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Invariant Algebraic Surfaces of Polynomial Vector Fields in Dimension Three

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Abstract

We discuss criteria for the nonexistence, existence and computation of invariant algebraic surfaces for three-dimensional complex polynomial vector fields, thus transferring a classical problem of Poincaré from dimension two to dimension three. Such surfaces are zero sets of certain polynomials which we call semi-invariants of the vector fields. The main part of the work deals with finding degree bounds for irreducible semi-invariants of a given polynomial vector field that satisfies certain properties for its stationary points at infinity. As a related topic, we investigate existence criteria and properties for algebraic Jacobi multipliers. Some results are stated and proved for polynomial vector fields in arbitrary dimension and their invariant hypersurfaces. In dimension three we obtain detailed results on possible degree bounds. Moreover by an explicit construction we show for quadratic vector fields that the conditions involving the stationary points at infinity are generic but they do not a priori preclude the existence of invariant algebraic surfaces. In an appendix we prove a result on invariant lines of homogeneous polynomial vector fields.

Keywords Poincaré problem · Darboux integrability · Jacobi multiplier · Nonassociative algebras

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