ON THE 16-TH HILBERT PROBLEM FOR DISCONTINUOUS PIECEWISE POLYNOMIAL HAMILTONIAN SYSTEMS

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Abstract. In this paper we study the maximum number of limit cycles of the discontinuous piecewise differential systems with two zones separated by the straight line $y = 0$, in $y \geq 0$ there is a polynomial Hamiltonian system of degree m, and in $y \leq 0$ there is a polynomial Hamiltonian system of degree n.

First for this class of discontinuous piecewise polynomial Hamiltonian systems, which are perturbation of a linear center, we provide a sharp upper bound for the maximum number of the limit cycles that can bifurcate from the periodic orbits of the linear center using the averaging theory up to any order.

After for the general discontinuous piecewise polynomial Hamiltonian systems we also give an upper bound for their maximum number of limit cycles in function of m and n. Moreover, this upper bound is reached for some degrees of m and n.

1. Introduction and statement of main results

One of the most studied problems in the qualitatitve theory of the differential equations in the plane is to identify the maximum number of limit cycles that can exhibit a given class of differential systems. Thus a famous and challenging question is the Hilbert's 16th problem [22], which was proposed in 1900. In the second part of this question, Hilbert asked what is the maximum number of limit cycles that planar polynomial differential system of a given degree may have. Since 1900 many researchers are dedicated to study this problem and some excellent results were obtained, see for instance the survey paper [26]. But this question is far from being answered up to now, even for quadratic polynomial differential systems. Let $H(n)$ be the maximum number of limit cycles that a planar polynomial differential system of degree n may have. Sometimes, $H(n)$ is called *Hilbert number*. As far as we are concerned, the existing results showed $H(2) \geq 4, H(3) \geq 13, H(4) \geq 21, H(5) \geq 33, H(6) \geq 44, H(7) \geq 65, H(8) \geq 76$, etc, see [15, 17, 20, 27, 19, 33, 37, 39]. For n sufficiently large it is known that

$$H(n) \geq \frac{(n+2)^2 \ln(n+2)}{2 \ln 2},$$

see [10, 26, 19].

In these last twenty years an increasing interest has appeared for studying the discontinuous piecewise smooth differential systems, stimulated by lots of nonsmooth or discontinuous phenomena that come from mechanical engineering with dry frictions, feedback control systems, electrical circuits with switches, neuron models, biology, see for example

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[1, 3, 7, 8, 32, 34, 35, 36] and the references therein. Furthermore, a particular interest is paid to the following discontinuous piecewise polynomial differential system

$$(\dot{x},\dot{y}) = \begin{cases} Z^{+}(x,y) & \text{if} \quad y \geq 0, \\ Z^{-}(x,y) & \text{if} \quad y \leq 0, \end{cases} \tag{1}$$

where $(x,y) \in \mathbb{R}^2$ and $Z^{\pm} : \mathbb{R}^2 \to \mathbb{R}^2$ are two real polynomial vector fields. Here the x-axis is called the *switching line* or the *discontinuity boundary*. Throughout this paper we define the vector field on the switching line by Z^+, or equivalently Z^-, if $Z^+ \equiv Z^-$ on the entire switching line, so that the orbits of system (1) can be defined as in continuous systems. However, if $Z^+ \neq Z^-$ on the switching line, we adopt the so-called *Filippov convention* to define the orbits of the discontinuous piecewise differential system, see [12, 25] for more details.

A *crossing periodic orbit* of a discontinuous piecewise polynomial differential systems (1) is a periodic orbit which intersect the swithching line at two crossing points. Here a point p in the switching line is said to be a *crossing point* if $Z_2^+(p)Z_2^-(p) > 0$, where $Z_2^{\pm}$ is the second coordinate of $Z^{\pm}$. A crossing periodic orbit isolated in the set all crossing periodic orbits is called a *crossing limit cycle*.

As in polynomial differential systems, we can also consider the Hilbert's 16th problem for discontinuous piecewise polynomial differential systems (1), i.e. what is the maximum number of crossing limit cycles that systems (1) can exhibit in function of the degrees of the polynomial vector fields $Z^{\pm}$. Compared with polynomial differential systems, the determination of the Hilbert number of discontinuous piecewise polynomial systems (1) is more difficult, even for the discontinuous piecewise linear differential systems. It was conjectured in [21] that a discontinuous piecewise linear differential system has at most 2 crossing limit cycles. A negative answer to this conjecture was given in [23] by a numerical example with 3 crossing limit cycles, the first proof with 3 crossing limit cycles was given in [30]. Up to now, the best result is still 3 for discontinuous piecewise linear systems, see [6, 13, 29] for more systems with three crossing limit cycles. Recently, it was proved in [11] that discontinuous piecewise quadratic differential systems of form (1) can have 16 crossing limit cycles through perturbing a quadratic isochronous center. Regarding discontinuous piecewise cubic systems, 18 crossing limit cycles were obtained in [18], but this number was updated as 24 in [16] later on.

We point out that the averaging theory, which has been extended recently for discontinuous piecewise smooth differential systems [4, 24, 28, 38], is an important tools to study crossing limit cycles bifurcating from a periodic annulus. For instance using the averaging theory it was proved in [9] that the cyclicity of a Hopf bifurcation for planar linear-quadratic discontinuous polynomial differential systems is at least 5. In [31] the maximum number of crossing limit cycles of discontinuous discontinuous piecewise quadratic and cubic polynomial perturbations of a linear center was computed using the averaging theory of order n for $n = 1, 2, 3, 4, 5$. In [11] the averaging theory of order two was applied to discontinuous piecewise quadratic perturbations of quadratic isochronous centers. More results about limit cycles bifurcating from the periodic annulus of a linear center can be found in [5].

It is well known that limit cycles cannot exist in polynomial Hamiltonian systems. But it is possible for discontinuous piecewise polynomial Hamiltonian systems to have crossing limit cycles as it was studied in [40], where a lower bound and an upper bound for the maximum number of small amplitude crossing limit cycles bifurcating from a non-smooth focus were provided. Motivated by the work [40], in this paper we further study the maximum number of crossing limit cycles of discontinuous piecewise polynomial Hamiltonian systems. This can be regarded as an extension of the Hilbert's 16th problem to discontinuous piecewise polynomial Hamiltonian systems.

Let $H^+(x,y)$ and $H^-(x,y)$ be two real polynomials of degree $m+1$ and $n+1$ given by

$$H^+(x,y) = \sum_{i+j=1}^{m+1} a_{ij}^+ x^i y^j, \qquad H^-(x,y) = \sum_{i+j=1}^{n+1} a_{ij}^- x^i y^j. \tag{2}$$

First we consider a discontinuous piecewise polynomial Hamiltonian perturbation of the linear center $\dot{x} = -y, \dot{y} = x$, namely

$$(\dot{x}, \dot{y}) = \begin{cases} (-y - \varepsilon H_y^+(x,y), x + \varepsilon H_x^+(x,y)) & \text{if } y \geq 0, \\ (-y - \varepsilon H_y^-(x,y), x + \varepsilon H_x^-(x,y)) & \text{if } y \leq 0, \end{cases} \tag{3}$$

where $\varepsilon \in \mathbb{R}$ is a perturbation parameter with $|\varepsilon| > 0$ sufficiently small, the subscripts x and y denote the derivatives with respect to x and y respectively. Notice that system (3) is a discontinuous piecewise polynomial Hamiltonian system with the first integrals

$$I^+(x,y) = \frac{1}{2}(x^2 + y^2) + \varepsilon H^+(x,y), \qquad I^-(x,y) = \frac{1}{2}(x^2 + y^2) + \varepsilon H^-(x,y).$$

In this paper our first goal is to study the maximum number of crossing limit cycles of system (3), which can bifurcate from the period annulus of the linear center $\dot{x} = -y, \dot{y} = x$.

Before stating the main theorem we give the following useful result.

Proposition 1. *For $|\varepsilon| > 0$ sufficiently small the number of crossing periodic orbits for the discontinuous piecewise polynomial Hamiltonian system* (3) *and for the discontinuous piecewise polynomial Hamiltonian system*

$$(\dot{x}, \dot{y}) = \begin{cases} (-y, x + \varepsilon H_x^+(x,0)) & \text{if } y \geq 0, \\ (-y, x + \varepsilon H_x^-(x,0)) & \text{if } y \leq 0, \end{cases} \tag{4}$$

are the same.

According to Proposition 1, it is enough to consider system (4) in order to obtain the maximum number of crossing limit cycles that system (3) can have for $|\varepsilon| > 0$ sufficiently small. Applying the averaging method, up to any order, to system (4) we obtain the next result.

Theorem 2. *For $|\varepsilon| > 0$ sufficiently small the maximum number of crossing limit cycles, bifurcating from the periodic orbits of the linear differential center $\dot{x} = -y$, $\dot{y} = x$, which can be obtained with the averaging theory for the discontinuous piecewise polynomial Hamiltonian system* (4) *with $m \leq n$ is at most*

$$\max\left\{ \left[\frac{n}{2}\right], \left[\frac{n-1}{2}\right] + \left[\frac{m}{2}\right] \right\},$$

where $[z]$ denote the integer part function of $z \in \mathbb{R}$. Moreover this maximum is achievable.

Our second goal deals with the maximum number of crossing limit cycles for the general discontinuous piecewise polynomial Hamiltonian system

$$(\dot{x}, \dot{y}) = \begin{cases} (-H_y^+(x,y), H_x^+(x,y)) & \text{if } \quad y > 0, \\ (-H_y^-(x,y), H_x^-(x,y)) & \text{if } \quad y < 0, \end{cases} \tag{5}$$

where $H^{\pm}(x,y)$ are given in (2). The following theorem provides an upper bound for a such maximum.

Theorem 3. *The discontinuous piecewise polynomial Hamiltonian system* (5) *with* $m \leq n$ *has at most*

$$\left[\frac{mn}{2}\right] \qquad \left(resp. \quad \left[\frac{n(n-1)}{2}\right]\right)$$

crossing limit cycles if $m < n$ *(resp.* $m = n$*).*

It is worth mentioning that Yang, Han and Huang in the proof of [40, Theorem 1.2] also observed that $[nm/2]$ is an upper bound for the maximum number of crossing limit cycles that discontinuous piecewise polynomial Hamiltonian system (5) can have. However, we provide a better upper bound for the case of $m = n$ in Theorem 3, because $[n(n-1)/2] \leq [n^2/2]$.

Regarding the lower bound, it follows from [40, Theorem 1.2] that $[(n-1)/2] + [m/2]$ small amplitude crossing limit cycles can bifurcate from a non-smooth focus if $m \leq n$, so that $[(n-1)/2] + [m/2]$ is a lower bound for the maximum number of crossing limit cycles that system (5) with $m \leq n$ can have. According to Theorem 2, we can update the lower bound as

$$\max\left\{\left[\frac{n}{2}\right], \left[\frac{n-1}{2}\right] + \left[\frac{m}{2}\right]\right\}.$$

Although the lower bounds given in [40, Theorem 1.2] and Theorem 2 are the same for $m \geq 2$, the methods used to obtain the bounds are different. Our method, based in the application of the averaging theory to discontinuous piecewise polynomial Hamiltonian perturbations of a linear center, allows to obtain large amplitude crossing limit cycles. Thus, in terms of amplitude, we give a new lower bound for the maximum number of large amplitude crossing limit cycles.

More importantly, combining Theorems 2 and 3 we directly get the exact maximum number of crossing limit cycles for some classes of discontinuous piecewise polynomial Hamiltonian systems (5) as it is stated in the following result.

Theorem 4. *Consider the discontinuous piecewise polynomial Hamiltonian system* (5) *with* $m \leq n$*. The following statements hold.*

- (i) *If* $m = 1$*, the maximum number of crossing limit cycles that system* (5) *can have is exactly* $[n/2]$.
- (ii) *If* $m = n = 2$*, the maximum number of crossing limit cycles that system* (5) *can have is exactly* 1.

This paper is organized as follows. Section 2 contains a brief review for the averaging theory of any order. In Section 3 we study the perturbation problem (3) and give the proofs of Proposition 1 and Theorem 2. Section 4 is devoted to the proof of Theorem 3.

2. Preliminaries

In this section we recall the averaging theory and some important theorems in order to prove our main results.

As a main tool studying limit cycle bifurcations the averaging theory has been generalized for discontinuous piecewise smooth differential systems, see [24, 38, 28]. In what follows we shall state the averaging method of any order by considering the perturbed discontinuous piecewise smooth differential system

$$(\dot{x},\dot{y}) = \begin{cases} (-y+\varepsilon f^{+}(x,y), x+\varepsilon g^{+}(x,y)) & \text{if} \quad y \geq 0, \\ (-y+\varepsilon f^{-}(x,y), x+\varepsilon g^{-}(x,y)) & \text{if} \quad y \leq 0, \end{cases} \tag{6}$$

where $f^{\pm}, g^{\pm} : \mathbb{R}^2 \to \mathbb{R}$ are real polynomials, and $\varepsilon \in \mathbb{R}$ is a perturbation parameter.

Using the polar coordinates $(x,y) = (r\cos\theta, r\sin\theta)$ and taking θ as the new independent variable, for any given $N \in \mathbb{N}^{+}$ we transform system (6) into

$$\frac{dr}{d\theta} = \begin{cases} \displaystyle\sum_{i=1}^{N} \varepsilon^i F_i^{+}(\theta,r) + \varepsilon^{N+1} R^{+}(\theta,r,\varepsilon) & \text{if} \quad 0 \leq \theta \leq \pi, \\ \displaystyle\sum_{i=1}^{N} \varepsilon^i F_i^{-}(\theta,r) + \varepsilon^{N+1} R^{-}(\theta,r,\varepsilon) & \text{if} \quad -\pi \leq \theta \leq 0, \end{cases} \tag{7}$$

where $F_i^{\pm} : [0,2\pi] \times (0,+\infty) \to \mathbb{R}$, $i = 1,2,...,N$, are analytical functions of period 2π in the variable θ and are given by

$$F_1^{\pm}(\theta,r) = \cos\theta f^{\pm}(r\cos\theta, r\sin\theta) + \sin\theta g^{\pm}(r\cos\theta, r\sin\theta),$$
$$F_i^{\pm}(\theta,r) = F_1^{\pm}(\theta,r)\left(-\frac{\cos\theta g^{\pm}(r\cos\theta, r\sin\theta) - \sin\theta f^{\pm}(r\cos\theta, r\sin\theta)}{r}\right)^{i-1}.$$

According to the results of [24] we get that the *averaged function* $\mathcal{F}_i(r) : (0,+\infty) \to \mathbb{R}$ *of order* $i = 1,2,...,N$ is

$$\mathcal{F}_i(r) = \frac{y_i^{+}(\pi,r) - y_i^{-}(-\pi,r)}{i!} \qquad i = 1,2,...,N. \tag{8}$$

The functions $y_i^{\pm} : [0,2\pi] \times (0,+\infty) \to \mathbb{R}$ are defined recurrently as

$$\begin{aligned} y_1^{\pm}(\theta,r) &= \int_0^{\theta} F_1^{\pm}(\varphi,r)d\varphi, \\ y_i^{\pm}(\theta,r) &= i!\int_0^{\theta}\Bigg(F_i^{\pm}(\varphi,r) + \\ &\qquad \sum_{l=1}^{i}\sum_{S_l} \frac{1}{b_1!b_2!2!^{b_2}\cdots b_l!l!^{b_l}} \frac{\partial^L}{\partial r^L} F_{i-l}^{\pm}(\varphi,r) \prod_{j=1}^{l} y_j^{\pm}(\varphi,r)^{b_j}\Bigg)d\varphi, \end{aligned} \tag{9}$$

where S_l is the set of all l-tuples of non-negative integers $(b_1,b_2,...,b_l)$ satisfying $b_1 + 2b_2 + \cdots + lb_l = l$, and $L = b_1 + b_2 + \cdots + b_l$. Moreover we are assuming that $F_0^{\pm} = 0$ in (9) for convenience.

The next theorem proved in [24] implies that we can study the zeros of these averaged functions in order to obtain crossing limit systems of system (6) bifurcating from the periodic annulus of the linear center $\dot{x} = -y, \dot{y} = x$.

Theorem 5. *Consider the discontinuous piecewise smooth differential system* (7). *Suppose that* i_0 *is the first integer such that* $\mathcal{F}_i = 0$ *for* $1 \leq j \leq i_0 - 1$ *and* $\mathcal{F}_{i_0} \neq 0$. *If* $\mathcal{F}_{i_0}(\rho) = 0$ *and* $\mathcal{F}'_{i_0}(\rho) \neq 0$ *for some* $\rho \in (0,+\infty)$, *then for* $|\varepsilon| > 0$ *sufficiently small there exists a* 2π*-periodic solution* $r(\theta,\varepsilon)$ *of system* (7) *such that* $r(0,\varepsilon) \to \rho$ *as* $\varepsilon \to 0$.

The following two theorems will be used to prove our main results, see [2] and [14] for their proofs.

Theorem 6 (**Descartes Theorem**). *Consider the real polynomial* $p(x) = a_{i_1 1}x^{i_1} + a_{i_2}x^{i_2} + \dots + a_{i_r}x^{i_r}$ *with* $0 = i_1 < i_2 < \dots < i_r$ *with* $r > 1$. *If* $a_{i_j}a_{i_{j+1}} < 0$, *we say that* a_{i_j} *and* $a_{i_{j+1}}$ *have a variation of sign. If the number of variations of signs is* $r_0 \in \{0, 1, 2, \dots, r-1\}$, *then the polynomial* $p(x)$ *has at most* r_0 *positive real roots. Furthermore, we can choose the coefficients of the polynomial* $p(x)$ *in such a way that* $p(x)$ *has exactly* $r-1$ *positive real roots.*

Theorem 7 (**Bezout Theorem**). *Let* $F(x, y)$ *and* $G(x, y)$ *be two real polynomials. If both polynomials do not share a non-trivial common factor, then the system of equations*

$$F(x, y) = 0, \qquad G(x, y) = 0$$

has at most $\deg F \deg G$ *solutions.*

3. Proofs of Proposition 1 and Theorem 2

In this section we study the maximum number of crossing limit cycles bifurcating from the period annulus of the linear center $\dot{x} = -y, \dot{y} = x$ via discontinuous piecewise polynomial Hamiltonian perturbations. We start with the proof of Proposition 1.

Proof of Proposition 1. Assume that system (3) for $|\varepsilon| > 0$ sufficiently small has a crossing limit cycle $\Gamma(\varepsilon)$. Then $\Gamma(\varepsilon) \to \Gamma_0$ as $\varepsilon \to 0$, where Γ_0 is a periodic orbit in the period annulus of the linear center $\dot{x} = -y, \dot{y} = x$. So $\Gamma(\varepsilon)$ has exactly two intersections with the x-axis, denoted by $(p_1(\varepsilon), 0)$ and $(p_2(\varepsilon), 0)$ with $p_1(\varepsilon) < 0 < p_2(\varepsilon)$. Let $(p_0, 0)$ be the intersection between Γ_0 and the positive x-axis, we have $p_1(\varepsilon) \to -p_0$ and $p_2(\varepsilon) \to p_0$ as $\varepsilon \to 0$.

We claim that system (4) also has a crossing limit cycle that intersects the x-axis at $(p_1(\varepsilon), 0)$ and $(p_2(\varepsilon), 0)$ for $|\varepsilon| > 0$ sufficiently small. In fact, since the first integrals of the left and right systems of (3) are

$$I^+(x, y) = \frac{1}{2}(x^2 + y^2) + \varepsilon H^+(x, y), \qquad I^-(x, y) = \frac{1}{2}(x^2 + y^2) + \varepsilon H^-(x, y),$$

respectively, the points $(p_1(\varepsilon), 0)$ and $(p_2(\varepsilon), 0)$ satisfy

$$I^+(p_1(\varepsilon), 0) = I^+(p_2(\varepsilon), 0), \qquad I^-(p_1(\varepsilon), 0) = I^-(p_2(\varepsilon), 0),$$

that is,

$$\begin{aligned} \frac{(p_1(\varepsilon))^2}{2} + \varepsilon H^+(p_1(\varepsilon), 0) &= \frac{(p_2(\varepsilon))^2}{2} + \varepsilon H^+(p_2(\varepsilon), 0), \\ \frac{(p_1(\varepsilon))^2}{2} + \varepsilon H^-(p_1(\varepsilon), 0) &= \frac{(p_2(\varepsilon))^2}{2} + \varepsilon H^-(p_2(\varepsilon), 0). \end{aligned} \tag{10}$$

On the other hand, the first integrals of the left and right systems of (4) are

$$I_0^+(x, y) = \frac{1}{2}(x^2 + y^2) + \varepsilon H^+(x, 0), \quad I_0^-(x, y) = \frac{1}{2}(x^2 + y^2) + \varepsilon H^-(x, 0),$$

respectively. Therefore it follows from (10) that

$$I_0^+(p_1(\varepsilon), 0) = I_0^+(p_2(\varepsilon), 0), \qquad I_0^-(p_1(\varepsilon), 0) = I_0^-(p_2(\varepsilon), 0).$$

Since $p_1(\varepsilon) \to -p_0$ and $p_2(\varepsilon) \to p_0$ as $\varepsilon \to 0$, for $|\varepsilon| > 0$ sufficiently small we get that $(p_1(\varepsilon), 0)$ and $(p_2(\varepsilon), 0)$ are crossing points of system (4), and they lie in the same orbit for both the left and right systems of (4). Consequently, for $|\varepsilon| > 0$ sufficiently small, system (4) also has a crossing limit cycle passing through $(p_1(\varepsilon), 0)$ and $(p_2(\varepsilon), 0)$. That is, this claim is proved.

Similarly we can prove that system (3) also has a crossing limit cycle if system (4) also has it. This ends the proof of Proposition 1. □

As it was indicated in Proposition 1 it is enough to consider system (4) in order to obtain the maximum number of crossing limit cycles that system (3) can have for $|\varepsilon| > 0$ sufficiently small. Next we will apply the averaging method of any order to system (4) in subsections 3.1, 3.2, 3.3.

For sake of simplicity, we write $H_0^{\pm}(x) = H^{\pm}(x, 0)$, i.e.

$$H_0^{\pm}(x) = \sum_{i=1}^{m+1} a_i^{\pm} x^i, \qquad H_0^{-}(x) = \sum_{i=1}^{n+1} a_i^{-} x^i,$$

and $a_i^{\pm} = a_{i0}^{\pm}$. More precisely, we write

$$H_0^{\pm}(x) = \sum_{j=0}^{k^{\pm}} a_{2j+1}^{\pm} x^{2j+1} + \sum_{j=0}^{l^{\pm}} a_{2j+2}^{\pm} x^{2j+2}, \tag{11}$$

where

$$\begin{aligned}
k^{+} &= \frac{m-1}{2}, & l^{+} &= \frac{m-1}{2} & &\text{if } m \text{ is odd,}\\
k^{+} &= \frac{m}{2}, & l^{+} &= \frac{m-2}{2} & &\text{if } m \text{ is even,}\\
k^{-} &= \frac{n-1}{2}, & l^{-} &= \frac{n-1}{2} & &\text{if } n \text{ is odd,}\\
k^{-} &= \frac{n}{2}, & l^{-} &= \frac{n-2}{2} & &\text{if } n \text{ is even.}
\end{aligned} \tag{12}$$

3.1. **First order averaging method.** Applying the first order averaging method to system (4) we have the following.

Proposition 8. *For $|\varepsilon| > 0$ sufficiently small the first order averaging method predicts at most $[n/2]$ crossing limit cycles for the discontinuous piecewise polynomial Hamiltonian system (4) with $m \le n$. Moreover this number is reachable.*

Proof. Writing the discontinuous piecewise polynomial Hamiltonian system (4) in the form of system (7) and using the expression (11), we get

$$\begin{aligned}
F_1^{\pm}(\theta, r) = {} & \sin\theta H_{0x}^{\pm}(r\cos\theta)\\
= {} & \sin\theta \textstyle\sum_{j=0}^{k^{\pm}} (2j+1) a_{2j+1}^{\pm} (r\cos\theta)^{2j} + \sin\theta \sum_{j=0}^{l^{\pm}} (2j+2) a_{2j+2}^{\pm} (r\cos\theta)^{2j+1}.
\end{aligned}$$

Then, according to (8) and (9), the first order averaged function is

$$
\begin{aligned}
\mathcal{F}_1(r) &= \int_0^{\pi} F_1^+(\theta, r)d\theta - \int_0^{-\pi} F_1^-(\theta, r)d\theta \\
&= 2\sum_{j=0}^{k^+} a_{2j+1}^+ r^{2j} - 2\sum_{j=0}^{k^-} a_{2j+1}^- r^{2j} \\
&= 2\sum_{j=0}^{k^+} (a_{2j+1}^+ - a_{2j+1}^-) r^{2j} - 2\sum_{j=1+k^+}^{k^-} a_{2j+1}^- r^{2j}.
\end{aligned} \tag{13}
$$

Here the last equality is due to $k^+ \le k^-$ under the assumption $m \le n$. Clearly $\mathcal{F}_1(r)$ is a polynomial of degree $2k^-$, and all odd order terms vanish. This means that $\mathcal{F}_1(r)$ has at most k^- positive real simple zeros. Since all coefficients are free, we can choose them such that this number is reachable. By Theorem 5 this proposition is proved because $k^- = [n/2]$ as defined in (12). □

3.2. **Second order averaging method.** In order to apply the second order averaging method to system (4), we have to take $\mathcal{F}_1 = 0$. From (13) this is equivalent to take

$$
\begin{aligned}
&a_{2j+1}^- = a_{2j+1}^+ \qquad \text{for} \quad 0 = 1, 2, ..., k^+, \text{ and} \\
&a_{2j+1}^- = 0 \qquad \text{for} \quad j = 1 + k^+, 2 + k^+, ..., k^-.
\end{aligned} \tag{14}
$$

In particular we have the following.

Proposition 9. *For $|\varepsilon| > 0$ sufficiently small the second order averaging method predicts at most $[(n-1)/2] + [m/2]$ crossing limit cycles for the discontinuous piecewise polynomial Hamiltonian system (4) with $m \le n$. Moreover this number is reachable.*

Proof. Writing the discontinuous piecewise polynomial Hamiltonian system (4) into the form of system (7) and using (11) and (14), we have

$$
\begin{aligned}
F_1^{\pm}(\theta, r) &= \sin\theta \left(R(\cos\theta, r) + S^{\pm}(\cos\theta, r)\right), \\
F_2^{\pm}(\theta, r) &= -\frac{\cos\theta\sin\theta}{r}\left(R(\cos\theta, r) + S^{\pm}(\cos\theta, r)\right)^2,
\end{aligned}
$$

where

$$
R(\cos\theta, r) = \sum_{j=0}^{k^+} (2j+1) a_{2j+1}^+ (r\cos\theta)^{2j}, \quad S^{\pm}(\cos\theta, r) = \sum_{j=0}^{l^{\pm}} (2j+2) a_{2j+2}^{\pm} (r\cos\theta)^{2j+1}.
$$

Then

$$
\begin{aligned}
y_1^{\pm}(\theta, r) &= \int_0^{\theta} F_1^{\pm}(\varphi, r)d\varphi \\
&= \int_0^{\theta} \sin\varphi (R(\cos\varphi, r) + S^{\pm}(\cos\varphi, r))d\varphi \\
&= -\int_0^{\theta} (R(\cos\varphi, r) + S^{\pm}(\cos\varphi, r))d\cos\varphi \\
&= \sum_{j=0}^{k^+} a_{2j+1}^+ r^{2j}(1 - (\cos\theta)^{2j+1}) + \sum_{j=0}^{l^{\pm}} a_{2j+2}^{\pm} r^{2j+1}(1 - (\cos\theta)^{2j+2}),
\end{aligned} \tag{15}
$$

and

$$\begin{aligned}\frac{\partial F_1^{\pm}(\theta,r)}{\partial r} &= \sin\theta\frac{\partial R(\cos\theta,r)}{\partial r}+\sin\theta\frac{\partial S^{\pm}(\cos\theta,r)}{\partial r}\\ &= \sin\theta\sum_{j=0}^{k^+}(2j)(2j+1)a_{2j+1}^{+}r^{2j-1}(\cos\theta)^{2j}+\\ &\quad \sin\theta\sum_{j=0}^{l^{\pm}}(2j+1)(2j+2)a_{2j+2}^{\pm}r^{2j}(\cos\theta)^{2j+1}.\end{aligned} \tag{16}$$

Therefore from (15) and (16) it follows that

$$\begin{aligned}y_2^+(\pi,r) = &\int_0^{\pi}\left(2F_2^+(\theta,r)+2\frac{\partial F_1^+(\theta,r)}{\partial r}y_1^+(\theta,r)\right)d\theta\\ =&-8\int_0^1\left(\sum_{j=0}^{k^+}(2j+1)a_{2j+1}^{+}r^{2j}s^{2j+1}\right)\left(\sum_{j=0}^{l^+}(2j+2)a_{2j+2}^{+}r^{2j}s^{2j+1}\right)ds\\ &-4\int_0^1\left(\sum_{j=0}^{k^+}a_{2j+1}^{+}r^{2j}s^{2j+1}\right)\left(\sum_{j=0}^{l^+}(2j+1)(2j+2)a_{2j+2}^{+}r^{2j}s^{2j+1}\right)ds\\ &+4\int_0^1\left(\sum_{j=0}^{k^+}(2j)(2j+1)a_{2j+1}^{+}r^{2j-1}s^{2j}\right)\left(\sum_{j=0}^{l^+}a_{2j+2}^{+}r^{2j+1}(1-s^{2j+2})\right)ds\\ &+4\int_0^1\left(\sum_{j=0}^{k^+}a_{2j+1}^{+}r^{2j}\right)\left(\sum_{j=0}^{k^+}(2j)(2j+1)a_{2j+1}^{+}r^{2j-1}s^{2j}\right)ds\\ =&-4\sum_{k=0}^{k^++l^+}\eta^+(k)r^{2k}+8\sum_{k=1}^{2k^+}\zeta(k)r^{2k-1},\end{aligned}$$

and similarly,

$$\begin{aligned}y_2^-(-\pi,r) &= \int_0^{-\pi}\left(2F_2^-(\theta,r)+2\frac{\partial F_1^-(\theta,r)}{\partial r}y_1^-(\theta,r)\right)d\theta\\ &=-4\sum_{k=0}^{k^++l^-}\eta^-(k)r^{2k}+2\sum_{k=1}^{2k^+}\zeta(k)r^{2k-1},\end{aligned}$$

where

$$\eta^{\pm}(k)=\sum_{\substack{i+j=k\\ 0\le i\le k^+,0\le j\le l^{\pm}}}(2j+2)a_{2i+1}^{+}a_{2j+2}^{\pm},\qquad \zeta(k)=\sum_{\substack{i+j=k\\ 0\le i,j\le k^+}}ja_{2i+1}^{+}a_{2j+1}^{+}.$$

Associate with (8) and (9), the above calculations yield the second order averaged function

$$\begin{aligned}\mathcal{F}_2(r) &= \frac{y_2^+(\pi,r)-y_2^-(-\pi,r)}{2!}\\ &= -2\sum_{k=0}^{k^++l^+}\eta^+(k)r^{2k}+2\sum_{k=0}^{k^++l^-}\eta^-(k)r^{2k}\\ &= -2\mathcal{F}_{21}(r)\mathcal{F}_{22}(r),\end{aligned} \tag{17}$$

where

$$\begin{aligned}\mathcal{F}_{21}(r) &= \sum_{i=0}^{k^+}a_{2i+1}^+r^{2i},\\ \mathcal{F}_{22}(r) &= \sum_{j=0}^{l^+}(2j+2)(a_{2j+2}^+-a_{2j+2}^-)r^{2j}+\sum_{j=1+l^+}^{l^-}(2j+2)(-a_{2j+2}^-)r^{2j}.\end{aligned} \tag{18}$$

Here $l^+\le l^-$ because we are assuming that $m\le n$. Observe that $\mathcal{F}_{21}(r)$ (resp. $\mathcal{F}_{22}(r)$) has at most k^+ (resp. l^-) variations of signs. Hence, it follows from Theorem 6 that $\mathcal{F}_{21}(r)$ (resp. $\mathcal{F}_{22}(r)$) has at most k^+ (resp. l^-) positive real zeros. Furthermore, since all coefficients of $\mathcal{F}_{21}(r)$ and $\mathcal{F}_{22}(r)$ can be chosen arbitrarily, by Theorem 6 again we can choose them in such a way that $\mathcal{F}_{21}(r)$ (resp. $\mathcal{F}_{22}(r)$) has exactly k^+ (resp. l^-) positive real simple zeros and these zeros of $\mathcal{F}_{21}(r)$ are different from the ones of $\mathcal{F}_{22}(r)$. Consequently, $\mathcal{F}_2(r)$ has at most k^++l^- positive real simple zeros from (17), and this number is reachable. Using Theorem 5, we conclude the proof of Proposition 9 because $l^-=[(n-1)/2]$ and $k^+=[m/2]$ as defined in (12). □

3.3. **Higher order averaging method.** Now we apply the higher order averaging method to system (4). To do this, in the next lemma we explore the values of the coefficients for which the first and second order averaged functions are identically zeros, i.e. $\mathcal{F}_1=\mathcal{F}_2=0$.

Lemma 10. *Consider the first and second order averaged functions $\mathcal{F}_1$ and $\mathcal{F}_2$ given in* (13) *and* (17) *respectively. Then $\mathcal{F}_1=\mathcal{F}_2=0$ if and only if condition* (14) *and one of the following conditions holds:*

(i) $a_{2i+1}^+=0$ *for* $i=0,1,...,k^+$,
(ii) $a_{2j+2}^-=a_{2j+2}^+$ *for* $j=0,1,...,l^+$ *and* $a_{2j+2}^-=0$ *for* $j=1+l^+,...,l^-$.

Proof. From the expression of $\mathcal{F}_1$ given in (13) we directly obtain that $\mathcal{F}_1=0$ if and only if condition (14) holds. On the other hand, it follows from (18) that $\mathcal{F}_{21}=0$ if and only if condition (i) holds, and $\mathcal{F}_{22}=0$ if and only if condition (ii) holds. Thus $\mathcal{F}_2=0$ if and only if condition (i) or (ii) holds because $\mathcal{F}_2(r)=-2\mathcal{F}_{21}(r)\mathcal{F}_{22}(r)$ as given in (17). This ends the proof the lemma. □

In order to calculate the higher order averaged functions for system (4) we need some technical lemmas.

Lemma 11. *Let $\tilde{F}_i^\pm(s,r):[-1,1]\times(0,+\infty)\to\mathbb{R}$ with $i\in\mathbb{N}^+$, be the functions given by*

$$\tilde{F}_i^\pm(s,r)=\frac{(-s)^{i-1}}{r^{i-1}}\left(\sum_{j=0}^{l^\pm}(2j+2)a_{2j+2}^\pm(rs)^{2j+1}\right)^i.$$

Then $\tilde{F}_i^{\pm}(s,r)$ are odd functions with respect to the variable s, i.e. $\tilde{F}_i^{\pm}(-s,r) = -\tilde{F}_i^{\pm}(s,r)$. Moreover $\tilde{F}_i^{\pm}(s,r)$ is $\mathcal{C}^\infty$ with respect to r and the derivatives $\partial^k \tilde{F}_i^{\pm}(s,r)/\partial r^k$ for $k = 1, 2, ...$ are also odd functions with respect to the variable s.

Proof. This can be obtained by a direct computations. □

Lemma 12. *Let $\tilde{F}_i^{\pm}(s,r)$ be the functions given in Lemma 11, and $\tilde{y}_i^{\pm}(s,r) : [-1,1] \times (0,+\infty) \to \mathbb{R}$ with $i \in \mathbb{N}^+$, be the functions*

$$\begin{aligned}\tilde{y}_1^{\pm}(s,r) &= -\int_1^s \tilde{F}_1^{\pm}(\tau,r)d\tau,\\ \tilde{y}_i^{\pm}(s,r) &= -i!\int_1^s \Bigg(\tilde{F}_i^{\pm}(\tau,r)+\\ &\qquad \sum_{l=1}^{i}\sum_{S_l}\frac{1}{b_1!b_2!2!^{b_2}\cdots b_l!l!^{b_l}}\frac{\partial^L}{\partial r^L}\tilde{F}_{i-l}^{\pm}(\tau,r)\prod_{j=1}^{l}\tilde{y}_j^{\pm}(\tau,r)^{b_j}\Bigg)d\tau,\end{aligned}$$

where S_l is the set of all l-tuples of non-negative integers $(b_1, b_2, ..., b_l)$ satisfying $b_1 + 2b_2 + \cdots + lb_l = l$, and $L = b_1 + b_2 + \cdots + b_l$, $\tilde{F}_0^{\pm} = 0$. Then $\tilde{y}_i^{\pm}(s,r)$ are even functions with respect to the variable s.

Proof. From Lemma 11 we know that $\tilde{F}_1^{\pm}(s,r)$ are odd functions with respect to s, i.e. $\tilde{F}_1^{\pm}(-s,r) = -\tilde{F}_1^{\pm}(s,r)$. Then using the transformation $\tau \to -\tau$ we have

$$\begin{aligned}\tilde{y}_1^{\pm}(-s,r) &= -\int_1^{-s}\tilde{F}_1^{\pm}(\tau,r)d\tau = -\int_{-1}^{s}\tilde{F}_1^{\pm}(\tau,r)d\tau\\ &= -\int_1^s\tilde{F}_1^{\pm}(\tau,r)d\tau - \int_{-1}^{1}\tilde{F}_1^{\pm}(\tau,r)d\tau\\ &= \tilde{y}_1^{\pm}(s,r) - \int_{-1}^{1}\tilde{F}_1^{\pm}(\tau,r)d\tau\\ &= \tilde{y}_1^{\pm}(s,r),\end{aligned}$$

where the second and fifth equalities are due to the oddness of $\tilde{F}_1^{\pm}(s,r)$. Hence $\tilde{y}_1^{\pm}(s,r)$ are even functions with respect to s. Again by Lemma 11 we get that

$$\begin{aligned}&\tilde{F}_2^{\pm}(s,r) + \sum_{l=1}^{2}\sum_{S_l}\frac{1}{b_1!b_2!2!^{b_2}\cdots b_l!l!^{b_l}}\frac{\partial^L}{\partial r^L}\tilde{F}_{2-l}^{\pm}(s,r)\prod_{j=1}^{l}\tilde{y}_j^{\pm}(s,r)^{b_j}\\ &\quad = \tilde{F}_2^{\pm}(s,r) + \frac{\partial}{\partial r}\tilde{F}_1^{\pm}(s,r)\tilde{y}_1^{\pm}(s,r)\end{aligned} \tag{19}$$

are odd functions with respect to s, where the assumption $\tilde{F}_0^{\pm} = 0$ is used. Since the integral of an odd function is always an even function, $\tilde{y}_2^{\pm}(s,r)$ are even functions with respect to s by (19) and the definition of $\tilde{y}_2^{\pm}(s,r)$. By the method of induction we finally obtain Lemma 12. □

Having these lemmas we can prove the following proposition.

Proposition 13. *If the discontinuous piecewise polynomial Hamiltonian system* (4) *satisfies* (14) *and the condition* (i) *of Lemma 10, then the averaged function of any order is identically zero.*

Proof. Writing the discontinuous piecewise polynomial Hamiltonian system (4) into the form of system (7) and using (11) and the assumptions of the proposition, we have

$$
\begin{aligned}
F_1^{\pm}(\theta,r) &= \sin\theta H_{0x}^{\pm}(r\cos\theta) = \sin\theta\sum_{j=0}^{l^{\pm}}(2j+2)a_{2j+2}^{\pm}(r\cos\theta)^{2j+1},\\
F_i^{\pm}(\theta,r) &= \sin\theta\frac{(-\cos\theta)^{i-1}}{r^{i-1}}\left(H_{0x}^{\pm}(r\cos\theta)\right)^{i}\\
&= \sin\theta\frac{(-\cos\theta)^{i-1}}{r^{i-1}}\left(\sum_{j=0}^{l^{\pm}}(2j+2)a_{2j+2}^{\pm}(r\cos\theta)^{2j+1}\right)^{i}\\
&= \sin\theta\tilde{F}_i^{\pm}(\cos\theta,r) \quad \text{for } i\in\mathbb{N}^{+},
\end{aligned}
\tag{20}
$$

where $\tilde{F}_i^{\pm}(\theta,r)$ is defined in Lemma 11.

We claim that for system (4)

$$
y_i^{\pm}(\theta,r) = \tilde{y}_i^{\pm}(\cos\theta,r) \quad \text{for } i\in\mathbb{N}^{+}, \tag{21}
$$

where $y_i^{\pm}(\theta,r)$ and $\tilde{y}_i^{\pm}(s,r)$ are defined in (9) and Lemma 12 respectively. In fact, the function $y_1^{\pm}(\theta,r)$ for system (4) is

$$
\begin{aligned}
y_1^{\pm}(\theta,r) &= \int_0^{\theta}F_1^{\pm}(\varphi,r)d\varphi = -\int_0^{\theta}\sum_{j=0}^{l^{\pm}}(2j+2)a_{2j+2}^{\pm}(r\cos\varphi)^{2j+1}d\cos\varphi\\
&= -\int_1^{\cos\theta}\sum_{j=0}^{l^{\pm}}(2j+2)a_{2j+2}^{\pm}(r\tau)^{2j+1}d\tau = -\int_1^{\cos\theta}\tilde{F}_1^{\pm}(\tau,r)d\tau\\
&= \tilde{y}_1^{\pm}(\cos\theta,r).
\end{aligned}
\tag{22}
$$

Thus (21) holds for $i=1$. Joining (20) and (22) we further get that the function $y_2^{\pm}(\theta,r)$ for system (4) is

$$
\begin{aligned}
y_2^{\pm}(\theta,r) &= 2\int_0^{\theta}\Bigg(F_2^{\pm}(\varphi,r)+\\
&\quad\sum_{l=1}^{2}\sum_{S_l}\frac{1}{b_1!b_2!2!^{b_2}\cdots b_l!l!^{b_l}}\frac{\partial^L}{\partial r^L}F_{2-l}^{\pm}(\varphi,r)\prod_{j=1}^{l}y_j^{\pm}(\varphi,r)^{b_j}\Bigg)d\varphi\\
&= 2\int_0^{\theta}\Bigg(\sin\varphi\tilde{F}_2^{\pm}(\cos\varphi,r)+\\
&\quad\sum_{l=1}^{2}\sum_{S_l}\frac{1}{b_1!b_2!2!^{b_2}\cdots b_l!l!^{b_l}}\frac{\partial^L}{\partial r^L}\Big(\sin\varphi\tilde{F}_{2-l}^{\pm}(\cos\varphi,r)\Big)\prod_{j=1}^{l}\tilde{y}_j^{\pm}(\cos\varphi,r)^{b_j}\Bigg)d\varphi\\
&= -2\int_1^{\cos\theta}\Bigg(\tilde{F}_2^{\pm}(\tau,r)+\\
&\quad\sum_{l=1}^{2}\sum_{S_l}\frac{1}{b_1!b_2!2!^{b_2}\cdots b_l!l!^{b_l}}\frac{\partial^L}{\partial r^L}\tilde{F}_{2-l}^{\pm}(\tau,r)\prod_{j=1}^{l}\tilde{y}_j^{\pm}(\tau,r)^{b_j}\Bigg)d\tau\\
&= \tilde{y}_2^{\pm}(\cos\theta,r),
\end{aligned}
$$

i.e. (21) holds for $i = 2$. By the method of induction we can obtain (21) for all $i \in \mathbb{N}^+$.

Consequently the above analysis implies that the averaged function $\mathcal{F}_i(r)$ of order $i \in \mathbb{N}^+$ for system (4) is

$$
\begin{aligned}
\mathcal{F}_i(r) &= \frac{y_i^+(\pi, r) - y_i^-(-\pi, r)}{i!} = \frac{\tilde{y}_i^+(-1, r) - \tilde{y}_i^-(-1, r)}{i!} \\
&= -\int_1^{-1} \Bigg(\tilde{F}_i^+(\tau, r) + \\
&\qquad \sum_{l=1}^{i} \sum_{S_l} \frac{1}{b_1! b_2! 2!^{b_2} \cdots b_l! l!^{b_l}} \frac{\partial^L}{\partial r^L} \tilde{F}_{i-l}^+(\tau, r) \prod_{j=1}^{l} \tilde{y}_j^+(\tau, r)^{b_j} \Bigg) d\tau \\
&\quad + \int_1^{-1} \Bigg(\tilde{F}_i^-(\tau, r) + \\
&\qquad \sum_{l=1}^{i} \sum_{S_l} \frac{1}{b_1! b_2! 2!^{b_2} \cdots b_l! l!^{b_l}} \frac{\partial^L}{\partial r^L} \tilde{F}_{i-l}^-(\tau, r) \prod_{j=1}^{l} \tilde{y}_j^-(\tau, r)^{b_j} \Bigg) d\tau.
\end{aligned} \tag{23}
$$

According to Lemmas 11 and 12, all integrands in (23) are both odd functions with respect to the variable τ, so that $\mathcal{F}_i$ is identically zero for all $i \in \mathbb{N}^+$. This proves the proposition. □

Regarding system (4) that satisfies (14) and the condition (ii) of Lemma 10, we can obtain the next result.

Proposition 14. *If the discontinuous piecewise polynomial Hamiltonian system* (4) *satisfies* (14) *and the condition* (ii) *of Lemma 10, then there exist no crossing limit cycles.*

Proof. Under the assumptions of proposition, we have $H_0^+(x) \equiv H_0^-(x)$ form their expressions given in (11). In this case system (4) is a smooth Hamiltonian system, so that there exist no limit cycles, also no crossing limit cycles. □

Now we are in a position to prove Theorem 2.

Proof of Theorem 2. According to Propositions 8 and 9, the first and second order averaging methods predict that the maximum number of crossing limit cycles that the discontinuous piecewise polynomial Hamiltonian system (4) with $m \leq n$ can have, bifurcating from the periodic orbits of the period annulus of the linear center, is at most

$$
\max\left\{ \left[\frac{n}{2}\right], \left[\frac{n-1}{2}\right] + \left[\frac{m}{2}\right] \right\}.
$$

Moreover this number is reachable. On the other hand, from Lemma 10, Propositions 13 and 14 it follows that the averaging methods of higher order than two do not produce more crossing limit cycles. This concludes the proof of Theorem 2. □

4. Proof of Theorem 3

We now prove Theorem 3. Suppose that system (5) has a crossing limit cycle, then it must intersect the x-axis at two different points, denoted by $(X, 0)$ and $(x, 0)$ with $X > x$.

Since $H^-(x,y)$ and $H^+(x,y)$ are the first integrals, the two points $(X,0)$ and $(x,0)$ must satisfy the system of equations

$$\begin{aligned} H^-(X,0) - H^-(x,0) &= (X-x)P(X,x) = 0, \\ H^+(X,0) - H^+(x,0) &= (X-x)Q(X,x) = 0. \end{aligned} \tag{24}$$

Here P and Q are two real polynomials of degree m and n respectively. Thus, from Theorem 7 we obtain that system (24) has at most mn real solutions with $X \neq x$. Since the solutions of (24) are symmetric, i.e. if (a,b) is a solution of $P(X,x) = Q(X,x) = 0$, then (b,a) is also a solution of it, the maximum number of crossing limit cycles for system (5) is at most $[mn/2]$.

We further consider the case of $m = n$. In this case we can write $P(X,x)$ and $Q(X,x)$ as

$$P(X,x) = P_{n-1}(X,x) + a_{n0}^- \sum_{i=0}^{n} X^i x^{n-i},$$

$$Q(X,x) = Q_{n-1}(X,x) + a_{n0}^+ \sum_{i=0}^{n} X^i x^{n-i},$$

respectively, where P_{n-1} and Q_{n-1} are two real polynomials of degree $n-1$. If $a_{n0}^- a_{n0}^+ = 0$, then system (24) has at most $n(n-1)$ solutions directly from Theorem 7. If $a_{n0}^- a_{n0}^+ \neq 0$, then (24) is equivalent to the system of equations

$$\begin{aligned} &P_{n-1}(X,x) + a_{n0}^- \sum_{i=0}^{n} X^i x^{n-i} = 0, \\ &a_{n0}^- Q_{n-1}(X,x) - a_{n0}^+ P_{n-1}(X,x) = 0. \end{aligned} \tag{25}$$

Thus system (25), or equivalently (24), has at most $n(n-1)$ solutions by Theorem 7 again. Finally, using the symmetry of the solutions of (24), we get that system (5) has at most $[n(n-1)/2]$ crossing limit cycles in the case of $m = n$.

From these last two paragraphs the proof of Theorem 3 follows.

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