Integrals, invariant manifolds, and degeneracy for central force problems in $\mathbb{R}^n$

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Since the notion of angular momentum is defined in any dimension by using the exterior product in $\mathbb{R}^n$, one would guess that central force problems in any dimension are completely integrable, as it is known for $n=2$ or 3. This is proved explicitly in this paper, by constructing n first integrals independent and involution: the energy and some combinations of the angular momentum components. It is shown that these problems are always reduced to a two-dimensional plane, and the invariant manifolds are topologically, the Cartesian product of those of the reduced problem times $n-2$ circle factors. It is also proved that for $n>2$ these problems are degenerate in the sense that their Hamiltonians do not satisfy one of the hypotheses of the Kolmogorov–Arnold–Moser theorem for persistence of invariant tori, when one considers small perturbations.

I. INTRODUCTION

In this paper we deal with the motion of a particle subject to a central force in an n-dimensional space considered as a Hamiltonian system with n degrees of freedom.

One would expect that this system is *completely integrable.* Here, we exhibit explicitly n first integrals of the system that are in involution with linearly independent gradients.

The structure of a completely integrable Hamiltonian system with n degrees of freedom is particularly simple. Given the first integrals $F_1,\ldots, F_n$ one considers the *invariant manifolds* $I_{c_1,\ldots,c_n}$ defined by $F_1 = c_1,\ldots, F_n = c_n$, where $(c_1,\ldots,c_n)$ is a regular value of the function $(F_1,\ldots,F_n)$. We prove for a central force problem in $\mathbb{R}^n$ that any connected component of the above invariant manifolds is topologically an n-dimensional torus $(S^1)^n$ or a cylinder $\mathbb{R} \times (S^1)^{n-1}$.

Near a compact component of an invariant manifold we can introduce canonical coordinates (x,y) such that the Hamiltonian H depends only on y, where x is the position and y the momentum. If the Hessian

$$\det\left(\frac{\partial^2 H}{\partial y_i\,\partial y_j}\right)\neq 0,$$

we say that H is *nondegenerate.* On the other hand, if the determinant

$$\begin{vmatrix} \dfrac{\partial^2 H}{\partial y_i \partial y_j} & \dfrac{\partial H}{\partial y_i} \\ \dfrac{\partial H}{\partial y_j} & 0 \end{vmatrix} \neq 0,$$

we say that H is *isoenergetically nondegenerate.* The importance of these two notions is that either one is necessary in order to apply the KAM theorem. Then we show that the Hamiltonian associated to a central force problem in $\mathbb{R}^n$ is degenerate and isoenergetically degenerate if $n>2$.

Recently, an interest has arisen for the completely integrable Hamiltonian systems with n degrees of freedom. Some of the more studied examples are the following ones: The Toda lattice with finitely many particles on a line;[1–3] the inverse square potential of Calogero;[4–6] the geodesic flow on an n-dimensional ellipsoid in $\mathbb{R}^{n+1}$;[7–9] the motion of a mass point on the n-dimensional sphere under the influence of a quadratic potential;[10] the n-dimensional simple harmonic oscillator;[11] the motion of a free n-dimensional rigid body about a fixed point.[12,13] This interest is partly justified by the fact that all this seemingly disconnected examples of integrable systems present some underlying common features, like group representations, isospectral deformation or symmetries. For more details, see Moser[14] and Adler and van Moerbeke.[15] The integrability problem from a more topological viewpoint has been studied by Fomenko.[16,17]

Notice that in this paper we focus our attention on the integrability of the specific Hamiltonian system asso-

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