

ON THE POINCARÉ-BENDIXSON FORMULA FOR PLANAR PIECEWISE SMOOTH VECTOR FIELDS

SHIMIN LI¹, CHANGJIAN LIU², JAUME LLIBRE³

ABSTRACT. The topological index, or simply the index, of an equilibrium point of a differential system is an integer which saves important information about the local phase portrait of the equilibrium.

There are mainly two ways to calculate the index of an isolated equilibrium point of a smooth vector field. First Poincaré and Bendixson proved that the index of an equilibrium point can be obtained from the number of hyperbolic and elliptic sectors that there are in a neighborhood of the equilibrium point, which is known as Poincaré-Bendixson formula for the topological index of an equilibrium point. Second several works contributed to the algebraic method of Cauchy's index for computing the index of an equilibrium point.

In this paper we extend the Poincaré-Bendixson formula to planar piecewise smooth vector fields. Applying this formula we define the index of generic codimension-one equilibria for piecewise smooth vector fields, including boundary equilibria, pseudo-equilibria and tangency points.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

A planar smooth differential system is defined by

$$(1) \quad \begin{aligned} \frac{dx}{dt} &= P(x, y), \\ \frac{dy}{dt} &= Q(x, y), \end{aligned}$$

where P and Q are C^r with $1 \leq r \leq \infty$. Let $Z(p) = (P(p), Q(p))$ be the smooth vector field at the point $p \in \mathbb{R}^2$ associated to system (1).

Suppose that $q \in \mathbb{R}^2$ is an isolated equilibrium point of the vector field (1), that is, $P(q) = Q(q) = 0$. One of the fundamental problems in the qualitative theory of planar differential systems is to provide a characterization of the local phase portraits in the vicinity of the equilibrium point q . The Jacobian matrix of system (1) at q is

$$(2) \quad DZ(q) := \begin{pmatrix} \frac{\partial P}{\partial x}(q) & \frac{\partial P}{\partial y}(q) \\ \frac{\partial Q}{\partial x}(q) & \frac{\partial Q}{\partial y}(q) \end{pmatrix}.$$

2010 *Mathematics Subject Classification.* 34C05, 34C07, 37G15.

Key words and phrases. Poincaré-Bendixson formula; Piecewise smooth vector fields; Regularization.