta, r)q_{k_2}^{\pm}(\theta, r)\cdots q_{k_{i-l}}^{\pm}(\theta, r)}{r^{i-l}}\right)$$

for $i = 1, 2, ..., N$, where $p_0^{\pm}(\theta, r) \equiv 0$ and

$$\begin{aligned} p_i^{\pm}(\theta, r) &= \cos\theta f_i^{\pm}(r\cos\theta, r\sin\theta) + \sin\theta g_i^{\pm}(r\cos\theta, r\sin\theta), \\ q_i^{\pm}(\theta, r) &= \sin\theta f_i^{\pm}(r\cos\theta, r\sin\theta) - \cos\theta g_i^{\pm}(r\cos\theta, r\sin\theta). \end{aligned} \tag{8}$$

Since $f_i^+(x,y)$ and $g_i^+(x,y)$ (respectively $f_i^-(x,y)$ and $g_i^-(x,y)$) are polynomials of degree n (respectively m), $p_i^+(\theta,r)$ and $q_i^+(\theta,r)$ (respectively $p_i^-(\theta,r)$ and $q_i^-(\theta,r)$) are also polynomials of degree n (respectively m) in the variable r with the coefficients that are polynomials in $\sin\theta$ and $\cos\theta$. Then

$$\widetilde{F}_i^{\pm}(\theta,r) := r^{i-2}F_i^{\pm}(\theta,r) - \frac{p_1^{\pm}(\theta)q_1^{\pm}(\theta)^{i-1}}{r} \tag{9}$$

are polynomials of degree $in-1$ and $im-1$ in r because $n,m \geq 1$, respectively. Here we are writing $p_1^{\pm}(\theta,0)$ and $q_1^{\pm}(\theta,0)$ as $p_1^{\pm}(\theta)$ and $q_1^{\pm}(\theta)$ to simplify the notation.

Let $h_i(q_1^{\pm}(\theta), q_1^{\pm}(0))$ be the function defined recurrently by

$$\begin{aligned} h_1(q_1^{\pm}(\theta), q_1^{\pm}(0)) =& q_1^{\pm}(\theta) - q_1^{\pm}(0), \\ h_i(q_1^{\pm}(\theta), q_1^{\pm}(0)) =& (i-1)!\left(q_1^{\pm}(\theta)^i - q_1^{\pm}(0)^i\right) \\ &+ i!\sum_{l=1}^{i-1}\sum_{S_l}\frac{M_{i-l}}{K}\int_0^{\theta} q_1^{\pm}(\varphi)^{i-l-1}\frac{dq_1^{\pm}(\varphi)}{d\varphi}\prod_{j=1}^{l} h_j(q_1^{\pm}(\varphi), q_1^{\pm}(0))^{b_j}d\varphi, \end{aligned} \tag{10}$$

for $i = 2,3,...,N$, where

$$M_{i-l} = \frac{(-1)^L(L+i-l-2)!}{(i-l-2)!},$$

and S_l, K, L, b_j are defined below (7).

We claim that

$$\tilde{y}_i^{\pm}(\theta,r) := r^{i-2}y_i^{\pm}(\theta,r) - \frac{h_i(q_1^{\pm}(\theta), q_1^{\pm}(0))}{r} \qquad i = 1,2,...,N, \tag{11}$$

are polynomials of degree $in-1$ and $im-1$ in r respectively, where $y_i^{\pm}(\theta,r)$ is defined in (7). This claim will be proved by induction and we only deal with $\tilde{y}_i^+(\theta,r)$, because a similar procedure can be applied to $\tilde{y}_i^-(\theta,r)$. To alleviate the notation, we drop the superscript $+$ in the proof of this claim. In fact, it follows from (7) and (9) that

$$\begin{aligned} y_1(\theta,r) &= \int_0^{\theta} F_1(\varphi,r)d\varphi = r\int_0^{\theta}\widetilde{F}_1(\varphi,r)d\varphi + \int_0^{\theta} p_1(\varphi)d\varphi \\ &= r\int_0^{\theta}\widetilde{F}_1(\varphi,r)d\varphi + q_1(\theta) - q_1(0). \end{aligned}$$

Thus $\tilde{y}_1(\theta,r) = \int_0^{\theta}\widetilde{F}_1(\varphi,r)d\varphi$ by the definitions of h_1 and $\tilde{y}_1$. Since $\widetilde{F}_1(\theta,r)$ is a polynomial of degree $n-1$ in r as it was stated below (9), so also $\tilde{y}_1(\theta,r)$ is a polynomial of the same degree, i.e. the claim holds for $i=1$.

To complete the proof of the claim, we next only need to prove it for $i = k$, provided that it holds for $i = 1,2,...,k-1$. According to (9), $\widetilde{F}_{k-l}(\theta,r)$ is a polynomial of degree $(k-l)n-1$ in r, and then a direct calculation yields

$$\partial^L F_{k-l}(\theta,r) = \frac{M_{k-l}p_1(\theta)q_1(\theta)^{k-l-1} + rP_{(k-l)n-1}(\theta,r)}{r^{L+k-l-1}}, \tag{12}$$

for $1 \leq k-l \leq k-1$. Here $P_{(k-l)n-1}(\theta,r)$ is a polynomial of degree $(k-l)n-1$ in r and its specific expression is neglected because it is not necessary in the next proof. Moreover we have

$$\begin{aligned}\prod_{j=1}^{l} y_j(\theta,r)^{b_j} &= \prod_{j=1}^{l}\left(\frac{h_j(q_1(\theta),q_1(0))+r\tilde{y}_j(\theta,r)}{r^{j-1}}\right)^{b_j}\\ &= \frac{1}{r^{l-L}}\prod_{j=1}^{l} h_j(q_1(\theta),q_1(0))^{b_j}+\frac{rP_{ln-1}(\theta,r)}{r^{l-L}},\end{aligned}\tag{13}$$

for $l \le k-1$, since we are assuming that the claim holds for $i = 1, 2, ..., k-1$ and $b_1+2b_2+\cdots+lb_l = l, b_1+b_2+\cdots+b_l = L$, where $P_{ln-1}(\theta,r)$ is a polynomial of degree $ln-1$ in r and we neglect the specific expression again. Joining (12) and (13) we obtain

$$\begin{aligned}&\int_0^\theta \sum_{l=1}^{k-1}\sum_{S_l}\frac{1}{K}\partial^L F_{k-l}(\varphi,r)\prod_{j=1}^{l} y_j(\varphi,r)^{b_j}d\varphi\\ =&\frac{rP_{kn-1}(\theta,r)}{r^{k-1}}+\frac{1}{r^{k-1}}\sum_{l=1}^{k-1}\sum_{S_l}\frac{M_{k-l}}{K}\int_0^\theta p_1(\varphi)q_1(\varphi)^{k-l-1}\prod_{j=1}^{l} h_j(q_1(\varphi),q_1(0))^{b_j}d\varphi,\end{aligned}\tag{14}$$

where $P_{kn-1}(\varphi,r)$ is a polynomial of degree $kn-1$ in r. By (7), (9), (14) and since we are assuming that $F_0^+ = 0$ in (7), it follows that

$$\begin{aligned}y_k(\theta,r) =&k!\int_0^\theta\left(F_k(\varphi,r)+\sum_{l=1}^{k-1}\sum_{S_l}\frac{1}{K}\partial^L F_{k-l}(\varphi,r)\prod_{j=1}^{l} y_j(\varphi,r)^{b_j}\right)d\varphi\\ =&\frac{k!}{r^{k-1}}\int_0^\theta r\widetilde{F}_k(\varphi,r)+p_1(\varphi)q_1(\varphi)^{k-1}d\varphi+\frac{k!rP_{kn-1}(\theta,r)}{r^{k-1}}\\ &+\frac{k!}{r^{k-1}}\sum_{l=1}^{k-1}\sum_{S_l}\frac{M_{k-l}}{K}\int_0^\theta p_1(\varphi)q_1(\varphi)^{k-l-1}\prod_{j=1}^{l} h_j(q_1(\varphi),q_1(0))^{b_j}d\varphi\\ =&\frac{k!}{r^{k-2}}\int_0^\theta \widetilde{F}_k(\varphi,r)d\varphi+\frac{k!P_{kn-1}(\theta,r)}{r^{k-2}}+\frac{h_k(q_1(\theta),q_1(0))}{r^{k-1}},\end{aligned}$$

because $dq_1(\theta)/d\theta = p_1(\theta)$. Thus

$$\tilde{y}_k(\theta,r) = k!\int_0^\theta \widetilde{F}_k(\varphi,r)d\varphi + k!P_{kn-1}(\theta,r)$$

from its definition given in (11). Since both $\widetilde{F}_k(\varphi,r)$ and $P_{kn-1}(\theta,r)$ are polynomials of degree $kn-1$ in r, we immediately get that $\tilde{y}_k(\theta,r)$ is also a polynomial of degree $kn-1$ in r. This ends the proof of the claim.

From (6) and (11) the averaged function of order N associated to system (2) is

$$\mathcal{F}_N(r) = \frac{y_N^+(\alpha,r)-y_N^-(\alpha-2\pi,r)}{N!} = \frac{1}{N!}\frac{\widetilde{\mathcal{F}}_N(r)}{r^{N-1}},\tag{15}$$

where

$$\widetilde{\mathcal{F}}_N(r) := r\tilde{y}_N^+(\alpha,r)+h_N(q_1^+(\alpha),q_1^+(0))-r\tilde{y}_N^-(\alpha-2\pi,r)-h_N(q_1^-(\alpha-2\pi),q_1^-(0)).$$

Since $\tilde{y}_N^+(\alpha,r)$ and $\tilde{y}_N^-(\alpha-2\pi,r)$ are polynomials of degree $Nn-1$ and $Nm-1$ respectively, $\widetilde{\mathcal{F}}_N(r)$ is of degree Nn due to $n \ge m$. This concludes that $\widetilde{\mathcal{F}}_N(r)$, or equivalently $\mathcal{F}_N(r)$, has at most Nn simple positive zeros, which directly gives statement (i) for $N = 1$ and statement (ii) for any N.

To obtain statement (i) for $N \geq 2$ we shall prove

$$h_N(q_1^+(\pi), q_1^+(0)) - h_N(q_1^-(-\pi), q_1^-(0)) = 0, \tag{16}$$

if $N \geq 2$ provided that $\mathcal{F}_1 = 0$. In fact, by (15) $\mathcal{F}_1 = 0$ implies

$$h_1(q_1^+(\pi), q_1^+(0)) - h_1(q_1^-(-\pi), q_1^-(0)) = 0.$$

Besides it follows from (8) that

$$q_1^+(\pi) = g_1^+(0,0), \quad q_1^-(-\pi) = g_1^-(0,0), \quad q_1^+(0) = -g_1^+(0,0), \quad q_1^-(0) = -g_1^-(0,0).$$

Thus together with the definition of h_1 given in (10), we get $g_1^+(0,0) = g_1^-(0,0)$, which implies

$$h_N(g_1^+(0,0), -g_1^+(0,0)) - h_N(g_1^-(0,0), -g_1^-(0,0)) = 0,$$

if $N \geq 2$, i.e. (16) holds. Consequently, $\widetilde{\mathcal{F}}_N(r) = r\tilde{y}_N^+(\pi, r) - r\tilde{y}_N^-(-\pi, r)$ and the averaged function in (15) becomes

$$\mathcal{F}_N(r) = \frac{1}{N!}\frac{\tilde{y}_N^+(\pi, r) - \tilde{y}_N^-(-\pi, r)}{r^{N-2}}, \tag{17}$$

if $N \geq 2$. As we have proved that $\tilde{y}_N^+(\theta, r)$ and $\tilde{y}_N^-(\theta, r)$ are polynomials of degree $Nn - 1$ and $Nm - 1$ in r respectively, the averaged function in (17) has at most $Nn - 1$ simple positive zeros because $n \geq m$, i.e. statement (ii) holds for $N \geq 2$. □

For the piecewise polynomial differential system (2) Proposition 6 provides upper bounds for the maximum number of simple positive zeros of the corresponding averaged functions of order N. Next we study the realitzation of these upper bounds for $N = 1, 2$.

Proposition 7. *Consider the piecewise polynomial differential system*

$$(\dot{x}, \dot{y}) = \begin{cases} \left(-y + \varepsilon \sum_{i=0}^{n} a_i y^i, x + \varepsilon \sum_{i=0}^{n} b_i y^i\right) & \text{if} \quad (x, y) \in \Sigma_\alpha^+, \\ (-y, x) & \text{if} \quad (x, y) \in \Sigma_\alpha^-. \end{cases} \tag{18}$$

For any given $\alpha \in (0, \pi]$ and $n \geq 1$ there exists a choice of the parameters a_i and b_i such that the first order averaged function associated to (18) *has exactly n simple positive zeros.*

Proof. Using the polar coordinates we write system (18) in the form (5) with

$$F_1^+(\theta, r) = \sum_{i=0}^{n} a_i \cos\theta \sin^i \theta r^i + \sum_{i=0}^{n} b_i \sin^{i+1} \theta r^i, \qquad F_1^-(\theta, r) = 0.$$

Then according to the definition in (6), the first order averaged function is

$$\mathcal{F}_1(r) = \int_0^\pi F_1^+(\theta, r) d\theta = \sum_{i=0}^{n} b_i \mathcal{S}_{i+1}(\pi) r^i$$

if $\alpha = \pi$. Thus $\mathcal{F}_1(r)$ is a complete polynomial of degree n because $\mathcal{S}_{i+1}(\pi) \neq 0$, see (iii) of Lemma 5, which implies that we can choose b_i in such a way that $\mathcal{F}_1(r)$ has exactly n simple positive zeros. If $\alpha \in (0, \pi)$, then $\mathcal{F}_1(r)$ becomes

$$\mathcal{F}_1(r) = \sum_{i=0}^{n} \left(\frac{a_i}{i+1} \sin^{i+1} \alpha + b_i \mathcal{S}_{i+1}(\alpha)\right) r^i.$$

In this case it is also a complete polynomial of degree n because $\sin\alpha \neq 0$ for $\alpha \in (0, \pi)$. Again we can choose a_i in such that $\mathcal{F}_1(r)$ has exactly n simple positive zeros. □

Proposition 8. *Consider the piecewise polynomial differential system*

$$(19)\qquad (\dot{x},\dot{y})=\begin{cases}\begin{pmatrix} -y+\varepsilon\displaystyle\sum_{i=0}^{n}(a_ix^i+b_iy^i+c_ixy^{i-1}) \\ x+\varepsilon\displaystyle\sum_{i=0}^{n}d_iy^i+\varepsilon^2\sum_{i=0}^{n}e_iy^i\end{pmatrix}^{\top} & \text{if}\quad (x,y)\in\Sigma_\pi^+,\\ (-y,x) & \text{if}\quad (x,y)\in\Sigma_\pi^-,\end{cases}$$

where $n\geq 1$*,*

$$(20)\qquad d_0=0,\qquad c_i=-i\varrho_i a_i-id_i,$$

for $i=0,1,...,n$ *and either* $\varrho_i=1$ *if* i *is odd, or* $\varrho_i=0$ *if* i *is even. For any given* n *there exists a choice of the parameters* a_i,b_i*,* d_i *and* e_i *such that the second order averaged function associated to* (19) *has exactly* $2n-1$ *simple positive zeros.*

Proof. Using polar coordinates system (19) with (20) writes in the form (5) with

$$(21)\qquad \begin{aligned} &F_1^+(\theta,r)=\sum_{i=0}^{n}\mu_i(\theta)r^i, && F_2^+(\theta,r)=\sum_{i=0}^{n}e_i\sin^{i+1}\theta r^i+\frac{1}{r}\sum_{i=0}^{n}\mu_i(\theta)r^i\sum_{i=0}^{n}\nu_i(\theta)r^i,\\ &F_1^-(\theta,r)=0, && F_2^-(\theta,r)=0,\end{aligned}$$

where

$$(22)\qquad \begin{aligned}\mu_i(\theta)&=a_i(\cos^{i+1}\theta-i\varrho_i\cos^2\theta\sin^{i-1}\theta)+b_i\cos\theta\sin^i\theta+d_i(\sin^{i+1}\theta-i\cos^2\theta\sin^{i-1}\theta),\\ \nu_i(\theta)&=a_i(\sin\theta\cos^i\theta-i\varrho_i\cos\theta\sin^i\theta)+b_i\sin^{i+1}\theta-d_i(i+1)\cos\theta\sin^i\theta.\end{aligned}$$

From (i) and (iii) of Lemma 5 we have

$$\begin{aligned}&\int_0^\pi \cos^{i+1}\theta-i\varrho_i\cos^2\theta\sin^{i-1}\theta d\theta=0 &&\text{for}\quad i=0,1,2,...,n,\\ &\int_0^\pi \sin^{i+1}\theta-i\cos^2\theta\sin^{i-1}\theta d\theta=0 &&\text{for}\quad i=1,2,...,n,\end{aligned}$$

together with the definition in (6), we obtain that the first order averaged function vanishes.

By (6) and (21) the second order averaged function is

$$\mathcal{F}_2(r)=\int_0^\pi F_2^+(\theta,r)+\partial F_1^+(\theta,r)\int_0^\theta F_1^+(\varphi,r)d\varphi d\theta=\sum_{i=0}^{n}e_i\mathcal{S}_{i+1}(\pi)r^i+\frac{1}{r}\sum_{i=0}^{2n}V_ir^i,$$

where

$$(23)\qquad V_i=\sum_{\substack{i_1+i_2=i\\ 0\leq i_1,i_2\leq n}}\int_0^\pi \mu_{i_1}(\theta)\nu_{i_2}(\theta)+i_1\mu_{i_1}(\theta)\int_0^\theta\mu_{i_2}(\varphi)d\varphi d\theta.$$

It is easy to obtain that $V_0=(a_0+b_0)^2\int_0^\pi\cos\theta\sin\theta d\theta=0$ because $d_0=0$. Hence $\mathcal{F}_2(r)$ becomes

$$(24)\qquad \mathcal{F}_2(r)=\sum_{i=0}^{n}e_i\mathcal{S}_{i+1}(\pi)r^i+\sum_{i=1}^{2n}V_ir^{i-1},$$

and it has at most $2n-1$ simple positive zeros. To obtain the realization of this number, we need more information on V_i. In particular, by (22) and (23) we can write V_i in the form

$$
V_i = \sum_{\substack{i_1+i_2=i\\ 0\le i_1\le i_2\le n}} \xi_1^i(i_1,i_2)a_{i_1}a_{i_2} + \xi_2^i(i_1,i_2)b_{i_1}b_{i_2} + \xi_3^i(i_1,i_2)d_{i_1}d_{i_2}
+ \sum_{\substack{i_1+i_2=i\\ 0\le i_1,i_2\le n}} \xi_4^i(i_1,i_2)a_{i_1}b_{i_2} + \xi_5^i(i_1,i_2)a_{i_1}d_{i_2} + \xi_6^i(i_1,i_2)b_{i_1}d_{i_2}, \tag{25}
$$

and the following four statements will be proved,

(a) $\xi_2^i(i_1,i_2)=\xi_3^i(i_1,i_2)=\xi_6^i(i_1,i_2)=0$ for any i_1 and i_2;
(b) if $i_1=i_2=i/2\ge 3$ is odd, then $\xi_4^i(i/2,i/2)<0$ and $\xi_4^{i-1}(i/2,i/2-1)<0$;
(c) if $i_1=i_2=i/2\ge 2$ is even, then $\xi_5^i(i/2,i/2)<0$ and $\xi_5^{i-1}(i/2,i/2-1)<0$;
(d) if i is even, then $\xi_1^i(i/2,i/2)=0$.

Regarding statement (a), $\xi_2^i(i_1,i_2),\xi_3^i(i_1,i_2)$ and $\xi_6^i(i_1,i_2)$ are the coefficients of $b_{i_1}b_{i_2}, d_{i_1}d_{i_2}$ and $b_{i_1}d_{i_2}$ respectively. Then, according to (22) and (23), we get

$$
\xi_2^i(i_1,i_2) = \int_0^\pi \cos\theta\sin^{\frac{i}{2}}\theta\sin^{\frac{i}{2}+1}\theta + \frac{i}{2}\cos\theta\sin^{\frac{i}{2}}\theta\int_0^\theta \cos\varphi\sin^{\frac{i}{2}}\varphi d\varphi d\theta = 0,
$$

and

$$
\begin{aligned}
\xi_3^i(i_1,i_2) =& \int_0^\pi\Big(\sin^{\frac{i}{2}+1}\theta - \frac{i}{2}\cos^2\theta\sin^{\frac{i}{2}-1}\theta\Big)\Big(-\frac{i+2}{2}\cos\theta\sin^{\frac{i}{2}}\theta\Big)\\
&+\frac{i}{2}\Big(\sin^{\frac{i}{2}+1}\theta - \frac{i}{2}\cos^2\theta\sin^{\frac{i}{2}-1}\theta\Big)\Big(\int_0^\theta \sin^{\frac{i}{2}+1}\varphi - \frac{i}{2}\cos^2\varphi\sin^{\frac{i}{2}-1}\varphi d\varphi\Big)d\theta\\
=&\frac{i}{2}\int_0^\pi\Big(\sin^{\frac{i}{2}+1}\theta - \frac{i}{2}\cos^2\theta\sin^{\frac{i}{2}-1}\theta\Big)\Big(\int_0^\theta \sin^{\frac{i}{2}+1}\varphi - \frac{i}{2}\cos^2\varphi\sin^{\frac{i}{2}-1}\varphi d\varphi\Big)d\theta\\
=&-\frac{i}{2}\int_0^\pi\Big(\sin^{\frac{i}{2}+1}\theta - \frac{i}{2}\cos^2\theta\sin^{\frac{i}{2}-1}\theta\Big)\cos\theta\sin^{\frac{i}{2}}\theta d\theta = 0,
\end{aligned}
$$

if $i_1=i_2=i/2$, while if $i_1\ne i_2$ then

$$
\begin{aligned}
\xi_2^i(i_1,i_2) =& \int_0^\pi \cos\theta\sin^{i_1}\theta\sin^{i_2+1}\theta + i_1\cos\theta\sin^{i_1}\theta\int_0^\theta\cos\varphi\sin^{i_2}\varphi d\varphi d\theta\\
&+\int_0^\pi \cos\theta\sin^{i_2}\theta\sin^{i_1+1}\theta + i_2\cos\theta\sin^{i_2}\theta\int_0^\theta\cos\varphi\sin^{i_1}\varphi d\varphi d\theta = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\xi_3^i(i_1,i_2) =& \int_0^\pi \big(\sin^{i_1+1}\theta - i_1\cos^2\theta\sin^{i_1-1}\theta\big)\big(-(i_2+1)\cos\theta\sin^{i_2}\theta\big) \\
&+ i_1\,\big(\sin^{i_1+1}\theta - i_1\cos^2\theta\sin^{i_1-1}\theta\big)\left(\int_0^\theta \sin^{i_2+1}\varphi - i_2\cos^2\varphi\sin^{i_2-1}\varphi d\varphi\right) d\theta \\
&+ \int_0^\pi \big(\sin^{i_2+1}\theta - i_2\cos^2\theta\sin^{i_2-1}\theta\big)\big(-(i_1+1)\cos\theta\sin^{i_1}\theta\big) \\
&+ i_2\,\big(\sin^{i_2+1}\theta - i_2\cos^2\theta\sin^{i_2-1}\theta\big)\left(\int_0^\theta \sin^{i_1+1}\varphi - i_1\cos^2\varphi\sin^{i_1-1}\varphi d\varphi\right) d\theta \\
=& - i_1\int_0^\pi \big(\sin^{i_1+1}\theta - i_1\cos^2\theta\sin^{i_1-1}\theta\big)\cos\theta\sin^{i_2}\theta d\theta \\
&- i_2\int_0^\pi \big(\sin^{i_2+1}\theta - i_2\cos^2\theta\sin^{i_2-1}\theta\big)\cos\theta\sin^{i_1}\theta d\theta = 0.
\end{aligned}
$$

Here we used $\cos^2\theta = 1-\sin^2\theta$ and (iv) of Lemma 5 in the computation of $\xi_3^i(i_1,i_2)$. Furthermore

$$
\begin{aligned}
\xi_6^i(i_1,i_2) =& \int_0^\pi \cos\theta\sin^{i_1}\theta\,\big(-(i_2+1)\cos\theta\sin^{i_2}\theta\big) \\
&+ i_1\cos\theta\sin^{i_1}\theta\int_0^\theta \sin^{i_2+1}\varphi - i_2\cos^2\varphi\sin^{i_2-1}\varphi d\varphi d\theta \\
&+ \int_0^\pi \big(\sin^{i_2+1}\theta - i_2\cos^2\theta\sin^{i_2-1}\theta\big)\sin^{i_1+1}\theta \\
&+ i_2\,\big(\sin^{i_2+1}\theta - i_2\cos^2\theta\sin^{i_2-1}\theta\big)\int_0^\theta \cos\varphi\sin^{i_1}\varphi d\varphi d\theta \\
=& -(i_1+i_2+1)\int_0^\pi \cos^2\theta\sin^{i_1+i_2}\theta d\theta \\
&+ \frac{i_1+i_2+1}{i_1+1}\int_0^\pi \sin^{i_1+i_2+2}\theta - i_2\cos^2\theta\sin^{i_1+i_2}\theta d\theta \\
=& \frac{(i_1+i_2+1)(i_1+i_2+2)}{i_1+1}\int_0^\pi \sin^{i_1+i_2+2}\theta d\theta - \frac{(i_1+i_2+1)^2}{i_1+1}\int_0^\pi \sin^{i_1+i_2}\theta d\theta \\
=& 0,
\end{aligned}
$$

where we used $\cos^2\theta = 1-\sin^2\theta$ and (iv) of Lemma 5 in the second equality, and the last equality is a straightway application of (iii) of Lemma 5. These computations conclude statement (a).

Now we prove statement (b). From (25) we know that $\xi_4^i(i/2, i/2)$ and $\xi_4^{i-1}(i/2, i/2-1)$ are the coefficients of $a_{\frac{i}{2}}b_{\frac{i}{2}}$ and $a_{\frac{i}{2}}b_{\frac{i}{2}-1}$ respectively. Since $i/2$ is odd, we get $\varrho_{\frac{i}{2}} = 1$ from its definition given in Proposition 8. Thus, using (22) and (23), for $i/2 \geq 3$ we get

$$
\begin{aligned}
\xi_4^i(i/2, i/2) = & \int_0^{\pi} \left(\cos^{\frac{i}{2}+1}\theta - \frac{i}{2}\cos^2\theta \sin^{\frac{i}{2}-1}\theta\right) \sin^{\frac{i}{2}+1}\theta d\theta \\
& + \frac{i}{2}\left(\cos^{\frac{i}{2}+1}\theta - \frac{i}{2}\cos^2\theta \sin^{\frac{i}{2}-1}\theta\right) \int_0^{\theta}\left(\cos\varphi \sin^{\frac{i}{2}}\varphi\right) d\varphi d\theta \\
& + \int_0^{\pi} \cos\theta \sin^{\frac{i}{2}}\theta \left(\sin\theta \cos^{\frac{i}{2}}\theta - \frac{i}{2}\cos\theta \sin^{\frac{i}{2}}\theta\right) d\theta \\
& + \frac{i}{2}\cos\theta \sin^{\frac{i}{2}}\theta \left(\int_0^{\theta} \cos^{\frac{i}{2}+1}\varphi - \frac{i}{2}\cos^2\varphi \sin^{\frac{i}{2}-1}\varphi d\varphi\right) d\theta \\
= & 2\int_0^{\pi} \cos^{\frac{i}{2}+1}\theta \sin^{\frac{i}{2}+1}\theta - \frac{i}{2}\cos^2\theta \sin^i\theta d\theta \\
& + \frac{i}{2}\int_0^{\pi} \cos\theta \sin^{\frac{i}{2}}\theta d\theta \int_0^{\pi} \cos^{\frac{i}{2}+1}\theta - \frac{i}{2}\cos^2\theta \sin^{\frac{i}{2}-1}\theta d\theta \\
= & \frac{\pi}{2^{\frac{i}{2}}}\left(\frac{\frac{i}{2}!!}{(\frac{i}{2}+1)!!} - \frac{i}{2}\frac{(i-1)!!}{(\frac{i}{2}+1)!}\right) < 0,
\end{aligned}
$$

where we used the integration by parts method in the second equality and statements (iii)(v) of Lemma 5 are applied to the third one. In a similar way we have

$$
\begin{aligned}
\xi_4^{i-1}(i/2, i/2-1) = & \int_0^{\pi} \left(\cos^{\frac{i}{2}+1}\theta - \frac{i}{2}\cos^2\theta \sin^{\frac{i}{2}-1}\theta\right) \sin^{\frac{i}{2}}\theta d\theta \\
& + \frac{i}{2}\left(\cos^{\frac{i}{2}+1}\theta - \frac{i}{2}\cos^2\theta \sin^{\frac{i}{2}-1}\theta\right) \int_0^{\theta}\left(\cos\varphi \sin^{\frac{i}{2}-1}\varphi\right) d\varphi d\theta \\
& + \int_0^{\pi} \cos\theta \sin^{\frac{i}{2}-1}\theta \left(\sin\theta \cos^{\frac{i}{2}}\theta - \frac{i}{2}\cos\theta \sin^{\frac{i}{2}}\theta\right) d\theta \\
& + (\frac{i}{2}-1)\cos\theta \sin^{\frac{i}{2}-1}\theta \left(\int_0^{\theta} \cos^{\frac{i}{2}+1}\varphi - \frac{i}{2}\cos^2\varphi \sin^{\frac{i}{2}-1}\varphi d\varphi\right) d\theta \\
= & 2\int_0^{\pi} \cos^{\frac{i}{2}+1}\theta \sin^{\frac{i}{2}}\theta - \frac{i}{2}\cos^2\theta \sin^{i-1}\theta d\theta \\
& + \int_0^{\pi} \left(\cos^{\frac{i}{2}+1}\theta - \frac{i}{2}\cos^2\theta \sin^{\frac{i}{2}-1}\theta\right) \int_0^{\theta}\left(\cos\varphi \sin^{\frac{i}{2}-1}\varphi\right) d\varphi d\theta \\
= & (2+\frac{2}{i})\int_0^{\pi} \cos^{\frac{i}{2}+1}\theta \sin^{\frac{i}{2}}\theta - \frac{i}{2}\cos^2\theta \sin^{i-1}\theta d\theta \\
= & (2+\frac{2}{i})\left(\frac{\frac{i}{2}!!(\frac{i}{2}-1)!!}{(i-2)!!} - \frac{i}{2}\right)\int_0^{\pi} \cos^2\theta \sin^{i-1}\theta d\theta \\
= & (i+1)\left(2^{1-\frac{i}{2}} - 1\right)\int_0^{\pi} \cos^2\theta \sin^{i-1}\theta d\theta < 0,
\end{aligned}
$$

for $i/2 \geq 3$, where we used the integration by parts method again in the second equality and the fourth one is due to (vi) of Lemma 5. So statement (b) holds.

We see that $\xi_5^i(i/2, i/2)$ and $\xi_5^{i-1}(i/2, i/2-1)$ are the coefficients of $a_{\frac{i}{2}}d_{\frac{i}{2}}$ and $a_{\frac{i}{2}}d_{\frac{i}{2}-1}$ respectively. Moreover, $\varrho_{\frac{i}{2}} = 0$ because we are assuming that $i/2$ is even in statement (c).

Thus, using (22) and (23) again, for $i/2 \geq 2$ we get

$$\begin{aligned}
\xi_5^i(i/2,i/2) =& \int_0^\pi \cos^{\frac{i}{2}+1}\theta \left(-\frac{i+2}{2}\cos\theta\sin^{\frac{i}{2}}\theta\right)d\theta \\
&+ \frac{i}{2}\cos^{\frac{i}{2}+1}\theta\left(\int_0^\theta \sin^{\frac{i}{2}+1}\varphi - \frac{i}{2}\cos^2\varphi\sin^{\frac{i}{2}-1}\varphi d\varphi\right)d\theta \\
&+ \int_0^\pi \left(\sin^{\frac{i}{2}+1}\theta - \frac{i}{2}\cos^2\theta\sin^{\frac{i}{2}-1}\theta\right)\sin\theta\cos^{\frac{i}{2}}\theta d\theta \\
&+ \frac{i}{2}\left(\sin^{\frac{i}{2}+1}\theta - \frac{i}{2}\cos^2\theta\sin^{\frac{i}{2}-1}\theta\right)\int_0^\theta \cos^{\frac{i}{2}+1}\varphi d\varphi d\theta \\
=& -(i+1)\int_0^\pi \cos^{\frac{i}{2}+2}\theta\sin^{\frac{i}{2}}\theta d\theta + \int_0^\pi \cos^{\frac{i}{2}}\theta\sin^{\frac{i}{2}+2}\theta d\theta \\
&+ \frac{i}{2}\int_0^\pi \cos^{\frac{i}{2}+1}\theta d\theta\int_0^\pi \sin^{\frac{i}{2}+1}\theta - \frac{i}{2}\cos^2\theta\sin^{\frac{i}{2}-1}\theta d\theta \\
=& -(i+2)\int_0^\pi \cos^{\frac{i}{2}+2}\theta\sin^{\frac{i}{2}}\theta d\theta + \int_0^\pi \cos^{\frac{i}{2}}\theta\sin^{\frac{i}{2}}\theta d\theta \\
=& -\frac{i}{2}\int_0^\pi \cos^{\frac{i}{2}}\theta\sin^{\frac{i}{2}}\theta d\theta < 0,
\end{aligned}$$

where we used the integration by parts method in the second equality and the third equality is due to $\int_0^\pi \cos^{\frac{i}{2}+1}\theta d\theta = 0$ for $i/2$ even, and the last equality is obtained by (vi) of Lemma 5. Applying these techniques and (iv) of Lemma 5 to $\xi_5^{i-1}(i/2, i/2-1)$ we get

$$\begin{aligned}
\xi_5^{i-1}(i/2,i/2-1) =& \int_0^\pi \cos^{\frac{i}{2}+1}\theta\left(-\frac{i}{2}\cos\theta\sin^{\frac{i}{2}-1}\theta\right)d\theta \\
&+ \frac{i}{2}\cos^{\frac{i}{2}+1}\theta\left(\int_0^\theta \sin^{\frac{i}{2}}\varphi - (\frac{i}{2}-1)\cos^2\varphi\sin^{\frac{i}{2}-2}\varphi d\varphi\right)d\theta \\
&+ \int_0^\pi\left(\sin^{\frac{i}{2}}\theta - (\frac{i}{2}-1)\cos^2\theta\sin^{\frac{i}{2}-2}\theta\right)\sin\theta\cos^{\frac{i}{2}}\theta d\theta \\
&+ (\frac{i}{2}-1)\left(\sin^{\frac{i}{2}}\theta - (\frac{i}{2}-1)\cos^2\theta\sin^{\frac{i}{2}-2}\theta\right)\int_0^\theta \cos^{\frac{i}{2}+1}\varphi d\varphi d\theta \\
=& -(i-1)\int_0^\pi \cos^{\frac{i}{2}+2}\theta\sin^{\frac{i}{2}-1}\theta d\theta + \int_0^\pi \cos^{\frac{i}{2}}\theta\sin^{\frac{i}{2}+1}\theta d\theta \\
&+ \int_0^\pi \cos^{\frac{i}{2}+1}\theta\left(\int_0^\theta \sin^{\frac{i}{2}}\varphi - (\frac{i}{2}-1)\cos^2\varphi\sin^{\frac{i}{2}-2}\varphi d\varphi\right)d\theta \\
=& -i\int_0^\pi \cos^{\frac{i}{2}+2}\theta\sin^{\frac{i}{2}-1}\theta d\theta + \int_0^\pi \cos^{\frac{i}{2}}\theta\sin^{\frac{i}{2}+1}\theta d\theta \\
=& -(i+1)\int_0^\pi \cos^{\frac{i}{2}}\theta\sin^{\frac{i}{2}+1}\theta d\theta < 0.
\end{aligned}$$

In summary statement (c) is proved.

Finally, since $\xi_1^i(i/2,i/2)$ is the coefficient of $a_{\frac{i}{2}}a_{\frac{i}{2}}$, statement (d) follows from the following computation

$$\begin{aligned}
\xi_1^i(i/2,i/2) =& \int_0^\pi \left(\cos^{\frac{i}{2}+1}\theta - \frac{i}{2}\varrho_{\frac{i}{2}}\cos^2\theta\sin^{\frac{i}{2}-1}\theta\right)\left(\sin\theta\cos^{\frac{i}{2}}\theta - \frac{i}{2}\varrho_{\frac{i}{2}}\cos\theta\sin^{\frac{i}{2}}\theta\right)\\
&+\frac{i}{2}\left(\cos^{\frac{i}{2}+1}\theta - \frac{i}{2}\varrho_{\frac{i}{2}}\cos^2\theta\sin^{\frac{i}{2}-1}\theta\right)\left(\int_0^\theta \cos^{\frac{i}{2}+1}\varphi - \frac{i}{2}\varrho_{\frac{i}{2}}\cos^2\varphi\sin^{\frac{i}{2}-1}\varphi\right)d\theta\\
=& \int_0^\pi \left(\cos^{\frac{i}{2}+1}\theta - \frac{i}{2}\varrho_{\frac{i}{2}}\cos^2\theta\sin^{\frac{i}{2}-1}\theta\right)\left(\sin\theta\cos^{\frac{i}{2}}\theta - \frac{i}{2}\varrho_{\frac{i}{2}}\cos\theta\sin^{\frac{i}{2}}\theta\right)d\theta\\
&+\frac{i}{4}\left(\int_0^\pi \cos^{\frac{i}{2}+1}\theta - \frac{i}{2}\varrho_{\frac{i}{2}}\cos^2\theta\sin^{\frac{i}{2}-1}\theta d\theta\right)^2\\
=& \int_0^\pi \left(\cos^{\frac{i}{2}+1}\theta - \frac{i}{2}\varrho_{\frac{i}{2}}\cos^2\theta\sin^{\frac{i}{2}-1}\theta\right)\left(\sin\theta\cos^{\frac{i}{2}}\theta - \frac{i}{2}\varrho_{\frac{i}{2}}\cos\theta\sin^{\frac{i}{2}}\theta\right)d\theta\\
=& -\frac{2}{i+2}\int_0^\pi \cos^{\frac{i}{2}+1}\theta + \frac{i}{2}\varrho_{\frac{i}{2}}\sin^{\frac{i}{2}+1}\theta d\left(\cos^{\frac{i}{2}+1}\theta + \frac{i}{2}\varrho_{\frac{i}{2}}\sin^{\frac{i}{2}+1}\theta\right)\\
&-\int_0^\pi \frac{i}{2}\varrho_{\frac{i}{2}}\sin^{\frac{i}{2}-1}\theta\left(\sin\theta\cos^{\frac{i}{2}}\theta - \frac{i}{2}\varrho_{\frac{i}{2}}\cos\theta\sin^{\frac{i}{2}}\theta\right)d\theta\\
=& -\frac{i}{2}\varrho_{\frac{i}{2}}\int_0^\pi \sin^{\frac{i}{2}}\theta\cos^{\frac{i}{2}}\theta d\theta = 0.
\end{aligned}$$

Here the third equality is obtained by a direct computation using (i)(iii) of Lemma 5, the last equality is obtain by joining (v) of Lemma 5 and the fact that $\varrho_{\frac{i}{2}} = 0$ if $\frac{i}{2}$ is even.

Under statements (a) and (d) the expression of V_i can be simplified as follows

$$V_i = \sum_{\substack{i_1+i_2=i\\0\le i_1<i_2\le n}} \xi_1^i(i_1,i_2)a_{i_1}a_{i_2} + \sum_{\substack{i_1+i_2=i\\0\le i_1,i_2\le n}} \xi_4^i(i_1,i_2)a_{i_1}b_{i_2} + \xi_5^i(i_1,i_2)a_{i_1}d_{i_2}. \tag{26}$$

Assuming that n is odd, we can choose the parameters a_i, b_i, d_i and e_i to produce $2n-1$ simple positive zeros following the next procedure. We start by letting all parameters equal zero except $a_n = b_n = 1$, and we continue adding the rest of parameters such that $a_nb_{n-1} < 0$, $a_{n-1}d_{n-1} > 0$, $a_{n-1}d_{n-2} < 0$, $a_{n-2}b_{n-2} > 0$, $a_{n-2}b_{n-3} < 0$,..., $e_n < 0, e_{n-1} > 0, ..., e_1 < 0, e_0 > 0$ in the next each step. Here the total number of steps is $2n-1$. From (26) and statement (b) we get $\mathcal{F}_2(r) = V_{2n}r^{2n-1}$ with $V_{2n} = \xi_4^{2n}(n,n)a_nb_n < 0$ when all parameters are zero except a_n and b_n. In the first step we add the parameter $b_{n-1} < 0$, which satisfies $a_nb_{n-1} < 0$, and then from (26) and statements (b) the function $\mathcal{F}_2(r)$ is becomes

$$\mathcal{F}_2(r) = V_{2n}r^{2n-1} + V_{2n-1}r^{2n-2}, \tag{27}$$

where $V_{2n} = \xi_4^{2n}(n,n)a_nb_n < 0$ and $V_{2n-1} = \xi_4^{2n-1}(n,n-1)a_nb_{n-1} > 0$. Thus one simple positive zero bifurcates from $r = 0$ by choosing $b_{n-1} < 0$ in such a way that $|V_{2n-1}| \ll |V_{2n}|$. In the second step we further update $\mathcal{F}_2(r)$ adding the parameters a_{n-1} and d_{n-1} satisfying $a_{n-1}d_{n-1} > 0$. In this case (26) and statement (c) imply that

$$\mathcal{F}_2(r) = V_{2n}r^{2n-1} + V_{2n-1}r^{2n-2} + V_{2n-2}r^{2n-3}, \tag{28}$$

with $V_{2n-2} = \xi_4^{2n-2}(n-1,n-1)a_{n-1}b_{n-1} + \xi_5^{2n-2}(n-1,n-1)a_{n-1}d_{n-1}$. Notice that V_{2n} and V_{2n-1} in (28) may be different from the ones in (27) after adding the new parameters a_{n-1} and d_{n-1}. We use the same notations only for the sake of simplification. Choosing a_{n-1} and d_{n-1} such that $V_{2n-2} < 0$ and $|V_{2n-2}| \ll |V_{2n-1}|$, we get the second simple positive zero bifurcating from $r=0$. Continue to add the parameters d_{n-2} with $a_{n-1}d_{n-2}<0$ and then

$$\mathcal{F}_2(r)=V_{2n}r^{2n-1}+V_{2n-1}r^{2n-2}+V_{2n-2}r^{2n-3}+V_{2n-3}r^{2n-4},$$

where $V_{2n-3}=\xi_5^{2n-3}(n-1,n-2)a_{n-1}d_{n-2}>0$ by (26) and statement (c). Thus, in the third step the third simple positive zero occurs when $|V_{2n-3}|\ll|V_{2n-2}|$. Following the procedure, we finally obtain that $2n-1$ simple positive zeros bifurcate from $r=0$. We note that it is reasonable to add the parameters e_i for $i=n,...,1,0$ from step $n+1$ to $2n$, because $\mathcal{S}_{i+1}(\pi)\neq 0$ in (24).

A similar procedure can be applied to the case where n is even, adding the parameters in the following order $a_nd_n>0$, $a_nd_{n-1}<0$, $a_{n-1}b_{n-1}>0$, $a_{n-1}b_{n-2}<0$, $a_{n-2}d_{n-2}>0$, $a_{n-2}d_{n-3}<0$,..., $e_n>0$, $e_{n-1}<0$,...,$e_1<0$, $e_0>0$ in the next $2n-1$ steps. This ends the proof of Proposition 8. □

The procedure of choosing parameters in the proof of Proposition 8 relies on statements (a)-(d). As an example we give the explicit expression of $\mathcal{F}_2(r)$ for system (19) with $n=5$,

$$\begin{aligned}\mathcal{F}_2(r)=&-\frac{25\pi}{128}a_5b_5r^9-\frac{2}{945}(55a_4a_5+360a_5b_4+96a_4d_5)r^8-\frac{\pi}{128}(40a_5b_3+15a_3b_5+12a_4d_4)r^7\\&+\frac{2}{105}(15a_3a_4-55a_2a_5-70a_5b_2-24a_3b_4-24a_4d_3-16a_2d_5)r^6\\&-\frac{\pi}{16}(10a_5b_1+3a_3b_3+4a_4d_2+2a_2d_4-5e_5)r^5\\&-\frac{2}{15}(3a_2a_3+25a_0a_5+25a_5b_0+6a_3b_2+12a_4d_1+4a_2d_3-8e_4)r^4\\&-\frac{\pi}{8}(3a_3b_1+2a_2d_2-3e_3)r^3-\frac{2}{3}(3a_0a_3+3a_3b_0+2a_2d_1-2e_2)r^2+\frac{1}{2}e_1\pi r+2e_0.\end{aligned}$$

It is easy to see that the above $\mathcal{F}_2(r)$ satisfies all statements (a)-(d), and in each step we can add the parameters in the order a_5b_5, a_5b_4, a_4d_4, a_4d_3, e_5, e_4, e_3, e_2, e_1, e_0 to produce 9 simple positive zeros.

Proposition 9. *Consider the piecewise polynomial differential system*

$$(\dot{x},\dot{y})=\begin{cases}\begin{pmatrix}-y+\varepsilon\sum_{i=0}^{n}(a_ix^{i-1}y+b_iy^i)\\ x+\varepsilon\sum_{i=0}^{n}c_ix^i+\varepsilon^2\sum_{i=0}^{n}d_iy^i\end{pmatrix}^{\top} & \text{if}\quad (x,y)\in\Sigma_\alpha^+,\\ (-y-\varepsilon b_0,x-\varepsilon b_0) & \text{if}\quad (x,y)\in\Sigma_\alpha^-,\end{cases}\tag{29}$$

where $\alpha\in(0,\pi/2]$, $n\geq 1$, $a_0=0, c_i=\varpi_ib_i-a_i$ with $i=0,1,2,...,n$,

$$\varpi_0=\frac{2\sin\alpha-\cos\alpha+1}{\cos\alpha-1},\qquad \varpi_i=\frac{\sin^{i+1}\alpha}{\cos^{i+1}\alpha-1}\qquad i\geq 1.\tag{30}$$

For any given α and n there exists a choice of the parameters a_i,b_i and d_i such that the second order averaged function associated to (29) *has exactly $2n$ simple positive zeros.*

Proof. Writing system (29) in the form (5) and using the condition $c_i = \varpi_i b_i - a_i$ we obtain

$$F_1^+(\theta,r)=\sum_{i=0}^{n}\zeta_i(\theta)r^i,\qquad F_2^+(\theta,r)=\sum_{i=0}^{n}d_i\sin^{i+1}\theta r^i+\frac{1}{r}\sum_{i=0}^{n}\zeta_i(\theta)r^i\sum_{i=0}^{n}\eta_i(\theta)r^i,$$
$$F_1^-(\theta,r)=-b_0(\cos\theta+\sin\theta),\quad F_2^-(\theta,r)=\frac{1}{r}b_0^2(\sin^2\theta-\cos^2\theta),$$

where

$$\zeta_i(\theta)=b_i(\varpi_i\sin\theta\cos^i\theta+\cos\theta\sin^i\theta),$$
$$\eta_i(\theta)=a_i\cos^{i-1}\theta+b_i(\sin^{i+1}\theta-\varpi_i\cos^{i+1}\theta).$$

From (6) and (30) the first order averaged function is

$$\begin{aligned}\mathcal{F}_1(r)&=\sum_{i=0}^{n}\int_0^{\alpha}\zeta_i(\theta)d\theta r^i+b_0\int_0^{\alpha-2\pi}\cos\theta+\sin\theta d\theta\\&=\sum_{i=0}^{n}\frac{\varpi_i(1-\cos^{i+1}\alpha)+\sin^{i+1}\alpha}{i+1}b_ir^i+b_0(\sin\alpha-\cos\alpha+1)=0.\end{aligned}$$

Moreover the second order averaged function is

$$\begin{aligned}\mathcal{F}_2(r)&=\int_0^{\alpha}F_2^+(\theta,r)+\partial F_1^+(\theta,r)\int_0^{\theta}F_1^+(\varphi,r)d\varphi d\theta\\&\quad-\int_0^{\alpha-2\pi}F_2^-(\theta,r)+\partial F_1^-(\theta,r)\int_0^{\theta}F_1^-(\varphi,r)d\varphi d\theta\\&=\sum_{i=0}^{n}d_i\mathcal{S}_{i+1}(\alpha)r^i+\frac{1}{r}\sum_{i=0}^{2n}W_ir^i+\frac{1}{r}b_0^2\sin\alpha\cos\alpha,\end{aligned}\tag{31}$$

where

$$\begin{aligned}W_i&=\sum_{\substack{i_1+i_2=i\\0\le i_1,i_2\le n}}\int_0^{\alpha}\zeta_{i_1}(\theta)\eta_{i_2}(\theta)+i_1\zeta_{i_1}(\theta)\int_0^{\theta}\zeta_{i_2}(\varphi)d\varphi d\theta\\&=\sum_{\substack{i_1+i_2=i\\0\le i_1,i_2\le n}}\kappa_1^i(i_1,i_2)b_{i_1}a_{i_2}+\kappa_2^i(i_1,i_2)b_{i_1}b_{i_2},\end{aligned}$$

and

$$\begin{aligned}\kappa_1^i(i_1,i_2)&=\int_0^{\alpha}(\varpi_{i_1}\sin\theta\cos^{i_1}\theta+\cos\theta\sin^{i_1}\theta)\cos^{i_2-1}\theta d\theta,\\\kappa_2^i(i_1,i_2)&=\int_0^{\alpha}(\varpi_{i_1}\sin\theta\cos^{i_1}\theta+\cos\theta\sin^{i_1}\theta)(\sin^{i_2+1}\theta-\varpi_{i_2}\cos^{i_2+1}\theta)\\&\quad+i_1(\varpi_{i_1}\sin\theta\cos^{i_1}\theta+\cos\theta\sin^{i_1}\theta)\int_0^{\theta}(\varpi_{i_2}\sin\theta\cos^{i_2}\theta+\cos\theta\sin^{i_2}\theta)d\theta.\end{aligned}$$

Clearly $r\mathcal{F}_2(r)$ is a polynomial of degree $2n$, implying that it has at most $2n$ simple positive zeros. To get the realization we need more information on the coefficient W_i. In particular we have the following statements for $\alpha\in(0,\pi/2]$

(e) $\kappa_2^0(0,0)+\sin\alpha\cos\alpha\neq 0$;
(f) $\kappa_1^i(i_1,i_2)\neq 0$ when $i_1\ge 2$ and $i_2\ge 2$.

Indeed, from the expression of $\kappa_2^i(i_1,i_2)$ we directly obtain

$$\kappa_2^0(0,0)=\int_0^\alpha(\varpi_0\sin\theta+\cos\theta)(\sin\theta-\varpi_0\cos\theta)d\theta=-\varpi_0\sin\alpha\cos\alpha+\frac{1-\varpi_0^2}{2}\sin^2\alpha,$$

together with (30), which imply that

$$\kappa_2^0(0,0)+\sin\alpha\cos\alpha=(1-\varpi_0)\sin\alpha\left(\cos\alpha+\frac{1+\varpi_0}{2}\sin\alpha\right)=(\varpi_0-1)\sin\alpha\neq 0,$$

for $\alpha\in(0,\pi/2]$, i.e. statement (e) holds.

Using the integration by parts method and (30) we get

$$\begin{aligned}\kappa_1^i(i_1,i_2)=&\frac{i_2-1}{i_1+1}\int_0^\alpha(\sin^{i_1+1}\theta-\varpi_{i_1}\cos^{i_1+1}\theta+\varpi_{i_1})\cos^{i_2-2}\theta\sin\theta d\theta\\ =&\frac{i_2-1}{i_1+1}\int_0^\alpha\psi_{i_1}(\theta)(\cos^{i_1+1}\theta-1)\cos^{i_2-2}\theta\sin\theta d\theta,\end{aligned}$$

where

$$\psi_{i_1}(\theta)=\frac{\sin^{i_1+1}\theta}{\cos^{i_1+1}\theta-1}-\varpi_{i_1}=\frac{\sin^{i_1+1}\theta}{\cos^{i_1+1}\theta-1}-\frac{\sin^{i_1+1}\alpha}{\cos^{i_1+1}\alpha-1}.$$

Since

$$\frac{d\psi_{i_1}(\theta)}{d\theta}=(i_1+1)\sin^{i_1}\theta\frac{\cos^{i_1}\theta-\cos\theta}{(\cos^{i_1+1}\theta-1)^2},$$

$\psi_{i_1}(\theta)$ is strictly decreasing in $(0,\pi/2]$ when $i_1\geq 2$, so that $\psi_{i_1}(\theta)>\psi_{i_1}(\alpha)=0$ for $\theta\in(0,\alpha)$ and $\alpha\in(0,\pi/2]$. Consequently $\kappa_1^i(i_1,i_2)<0$ when $i_1\geq 2$ and $i_2\geq 2$, because the integrant is non-positive and is not identically zero, i.e. statement (f) holds.

Consider the polynomial $r\mathcal{F}_2(r)$ where $\mathcal{F}_2(r)$ is given in (31). The constant term is

$$W_0+b_0^2\sin\alpha\cos\alpha=b_0^2(\kappa_2^0(0,0)+\sin\alpha\cos\alpha),$$

because $a_0=0$, and thus we can rewrite the polynomial as

$$r\mathcal{F}_2(r)=b_0^2(\kappa_2^0(0,0)+\sin\alpha\cos\alpha)+\sum_{i=1}^{n+1}(d_{i-1}\mathcal{S}_i(\alpha)+W_i)\,r^i+\sum_{n+2}^{2n}W_ir^i.$$

Fixing $b_1=b_2=\cdots=b_n=1$, we find that W_i with $i=n+2,n+3,...,2n$ only contains the parameters a_{i-n}, a_{i-n+1}, ..., a_n, and then W_i has one less parameter than W_{i-1}. Moreover W_i with $i=1,2,...,n+1$ is independent of d_j, $j=0,1,2,...,n$. Therefore joining statements (e), (f) and the fact that $\mathcal{S}_i(\alpha)>0$ for $\alpha\in(0,\pi/2]$, we get that all coefficients of $r\mathcal{F}_2(r)$ can be chosen arbitrarily, i.e. it is a complete polynomial of degree $2n$. This concludes the proof of Proposition 9. □

Finally, combining these propositions we can prove Theorem 1.

Proof of Theorem 1. By the averaging theory statement (i) follows from statement (i) of Proposition 6, and Propositions 7 and 8. Statement (ii) follows from statement (ii) of Proposition 6, and Propositions 7 and 9. □

4. Proof of Theorem 2

In this section we focus on the piecewise polynomial Liénard system (3) with $N=2$ providing the proof of Theorem 2. In this section we take

$$f_1^+(x)=\sum_{i=0}^{n}a_i^+x^i,\qquad f_2^+(x)=\sum_{i=0}^{n}b_i^+x^i,\qquad f_1^-(x)=\sum_{i=0}^{m}a_i^-x^i,\qquad f_2^-(x)=\sum_{i=0}^{m}b_i^-x^i.$$

First we deal with the case of $\alpha=\pi$ where the switching boundary is a straight line.

Proposition 10. *Consider the piecewise polynomial Liénard system* (3) *with $\alpha=\pi$ and $N=2$. For any given $n\geq m\geq 1$ the first* (*respectively second*) *order averaged function has at most $[(n-1)/2]$* (*respectively $n+[(m-1)/2]$*) *simple positive zeros. Moreover, these upper bounds are reached.*

Proof. For sake of convenience we rewrite $f_1^{\pm}(x)$ as

$$f_1^{\pm}(x)=\sum_{j=0}^{k^{\pm}}a_{2j+1}^{\pm}x^{2j+1}+\sum_{j=0}^{l^{\pm}}a_{2j}^{\pm}x^{2j},$$

where $k^+=[(n-1)/2]$, $k^-=[(m-1)/2]$, $l^+=[n/2]$ and $l^-=[m/2]$. Clearly, $k^+\geq k^-$ and $l^+\geq l^-$ due to $n\geq m$. Using polar coordinates system (3) becomes (5) with

$$\begin{aligned}
F_1^{\pm}(\theta,r)&=\cos\theta\left(R^{\pm}(\cos\theta,r)+S^{\pm}(\cos\theta,r)\right),\\
F_2^{+}(\theta,r)&=\sum_{i=0}^{n}b_i^+\cos^{i+1}\theta r^i+\frac{\cos\theta\sin\theta}{r}\left(R^{+}(\cos\theta,r)+S^{+}(\cos\theta,r)\right)^2,\\
F_2^{-}(\theta,r)&=\sum_{i=0}^{m}b_i^-\cos^{i+1}\theta r^i+\frac{\cos\theta\sin\theta}{r}\left(R^{-}(\cos\theta,r)+S^{-}(\cos\theta,r)\right)^2,
\end{aligned}$$

where

$$R^{\pm}(\cos\theta,r)=\sum_{j=0}^{k^{\pm}}a_{2j+1}^{\pm}\cos^{2j+1}\theta r^{2j+1},\qquad S^{\pm}(\cos\theta,r)=\sum_{j=0}^{l^{\pm}}a_{2j}^{\pm}\cos^{2j}\theta r^{2j}.$$

By the definition in (6) and statement (i) of Lemma 5, the first order averaged function is

$$\mathcal{F}_1(r)=\sum_{j=0}^{k^+}a_{2j+1}^{+}\mathcal{C}_{2j+2}(\pi)r^{2j+1}+\sum_{j=0}^{k^-}a_{2j+1}^{-}\mathcal{C}_{2j+2}(\pi)r^{2j+1}.$$

Due to $\mathcal{C}_{2j+2}(\pi)\neq 0$, all coefficients of $\mathcal{F}_1(r)$ can be chosen arbitrarily. By Theorem 4 and $k^+\geq k^-$, eventually, $\mathcal{F}_1(r)$ has at most $k^+=[(n-1)/2]$ simple positive zeros and there is a choice of the parameters $a_i^{\pm}$ such that the maximum is reached.

In order to compute the second order averaged function, we have to take $\mathcal{F}_1=0$, which is obviously equivalent to

$$a_{2j+1}^{+}=-a_{2j+1}^{-}\qquad \text{for } j=0,1,...,k^-,\qquad a_{2j+1}^{+}=0\qquad \text{for } j=k^-+1,...,k^+, \tag{32}$$

because $k^+ > k^-$ and $\mathcal{C}_{2j+2}(\pi) \neq 0$. Notice that this condition implies $R^+(\cos\theta, r) \equiv -R^-(\cos\theta, r)$. We compute the following integrals,

$$\begin{aligned}
\int_0^{\pi} F_2^+(\theta,r)d\theta &= \sum_{i=0}^{n} \mathcal{C}_{i+1}(\pi) b_i^+ r^i + \frac{1}{r}\int_0^{\pi} \cos\theta\sin\theta\left(R^+(\cos\theta,r)+S^+(\cos\theta,r)\right)^2 d\theta \\
&= \sum_{i=0}^{n} \mathcal{C}_{i+1}(\pi) b_i^+ r^i + \frac{1}{r}\int_{-1}^{1} x\left(R^+(x,r)+S^+(x,r)\right)^2 dx \\
&= \sum_{i=0}^{n} \mathcal{C}_{i+1}(\pi) b_i^+ r^i + \frac{4}{r}\int_0^{1} xR^+(x,r)S^+(x,r)dx \\
&= \sum_{i=0}^{n} \mathcal{C}_{i+1}(\pi) b_i^+ r^i + 4\sum_{j=0}^{k^++l^+} \sum_{\substack{j_1+j_2=j \\ 0\le j_1\le k^+, 0\le j_2\le l^+}} \frac{1}{2j+3} a_{2j_1+1}^+ a_{2j_2}^+ r^{2j},
\end{aligned}$$

$$\int_0^{-\pi} F_2^-(\theta,r)d\theta = -\sum_{i=0}^{m} \mathcal{C}_{i+1}(\pi) b_i^- r^i + 4\sum_{j=0}^{k^-+l^-} \sum_{\substack{j_1+j_2=j \\ 0\le j_1\le k^-, 0\le j_2\le l^-}} \frac{1}{2j+3} a_{2j_1+1}^- a_{2j_2}^- r^{2j},$$

and

$$\begin{aligned}
\int_0^{\pm\pi} \partial F_1^{\pm}(\theta,r)\int_0^{\theta} F_1^{\pm}(\varphi,r)d\varphi d\theta =& \int_0^{\pm\pi} \cos\theta\partial R^{\pm}(\cos\theta,r)\int_0^{\theta}\cos\varphi R^{\pm}(\cos\varphi,r)d\varphi d\theta \\
&+ \int_0^{\pm\pi} \cos\theta\partial R^{\pm}(\cos\theta,r)\int_0^{\theta}\cos\varphi S^{\pm}(\cos\varphi,r)d\varphi d\theta \\
&+ \int_0^{\pm\pi} \cos\theta\partial S^{\pm}(\cos\theta,r)\int_0^{\theta}\cos\varphi R^{\pm}(\cos\varphi,r)d\varphi d\theta \\
=& \sum_{j=0}^{2k^{\pm}} \sum_{\substack{j_1+j_2=j \\ 0\le j_1\le k^{\pm}, 0\le j_2\le k^{\pm}}} (2j_1+1)\mathcal{C}_{2j_1+2,2j_2+2}(\pi) a_{2j_1+1}^{\pm} a_{2j_2+1}^{\pm} r^{2j+1} \\
&+ \sum_{j=0}^{k^{\pm}+l^{\pm}} \sum_{\substack{j_1+j_2=j \\ 0\le j_1\le k^{\pm}, 0\le j_2\le l^{\pm}}} (2j_1+1)\mathcal{C}_{2j_1+2,2j_2+1}(\pi) a_{2j_1+1}^{\pm} a_{2j_2}^{\pm} r^{2j} \\
&+ \sum_{j=0}^{k^{\pm}+l^{\pm}} \sum_{\substack{j_1+j_2=j \\ 0\le j_1\le k^{\pm}, 0\le j_2\le l^{\pm}}} 2j_2\mathcal{C}_{2j_2+1,2j_1+2}(\pi) a_{2j_1+1}^{\pm} a_{2j_2}^{\pm} r^{2j}.
\end{aligned}$$

The above computations together with the definition (6) and the condition (32), yield the second order averaged function

$$\begin{aligned}
\mathcal{F}_2(r) =& \sum_{i=0}^{n} \mathcal{C}_{i+1}(\pi) b_i^+ r^i + \sum_{j=0}^{k^-+l^+} \sum_{\substack{j_1+j_2=j \\ 0\le j_1\le k^-, 0\le j_2\le l^+}} \vartheta_{j_1,j_2} a_{2j_1+1}^+ a_{2j_2}^+ r^{2j} \\
&+ \sum_{i=0}^{m} \mathcal{C}_{i+1}(\pi) b_i^- r^i + \sum_{j=0}^{k^-+l^-} \sum_{\substack{j_1+j_2=j \\ 0\le j_1\le k^-, 0\le j_2\le l^-}} \vartheta_{j_1,j_2} a_{2j_1+1}^+ a_{2j_2}^- r^{2j},
\end{aligned}$$

where

$$\vartheta_{j_1,j_2}=\frac{4}{2j+3}+(2j_1+1)\mathcal{C}_{2j_1+2,2j_2+1}(\pi)+2j_2\mathcal{C}_{2j_2+1,2j_1+2}(\pi).$$

It follows from (i) of Lemma 5 that $\mathcal{C}_{i+1}(\pi)=0$ for i even and $\mathcal{C}_{i+1}(\pi)\neq 0$ for i odd, the first and third summations in $\mathcal{F}_2(r)$ contribute only to the odd terms r^{2j+1}, which are from $j=0$ to $[(n-1)/2]$ because $n\geq m$. Moreover the second and fourth ones contribute only to the even terms r^{2j}, which are from $j=0$ to k^-+l^+ because $l^+\geq l^-$. Then the total number of terms is at most $[(n-1)/2]+k^-+l^++2=n+[(m-1)/2]+1$ using the definitions of $k^-$and l^+. Therefore $\mathcal{F}_2(r)$ has at most $n+[(m-1)/2]$ simple positive zeros by Theorem 4.

To get the realization of this upper bound, we prove $\vartheta_{j_1,j_2}\neq 0$ if $j_2\geq j_1$ in what follows. In fact, by (i) and (x) of Lemma 5, ϑ_{j_1,j_2} can be simplified to

$$\vartheta_{j_1,j_2}=\frac{4}{2j+3}+(2j_1+1-2j_2)\mathcal{C}_{2j_1+2,2j_2+1}(\pi).$$

If $j_2=j_1$, we have $\vartheta_{j_1,j_1}>0$ directly, because $\mathcal{C}_{2j_1+2,2j_1+1}(\pi)>0$ from (viii) of Lemma 5. If $j_2=j_1+1$, we have the recursion formula

$$\vartheta_{j_1,j_1+1}=\frac{-4}{(2j_1+3)(4j_1+3)(4j_1+1)}+\frac{2j_1+1}{2j_1+3}\vartheta_{j_1-1,j_1},$$

using property (vii) of Lemma 5. Hence, by induction we can prove that $\vartheta_{j_1,j_1+1}>1/(2j_1+3)^2>0$ for $j_1\geq 0$, where $\vartheta_{0,1}=2/9>1/9$. If $j_2=j_1+2$, we have the recursion formula

$$\vartheta_{j_1,j_1+2}=\frac{4(14j_1+13)}{(2j_1+5)(2j_1+2)(4j_1+5)(4j_1+3)}+\frac{(2j_1+4)(2j_1+1)}{(2j_1+5)(2j_1+2)}\vartheta_{j_1-1,j_1+1},$$

using property (vii) of Lemma 5 again. By induction we can prove that $\vartheta_{j_1,j_1+2}<-1/(j_1+3/2)^2<0$, where $\vartheta_{0,2}=-74/75<-4/9$. If $j_2=j_1+k$ with $k\geq 2$ now, we have

$$\vartheta_{j_1,j_1+k}=\frac{-2(8j_1+4k^2+3)}{(2j_1+2k+1)(2k-3)(4j_1+2k+1)}+\frac{(2j_1+2k)(2k-1)}{(2j_1+2k+1)(2k-3)}\vartheta_{j_1,j_1+k-1}.$$

Since $\vartheta_{j_1,j_1+2}<0$ and we are assuming that $k\geq 2$, we finally get $\vartheta_{j_1,j_1+k}<0$ for $k\geq 2$ by induction again. In conclusion we have that $\vartheta_{j_1,j_2}\neq 0$ if $j_2\geq j_1$.

Let $a^-_{2j_2}=0$ for $j_2=0,1,...,l^-$ and $b^-_i=0$ for $i=0,1,...,m$. Joining the fact that $\mathcal{C}_{i+1}(\pi)=0$ if i is even, the polynomial $\mathcal{F}_2(r)$ reduces to

$$\mathcal{F}_2(r)=\sum_{j=0}^{k^+}\mathcal{C}_{2j+2}(\pi)b^+_{2j+1}r^{2j+1}+\sum_{j=0}^{k^-+l^+}\sum_{\substack{j_1+j_2=j\\0\leq j_1\leq k^-,0\leq j_2\leq l^+}}\vartheta_{j_1,j_2}a^+_{2j_1+1}a^+_{2j_2}r^{2j}. \tag{33}$$

To obtain $n+[(m-1)/2]$ simple positive zeros, we now choose the parameters $a^+_{2j_1+1},a^+_{2j_2}$ and b^+_{2j+1} following the next procedure. We start letting all parameters equal zero except $a^+_0=a^+_1=1$, and then, in the next each step we add the rest of parameters one by one the order $b^+_1,a^+_2,b^+_3,a^+_4,...,b^+_{2k^+-1},a^+_{2k^+},b^+_{2k^++1},a^+_{2l^+},a^+_3,a^+_5,...,a^+_{2k^-+1}$ if n is even, while if n is odd, we need to interchange $a^+_{2l^+}$ and $b^+_{2k^++1}$ because $l^+=k^++1$ (respectively k^+) for n even (respectively odd). The sign of the added parameters must ensure that two adjacent terms have a variation of sign and it can be determined using the fact that $\mathcal{C}_{2j+2}(\pi)>0$ and $\vartheta_{j_1,j_2}>0$ (respectively <0) for $j_1\leq j_2\leq j_1+1$ (respectively $j_2\geq j_1+2$). More precisely, in the first step we add $b^+_1<0$ and one simple positive zero occurs, because $\mathcal{F}_2(r)=\mathcal{C}_2(\pi)b^+_1r+\vartheta_{0,0}$ with $\vartheta_{0,0}>0$ and $\mathcal{C}_2(\pi)>0$. The second one bifurcates from the infinity when we further add the parameter $a^+_2>0$ sufficiently small due to $\vartheta_{0,1}>0$. Then, due to $\mathcal{C}_4(\pi) > 0$, in the third step we add $b_3^+ < 0$ in the way that the coefficient of r^3 is far less than the one of r^2, so the third simple positive zero bifurcates from infinity. Repeating the above procedure with the specific order, we can obtain $n + [(m-1)/2]$ simple positive zeros. $\square$

As an example we give the explicit expression of $\mathcal{F}_2(r)$ in (33) for $n = 9$ and $m = 5$,

$$\begin{aligned}\mathcal{F}_2(r) = {} & 2a_0a_1 + \frac{\pi}{2}b_1 r + (\frac{2}{9}a_1a_2 + 2a_0a_3)r^2 + \frac{3\pi}{8}b_3r^3 + (\frac{14}{15}a_2a_3 - \frac{74}{75}a_1a_4 + 2a_0a_5)r^4 \\ & + \frac{5\pi}{16}b_5r^5 + (\frac{58}{525}a_3a_4 + \frac{26}{21}a_2a_5 - \frac{1426}{735}a_1a_6)r^6 + \frac{35\pi}{128}b_7r^7 \\ & + (\frac{578}{945}a_4a_5 - \frac{702}{1225}a_3a_6 - \frac{38914}{14175}a_1a_8)r^8 + \frac{63\pi}{256}b_9r^9 \\ & + (\frac{1774}{24255}a_5a_6 - \frac{84806}{72765}a_3a_8)r^{10} - \frac{381454}{945945}a_5a_8r^{12},\end{aligned}$$

where we drop the superscript $+$ to alleviate the notation. According to the last proof, we first let all parameters equal zero except $a_0 = a_1 = 1$ and 11 simple positive zeros can bifurcate from the infinity by perturbing the rest parameters following the order $b_1, a_2, b_3, a_4, b_5, a_6, b_7, a_8, b_9, a_3, a_5$.

Now we focus on $\alpha \in (0, \pi)$ where the switching boundary is non-regular.

Proposition 11. *Consider the piecewise polynomial Liénard system* (3) *with* $N = 2$. *For any given* $\alpha \in (0, \pi)$ *and* $n \geq m \geq 1$ *the first order averaged function has at most* n *simple positive zeros, which are reached, and the second order averaged function has at most* $\max\{2m-1, n\}$ (*respectively* $\max\{2m-2, n\}$) *simple positive zeros if* m *is odd* (*respectively even*).

Proof. In polar coordinates system (3) writes in the form (5) with

$$\begin{aligned} F_1^+(\theta, r) &= \sum_{i=0}^{n} a_i^+ \cos^{i+1}\theta r^i, \qquad F_1^-(\theta, r) = \sum_{i=0}^{m} a_i^- \cos^{i+1}\theta r^i, \\ F_2^+(\theta, r) &= \sum_{i=0}^{n} b_i^+ \cos^{i+1}\theta r^i + \frac{1}{r}\cos\theta\sin\theta\left(\sum_{i=0}^{n} a_i^+ \cos^i\theta r^i\right)^2, \\ F_2^-(\theta, r) &= \sum_{i=0}^{m} b_i^- \cos^{i+1}\theta r^i + \frac{1}{r}\cos\theta\sin\theta\left(\sum_{i=0}^{m} a_i^- \cos^i\theta r^i\right)^2. \end{aligned}$$

Thus the first order averaged function is

$$\mathcal{F}_1(r) = \sum_{i=0}^{n} a_i^+ \mathcal{C}_{i+1}(\alpha) r^j - \sum_{i=0}^{m} a_i^- \mathcal{C}_{i+1}(\alpha - 2\pi) r^j.$$

Since $\alpha \in (0, \pi)$, we know $\mathcal{C}_{i+1}(\alpha) \neq 0$ and $\mathcal{C}_{i+1}(\alpha - 2\pi) \neq 0$ for any $i \geq 0$ from (ii) of Lemma 5, so that all coefficients of $\mathcal{F}_1(r)$ are free, i.e. $\mathcal{F}_1(r)$ is a complete polynomial of degree n because $n \geq m$. Therefore $\mathcal{F}_1(r)$ has at most n simple positive zeros and there is a choice of the parameters $a_i^\pm$ such that the maximum is reached.

To get the second order averaged function, we have to take $\mathcal{F}_1 = 0$, which is equivalent to

$$a_i^+\mathcal{C}_{i+1}(\alpha) = a_i^-\mathcal{C}_{i+1}(\alpha-2\pi) \qquad \text{for} \quad i = 0,1,...,m, \text{ and}$$
$$a_i^+ = 0 \qquad \text{for} \quad i = m+1,...,n. \tag{34}$$

A direct computation yields the following integrals

$$\begin{aligned}
\int_0^\alpha F_2^+(\theta,r)d\theta &= \sum_{i=0}^n b_i^+\mathcal{C}_{i+1}(\alpha)r^i + \frac{1}{r}\int_0^\alpha \cos\theta\sin\theta\left(\sum_{i=0}^n a_i^+\cos^i\theta r^i\right)^2 d\theta \\
&= \sum_{i=0}^n b_i^+\mathcal{C}_{i+1}(\alpha)r^i - \frac{1}{r}\int_1^{\cos\alpha} x\left(\sum_{i=0}^n a_i^+x^ir^i\right)^2 dx \\
&= \sum_{i=0}^n b_i^+\mathcal{C}_{i+1}(\alpha)r^i - \frac{1}{r}\sum_{i=0}^{2n}\sum_{\substack{i_1+i_2=i\\0\le i_1,i_2\le n}}\frac{\cos^{i+2}\alpha-1}{i+2}a_{i_1}^+a_{i_2}^+r^i,
\end{aligned}$$

$$\int_0^{\alpha-2\pi} F_2^-(\theta,r)d\theta = \sum_{i=0}^m b_i^-\mathcal{C}_{i+1}(\alpha-2\pi)r^i - \frac{1}{r}\sum_{i=0}^{2m}\sum_{\substack{i_1+i_2=i\\0\le i_1,i_2\le m}}\frac{\cos^{i+2}\alpha-1}{i+2}a_{i_1}^-a_{i_2}^-r^i,$$

and

$$\int_0^\alpha \partial F_1^+(\theta,r)\int_0^\theta F_1^+(\varphi,r)d\varphi d\theta = \sum_{i=0}^{2n}\sum_{\substack{i_1+i_2=i\\0\le i_1,i_2\le n}} i_1\mathcal{C}_{i_1+1,i_2+1}(\alpha)a_{i_1}^+a_{i_2}^+r^{i-1},$$

$$\int_0^{\alpha-2\pi} \partial F_1^-(\theta,r)\int_0^\theta F_1^-(\varphi,r)d\varphi d\theta = \sum_{i=0}^{2m}\sum_{\substack{i_1+i_2=i\\0\le i_1,i_2\le m}} i_1\mathcal{C}_{i_1+1,i_2+1}(\alpha-2\pi)a_{i_1}^-a_{i_2}^-r^{i-1}.$$

Combining the above computations and the condition (34), we get the second order averaged function

$$\mathcal{F}_2(r) = \sum_{i=0}^n b_i^+\mathcal{C}_{i+1}(\alpha)r^i - \sum_{i=0}^m b_i^-\mathcal{C}_{i+1}(\alpha-2\pi)r^i + \frac{1}{r}\sum_{i=0}^{2m}\sum_{\substack{i_1+i_2=i\\0\le i_1,i_2\le m}}\omega_{i_1,i_2}a_{i_1}^+a_{i_2}^+r^i,$$

where

$$\begin{aligned}
\omega_{i_1,i_2} = &-\frac{\cos^{i+2}\alpha-1}{i+2} + i_1\mathcal{C}_{i_1+1,i_2+1}(\alpha) + \frac{\cos^{i+2}\alpha-1}{i+2}\frac{\mathcal{C}_{i_1+1}(\alpha)\mathcal{C}_{i_2+1}(\alpha)}{\mathcal{C}_{i_1+1}(\alpha-2\pi)\mathcal{C}_{i_2+1}(\alpha-2\pi)} \\
&- i_1\mathcal{C}_{i_1+1,i_2+1}(\alpha-2\pi)\frac{\mathcal{C}_{i_1+1}(\alpha)\mathcal{C}_{i_2+1}(\alpha)}{\mathcal{C}_{i_1+1}(\alpha-2\pi)\mathcal{C}_{i_2+1}(\alpha-2\pi)}.
\end{aligned}$$

Using (i) and (ii) of Lemma 5 we have $\mathcal{C}_{i+1}(\alpha) = \mathcal{C}_{i+1}(\alpha-2\pi)$ if i is even, and using (ix) of Lemma 5 we have $\mathcal{C}_{i_1+1,i_2+1}(\alpha) = \mathcal{C}_{i_1+1,i_2+1}(\alpha-2\pi)$ if i_2 is even. So $\omega_{i_1,i_2} = 0$ if both $i_1 \ge 0$ and $i_2 \ge 0$ are even. This implies that the constant term in the third summation of $\mathcal{F}_2(r)$ vanishes and the highest degree term r^{2m} also vanishes when m is even. Consequently due to $n \ge m$, $\mathcal{F}_2(r)$ is a polynomial of degree $\max\{2m-1,n\}$ (respectively $\max\{2m-2,n\}$) when m is odd (respectively even), in other words $\mathcal{F}_2(r)$ has at most $\max\{n,2m-1\}$ (respectively $\max\{n,2m-2\}$) simple positive zeros. □

Finally we can prove Theorem 2.

Proof of Theorem 2. Using the averaging theory it follows statement (i) of Theorem 2 from Proposition 10, and statement (ii) from Proposition 11. □

5. Conclusions

In this paper we extended the weak Hilbert'16 problem to discontinuous piecewise polynomial differential systems with two zones separated by the switching boundary Σ_α, either a straight line when $\alpha = \pi$ or a non-regular one when $\alpha \in (0, \pi)$. Here we allow that the degree of each subsystem in the two zones be different. More precisely, we studied the maximum number of crossing limit cycles bifurcating from the periodic annulus of the linear center $\dot{x} = -y, \dot{y} = -x$ when we perturb it inside this class of general piecewise polynomial differential systems. Depending on $\alpha = \pi$ or not, we provided upper bounds for the maximum number using the averaging method up to any order. Besides we also restricted the perturbation to the family of piecewise polynomial Liénard systems and some better upper bounds were given. As observed by many researchers, our results emphasize the importance of the shape of the switching boundary in the investigation of limit cycles.

We proved that all upper bounds obtained with the first order averaging method are reached. Regarding the upper bounds obtained with the second order averaging method, the main difficulty in the study of the realization, is to determine which terms of the corresponding averaged functions are not identically zero and then to give a suitable choice of parameters. Overcoming these difficulties, for $\alpha = \pi$ we proved the realization whatever the perturbation is inside the general piecewise polynomial family or piecewise Liénard one, while for $\alpha \in (0, \pi)$, the realization was obtained only in the case of the general piecewise polynomial perturbations with $\alpha \in (0, \pi/2]$. Hence, what about the general piecewise polynomial perturbations with $\alpha \in (\pi/2, \pi)$ and the piecewise Liénard perturbations with $\alpha \in (0, \pi)$? These need to be answered in the future.

Acknowledgements

The first author is partially suported by the grant CSC #201906240094 from the P.R. China. The second author is partially supported by the Ministerio de Ciencia, Innovación y Universidades, Agencia Estatal de Investigación grants MTM2016-77278-P (FEDER), the Agència de Gestió d'Ajuts Universitaris i de Recerca grant 2017SGR1617, and the H2020 European Research Council grant MSCA-RISE-2017-777911.

References

[1] V.I. Arnold, Ten problems, *Adv. Soviet Math.* **1** (1990), 1–8.
[2] I.S. Berezin, N.P. Zhidkov, *Computing Methods*, Reading, Mass.-London, 1965
[3] M. di Bernardo, C.J. Budd, A.R. Champneys, P. Kowalczyk, *Piecewise-Smooth Dynamical systems: Theory and Applications*, Applied Mathematical Sciences, Vol.163 (Springer Verlag, London), 2008.
[4] A. Buică, On the equivalence of the Melnikov functions method and the averaging method, *Qual. Theory Dyn. Syst.* **16** (2017), 547–560.
[5] C.A. Buzzi, M.F.S. Lima, J. Torregrosa, Limit cycles via higher order perturbations for some piecewise differential systems, *Physica D* **371** (2018), 28–47.
[6] C.A. Buzzi, C. Pessoa, J. Torregrosa, Piecewise linear perturbations of a linear center, *Discrete Contin. Dyn. Syst.* **9** (2013), 3915–3936.

[7] P.T. Cardin, J. Torregrosa, Limit cycles in planar piecewise linear differential systems with nonregular separation line, *Physica D* **337** (2016), 67–82.
[8] H. Chen, S. Duan, Y. Tang, J. Xie, Global dynamics of a mechanical system with dry friction, *J. Differential Equations* **265** (2018), 5490–5519.
[9] G. Dong, C. Liu, Note on limit cycles for m-piecewise discontinuous polynomial Liénard differential equations, *Z. Angew. Math. Phys.* **68**(2017), No. 97, 8 pp.
[10] A.F. Filippov, *Differential Equations with Discontinuous Righthand Sides*, Kluwer Academic Publishers, Dordrecht, 1988.
[11] E. Freire, E. Ponce, F. Torres, Canonical discontinuous planar piecewise linear systems, *SIAM J. Appl. Dyn. Syst.* **11**(2012), 181–211.
[12] A. Gasull, J. Torregrosa, A relation between small amplitude and big limit cycles, *Rocky. Mountain J. Math.* **31** (2001), 1277–1303.
[13] M. Han, V.G. Romanovski, X. Zhang, Equivalence of the Melnikov function method and the averaging method, *Qual. Theory Dyn. Syst.* **15** (2016), 471–479.
[14] I.D. Iliev, The number of limit cycles due to polynomial perturbations of the harmonic oscillator, *Math. Proc. Cambridge Philos. Soc.* **127** (1999), 317–322.
[15] J. Itikawa, J. Llibre, D.D. Novaes, A new result on averaging theory for a class of discontinuous planar differential systems with applications, *Rev. Mat. Iberoam.* **33** (2017), 1247–1265.
[16] F. Jiang, Z. Ji, Y. Wang, On the number of limit cycles of discontinuous Liénard polynomial differential systems, *Int. J. Bifur. Chaos* **28** (2018), 1850175.
[17] P. Kowalczyk, P. T. Piiroinen, Two-parameter sliding bifurcation of periodic solutions in a dry-friction oscillator, *Physica D* **237**(2008), 1053–1073.
[18] J. Li, Hilbert's 16th problem and bifurcations of planar polynomial vector fields, *Int. J. Bifur. Chaos* **13** (2003), 47–106.
[19] T. Li, H. Chen, X. Chen, Crossing periodic orbits of nonsmooth Liénard systems and applications, to appear, 2020.
[20] A. Lins Neto, W. de Melo, C.C. Pugh, On Liénard equations, in: Proc. Symp. Geom. and topol, in: Lectures Notes in Math., vol. 597, Springer-Verlag, 1977, pp. 335–357.
[21] J. Llibre, A.C. Mereu, D.D. Novaes, Averaging theory for discontinuous piecewise differential systems, *J. Differential Equations* **258** (2015), 4007–4032.
[22] J. Llibre, D.D. Novaes, C.A.B. Rodrigues, Averaging theory at any order for computing limit cycles of discontinuous piecewise differential systems with many zones, *Physica D* **353-354** (2017), 1–10.
[23] J. Llibre, E. Ponce, F. Torres, On the existence and uniqueness of limit cycles in Liénard differential equations allowing discontinuities, *Nonlinearity* **21**(2008), 2121–2142.
[24] J. Llibre, Y. Tang, Limit cycles of discontinuous piecewise quadratic and cubic polynomial perturbations of a linear center, *Discrete Contin. Dyn. Syst. Ser. B* **24** (2019), 1769–1784.
[25] J. Llibre, M.A. Teixeira, Limit cycles for m-piecewise discontinuous polynomial Liénard differential equations, *Z. Angew. Math. Phys.* **66** (2015), 51–66.
[26] O. Makarenkov, J.S.W. Lamb, Dynamics and bifurcations of nonsmooth systems: a survey, *Physica D* **241** (2012), 1826–1844.
[27] R.M. Martins, A.C. Mereu, Limit cycles in discontinuous classical Liénard equations, *Nonlin. Anal. Real World Appl.* **20** (2014), 67–73.
[28] D.D. Novaes, On nonsmooth perturbations of nondegenerate planar centers, *Publ. Mat.* Extra (2014), 395–420.
[29] J. A. Sanders, F. Verhulst and J. Murdock, *Averaging Methods in Nonlinear Dynamical Systems*, Second edition, Applied Mathematical Sciences 59, Springer, New York, 2007.
[30] L. Sheng, Limit cycles of a class of piecewise smooth Liénard systems, *Int. J. Bifur. Chaos* **26** (2016), 1650009.
[31] Y. Tian, M. Han, F. Xu, Bifurcations of small limit cycles in Liénard systems with cubic restoring terms, *J. Differential Equations* **267** (2019), 1561–1580.
[32] A. Tonnelier, W. Gerstner, W. Piecewise-linear differential equations and integrate-and-fire neurons: Insights from two-dimensional membrane models, *Phys. Rev. E* **67** (2003), 021908.
[33] L. Wei, X. Zhang, Averaging theory of arbitrary order for piecewise smooth differential systems and its application, *J. Dyn. Diff. Equat.* **30** (2018), 55–79.

[1] Department of Mathematics, Sichuan University, 610064 Chengdu, Sichuan, P.R. China

Email address: `t.l.scumath@gmail.com`

[2] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

Email address: jllibre@mat.uab.cat