LIMIT CYCLES OF DISCONTINUOUS PIECEWISE QUADRATIC PERTURBATIONS OF A LINEAR CENTER SEPARATED BY THE CURVE $y = x^n$

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Abstract. The study of the limit cycles of planar differential systems is one of the main problems in the qualitative theory of differential systems. These last years a big interest appeared for studying the limit cycles of the piecewise differential systems due to their many applications. Here we prove that the linear center $\dot{x} = y$, $\dot{y} = -x$, can produce at most 6 crossing limit cycles for $n \geq 4$ even and at most 7 crossing limit cycles for $n \geq 5$ odd using the averaging theory of first order, when it is perturbed by discontinuous piecewise differential systems formed by two pieces separated by the curve $y = x^n$ ($n \geq 4$), and having in each piece a quadratic polynomial differential system. Using the averaging theory of second order the perturbed system can be chosen in such way that it has $0, 1, 2, 3, 4, 5, 6, 7$ or 9 crossing limit cycles if $4 \leq n \leq 74$ is even and $0, 1, 2, 3, 4, 5, 6, 7, 9$ or 11 crossing limit cycles if $n \geq 76$ is even. The averaging theory of second order produces the same number of crossing limit cycles as the averaging theory of first order if $n \geq 5$ is odd. The main tools for proving our results are the new averaging theory developed for studying the crossing limit cycles of the discontinuous piecewise differential systems, and the theory for studying the zeros of a function using the extended Chebyshev systems.

1. Introduction and statement of the main results

The method of averaging has been one of the effectively analytical methods to detect the existence of limit cycles of the nonlinear differential equations. For the s-mooth differential equations the averaging theory can be found in some monographs (see [18, 19, 39]). In recent years the classical averaging theory for computing periodic solutions was developed rapidly (see [5, 16, 17, 27–29, 37]). Recently the averaging theory for computing periodic solutions has been developed for discontinuous piecewise differential systems (see [21, 26, 30, 42]).

It is well-known that a limit cycle of a differential system is an isolated periodic solution in the set of all periodic solutions of the system. Limit cycles have been a significant research topic in the qualitative theory of planar differential systems. The second part of the well-known Hilbert 16th problem (see [12, 23]) asks for the maximum number and relative positions of the limit cycles for the polynomial systems of degree n. To find an upper bound for the maximum number of limit cycles is an open problem but the possible relative positions were solved in [31].

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Also the Hilbert problem can be extended to nonsmooth differential systems with discontinuities because such differential systems are widely used in various fields, for example, control systems, economy, neuron system, electrical circuits and mechanics (see [3, 8, 13, 35, 40, 41]).

For a nonsmooth differential system we follow the Filippov's convention to define the vector fields on its discontinuous boundary (see [15]). Given a piecewise smooth system $\dot{x} = F_{\pm}(x)$, $x \in \Sigma_{\pm} := \mathbb{R}^n \setminus \Sigma$, where Σ is called the set of discontinuity or switching boundary, $\Sigma_{\pm}$ are open sets and $F_{\pm}$ are continuous functions in $\Sigma_{\pm}$ respectively. The vector field at each point of $\Sigma_{\pm}$ is defined by $F_{\pm}$, respectively. We call a point $x_0 \in \Sigma$ a crossing point if $F_{\pm}(x_0)$ point into Σ_+ or Σ_- simultaneously. In this case an orbit of the system near x_0 is a concatenation of the orbits of the two subsystems. The collection of all crossing points forms the crossing region. If a closed curve is formed by concatenating the orbits of the two subsystems and it intersects with Σ only at crossing points then we call the closed curve a crossing periodic orbit. The so-called crossing limit cycle is an isolated crossing periodic orbit in the set of all crossing periodic orbits of the system. There are many works studying the number of limit cycles of the discontinuous piecewise differential systems (see [6, 9–11, 14, 20, 25, 33, 43]).

Recently (see [1, 2, 7, 24, 26, 32, 38]) attentions were paid to the maximum number of crossing limit cycles for the perturbation of the linear center system $(\dot{x}, \dot{y})^T = (y, -x)^T$ by discontinuous piecewise polynomial differential systems with two zones of $\mathbb{R}^2$ separated by a discontinuous boundary Σ, i.e.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y + \displaystyle\sum_{i=1}^{m} \varepsilon^i f_i^{\pm}(x, y) \\ -x + \displaystyle\sum_{i=1}^{m} \varepsilon^i g_i^{\pm}(x, y) \end{pmatrix} \text{ if } (x, y) \in \Sigma_{\pm} := \mathbb{R}^2 \setminus \Sigma, \tag{1}$$

where $f_i^{\pm}$ and $g_i^{\pm}$ $(i = 1, \ldots, m)$ are real polynomials. In particular, interests were made to investigate system (1) with nonlinear switching boundary Σ. For $\Sigma = \{y = x^2\}$ Llibre, Mereu and Novaes [26] studied the number of crossing limit cycles for system (1) with quadratic perturbations and obtained that the maximum number of crossing limit cycles is 6 using the averaging theory up to order 2. Later on Buzzi, Llibre and Novaes [2] proved that the maximum number of crossing limit cycles is 7 for system (1) with linear perturbations and $\Sigma = \{y = x^3\}$ using the Melnikov functions up to order 2. For system (1) with quadratic perturbations and $\Sigma = \{y = x^n\}$ using the Melnikov function of order 1 Ramirez and Alves [38] obtained that the maximum number of crossing limit cycles is 4 if $n = 2$, and 6 if $n \geq 4$ even and, moreover, they gave an upper bound estimate of crossing limit cycles for n odd and higher order perturbations. Recently Andrade, Cespedes, Cruz and Novaes [1] considered the case $\Sigma = \{y = x^n\}$ $(n \geq 4)$ for system (1) with linear perturbations applying the higher order Melnikov method, and obtained that $H(2) \geq 4, H(3) \geq 8, H(n) \geq 7$, for $n \geq 4$ even, and $H(n) \geq 9$, for $n \geq 5$ odd, where $H(n)$ denotes an upper bound of the maximum number of crossing limit cycles for system (1) with linear perturbations and $\Sigma = \{y = x^n\}$.

In this paper we investigate the number of crossing limit cycles for system (1) with $\Sigma = \{y = x^n\}$ $(n \geq 4)$ using the averaging theory up to order 2 but with quadratic perturbations in a small parameter ε, i.e. the following system

$$(2)\qquad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} y + \varepsilon A_1(x,y) + \varepsilon^2 A_2(x,y) \\ -x + \varepsilon B_1(x,y) + \varepsilon^2 B_2(x,y) \end{pmatrix} \text{if } h_n(x,y) \geq 0, \\ \begin{pmatrix} y + \varepsilon C_1(x,y) + \varepsilon^2 C_2(x,y) \\ -x + \varepsilon D_1(x,y) + \varepsilon^2 D_2(x,y) \end{pmatrix} \text{if } h_n(x,y) \leq 0, \end{cases}$$

where $h_n(x,y) := y - x^n$ $(n \geq 4)$ and

$$A_i := \sum_{j+k=0}^{2} a_{ijk} x^j y^k, \qquad B_i := \sum_{j+k=0}^{2} b_{ijk} x^j y^k,$$
$$C_i := \sum_{j+k=0}^{2} c_{ijk} x^j y^k, \qquad D_i := \sum_{j+k=0}^{2} d_{ijk} x^j y^k,$$

for $i = 1, 2$.

Let $\mathcal{Q}_1^e$ be the set of conditions

$$(3)\qquad \begin{array}{lll} a_{110} = -d_{101}, & a_{111} = -b_{120} + c_{111} + d_{120}, & b_{100} = d_{100}, \\ b_{101} = d_{101}, & b_{102} = d_{102}, & c_{110} = -d_{101}, \end{array}$$

$\mathcal{Q}_1^o$ the set of conditions

$$(4)\qquad \begin{array}{l} a_{100} = c_{100}, \quad a_{110} = -b_{101} - c_{110} - d_{101}, \quad a_{120} = c_{120}, \\ a_{111} = -b_{120} + c_{111} + d_{120}, \quad a_{102} = -b_{111} + c_{102} + d_{111}, \\ b_{100} = d_{100}, \quad b_{102} = d_{102}, \end{array}$$

$\mathcal{Q}_2^e$ the set of conditions

$$(5)\qquad \begin{array}{l} a_{110} = b_{100} = b_{101} = b_{102} = c_{100} = c_{110} = c_{120} = d_{100} = d_{101} = d_{102} = 0, \\ a_{111} = -b_{120}, \qquad c_{101} = -d_{110}, \qquad c_{111} = -d_{120}, \quad c_{102} = -d_{111}, \end{array}$$

and $\mathcal{Q}_2^o$ the set of conditions

$$(6)\qquad \begin{array}{lll} \multicolumn{3}{l}{a_{100} = a_{120} = b_{100} = b_{102} = c_{100} = c_{110} = c_{120} = d_{100} = d_{101} = d_{102} = 0} \\ a_{110} = -b_{101}, & a_{111} = -b_{120}, & a_{102} = -b_{111}, \\ c_{101} = -d_{110}, & c_{111} = -d_{120}, & c_{102} = -d_{111}, \end{array}$$

where $\mathcal{Q}_2^e$ and $\mathcal{Q}_2^o$ are subsets of $\mathcal{Q}_1^e$ and $\mathcal{Q}_1^o$, respectively. Our main results are the following.

Theorem 1. *For $|\varepsilon|$ sufficiently small using the averaging theory of first order system (2) with $n \geq 4$ even (resp. $n \geq 5$ odd) has at most 6 (resp. 7) crossing limit cycles when the conditions $\mathcal{Q}_1^e$ (resp. $\mathcal{Q}_1^o$) do not hold. Moreover we can choose parameters $a_{ijk}, b_{ijk}, c_{ijk}$ and d_{ijk} ($i = 1, 2$ and $0 \leq j + k \leq 2$) such that system (2) with $n \geq 4$ even (resp. $n \geq 5$ odd) has exactly $0, 1, 2, 3, 4, 5$ or 6 (resp. $0, 1, 2, 3, 4, 5, 6$ or 7) crossing limit cycles.*

Theorem 2. *For $|\varepsilon|$ sufficiently small using the averaging theory of second order we can choose the parameters $a_{ijk}, b_{ijk}, c_{ijk}$ and d_{ijk} ($i = 1, 2$ and $0 \leq j + k \leq 2$) such that system (2) under the conditions $\mathcal{Q}_2^e$ with $4 \leq n \leq 74$ even (resp. $n \geq 76$ even) has $0, 1, 2, 3, 4, 5, 6, 7$ or 9 (resp. $0, 1, 2, 3, 4, 5, 6, 7, 9$ or 11) crossing limit cycles. For $n \geq 5$ odd the averaging theory of second order produces the same number of* *crossing limit cycles as the averaging theory of first order when the conditions Q_2^o hold.*

Note that as it is written in the proof of Theorem 2 we cannot determine whether the bound 8 for $n \geq 4$ even and the bound 10 for $n \geq 76$ even are reachable.

Theorems 1 and 2 are proved in Section 3.

2. Preliminaries

We state some necessary definitions and results, which are used in our proofs. Let A and D be open subsets of $\mathbb{R}^d$ and $\mathbb{S}^1 = \mathbb{R}/T$, where T is a positive period. Let $\chi_A(t,x)$ denote the characteristic function, i.e. $\chi_A(t,x) = 1$ (resp. 0) if $(t,x) \in A$ (resp. $\notin A$). Let S_j be a finite sequence of open disjoint subsets of $\mathbb{S}^1 \times D$ for $j = 1, 2, \ldots, M$ such that $\cup_{j=1}^M \overline{S}_j = \mathbb{S}^1 \times D$, as usual $\overline{S}_j$ denotes the closure of S_j. Moreover we use Σ to denote the union of boundaries of all $\overline{S}_j$ for $j = 1, 2, \ldots, M$. Given the following discontinuous piecewise differential system

$$\dot{x} = \varepsilon F_1(t,x) + \varepsilon^2 F_2(t,x) + \varepsilon^3 R(t,x,\varepsilon), \tag{7}$$

where

$$F_i := \sum_{j=1}^{M} \chi_{\bar{S}_j}(t,x) F_i^j(t,x) \text{ for } i = 1, 2, \qquad R := \sum_{j=1}^{M} \chi_{\bar{S}_j}(t,x) R_i^j(t,x,\varepsilon),$$

and $F_i^j : \mathbb{S}^1 \times D \to \mathbb{R}^d$, $R_i^j : \mathbb{S}^1 \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^d$ with $\varepsilon_0 > 0$, $i = 1, 2$ and $j = 1, \ldots, M$, are all continuous functions and are all T-periodic in the variable t. From [26] the averaged functions of orders one and two for system (7) are

$$f_1(z) = \int_0^T F_1(t,z)dt, \quad f_2(z) = \int_0^T \left(\frac{\partial F_1(t,z)}{\partial x} y_1(t,z) + F_2(t,z) \right) dt, \tag{8}$$

where $y_1(t,z) = \displaystyle\int_0^t F_1(s,z)ds$.

A point $p \in \Sigma$ is called a generic point of discontinuity if there exists a neighborhood U of p such that $S_p = U \cap \Sigma$ is a $\mathcal{C}^k$ embedded hypersurface.

The crossing hypothesis (CH) for system (7) is the following.

(CH) There exists an open bounded set $C \subset D$ such that for each $z \in \overline{C}$ the curve $\{(t,z) : t \in \mathbb{S}^1\}$ reaches transversely the set Σ only at generic points of discontinuity.

The assumption (CH) implies the following:

Lemma 3 ([26, Proposition 2]). *For $|\varepsilon| \neq 0$ sufficiently small, every solution of system (7) starting in C reaches the set of discontinuity Σ only at its crossing region.*

The followings are two averaging theorems of order up to 2 for system (7), where the notation $d_B(f_i, U, 0)$ $(i = 1, 2)$ denote the Brouwer degree, see [4] or the Appendix A of [26] for a definition.

Theorem 4 ([26, Theorem A])**.** *In addition to the crossing hypothesis (CH) assume the following conditions.*

(Ha1) *For $i = 1,2$ and $j = 1,2,\ldots,M$ the continuous functions F_i^j and R_i^j are locally Lipschitz with respect to x, and T-periodic with respect to the time t. Furthermore, for $j = 1,2,\ldots,M$, the boundaries of S_j are piecewise $\mathcal{C}^k$ embedded hypersurfaces with $k \geq 1$.*

(Ha2) *For $a^* \in C$ with $f_1(a^*) = 0$ there exists a neighborhood $U \subset C$ of a^* such that $f_1(z) \neq 0$ for all $z \in \overline{U}\backslash\{a^*\}$ and $d_B(f_1, U, 0) \neq 0$.*

Then for $|\varepsilon| \neq 0$ sufficiently small there exists a T-periodic solution $x(t,\varepsilon)$ of system (7) such that $x(0,\varepsilon) \to a^*$ as $\varepsilon \to 0$.

Theorem 5 ([26, Theorem B])**.** *Suppose that $f_1(z) \equiv 0$. In addition to the crossing hypothesis (CH) assume the following conditions.*

(Hb1) *For $j = 1,2,\ldots,M$ the functions $F_1^j(t,\cdot)$ are of class $\mathcal{C}^1$ for all $t \in \mathbb{R}$; for $j = 1,2,\ldots,M$ the functions $D_x F_1^j, F_2^j$ and R are locally Lipschitz with respect to x. Furthermore, for $j = 1,2,\ldots,M$, the boundaries of S_j are piecewise $\mathcal{C}^k$ embedded hypersurfaces with $k \geq 1$.*

(Hb2) *If $(t,z) \in \Sigma$ then $(0, y_1(t,z)) \in T_{(t,z)}\Sigma$.*

(Hb3) *For $a^* \in C$ with $f_2(a^*) = 0$, there exists a neighborhood $U \subset C$ of a^* such that $f_2(z) \neq 0$ for all $z \in \overline{U}\backslash\{a^*\}$ and $d_B(f_2, U, 0) \neq 0$.*

Then for $|\varepsilon| \neq 0$ sufficiently small there exists a T-periodic solution $x(t,\varepsilon)$ of system (7) such that $x(0,\varepsilon) \to a^*$ as $\varepsilon \to 0$.

It is known that if a function f is $\mathcal{C}^1$ then it is sufficient to check that the determinant of the Jacobian matrix $D(f)$ is non-zero in order to have that the Brouwer degree $d_B(f,U,0) \neq 0$, for more details see [34].

We shall need in the proofs of our results the theory on the ECT-systems. Let I denote a proper real interval of $\mathbb{R}$. An ordered set of real-valued functions $\mathcal{F} = [f_0(t), f_1(t), \ldots, f_n(t)]$ defined in the interval I is called an Extended Chebyshev system or simply ET-system in the interval I if and only if any nontrivial linear combination of f_i $(i = 0,1,\ldots,n)$ has at most n zeros counting multiplicities. Furthermore $\mathcal{F}$ becomes an Extended Complete Chebyshev system, or simply an ECT-system in the interval I if and only if for any $0 \leq k \leq n$, $(f_0, f_1, \ldots, f_k)$ is an ET-system. We can see the monograph [22] for more details. $\mathcal{F}$ is an ECT-system in the interval I if and only if $W(f_0, f_1, \ldots, f_k)(t) \neq 0$ in the interval I for $0 \leq k \leq n$, where $W(f_0, f_1, \ldots, f_k)(t)$ denotes the Wronskian of the functions $(f_0, f_1, \ldots, f_k)$ with respect to t, i.e.

$$W(f_0, f_1, \ldots, f_k)(t) = \begin{vmatrix} f_0(t) & f_1(t) & \cdots & f_k(t) \\ f_0'(t) & f_1'(t) & \cdots & f_k'(t) \\ \vdots & \vdots & \ddots & \vdots \\ f_0^{(k)}(t) & f_1^{(k)}(t) & \cdots & f_k^{(k)}(t) \end{vmatrix}.$$

The following two theorems give the maximum number of isolated zeros of the function $\Sigma_{i=0}^n \alpha_i f_i(t)$ in the case that some of the Wronskians vanish, where α_i are real numbers for $i = 0,1,\ldots,n$.

Theorem 6 ([36, Corollary 1.4])**.** *Let $\mathcal{F} = [f_0(t), f_1(t), \ldots, f_n(t)]$ be an ordered set of $\mathcal{C}^\infty$ functions defined in the interval $[a,b]$. Assume that all the Wronskians*

$W(f_0, f_1, \ldots, f_k)(t)$ do not vanish for $k = 0, 1, \ldots, n-1$, except $W(f_0, f_1, \ldots, f_n)(t)$, which has exactly one zero in the interval (a, b) and this zero is simple. Then any linear combination of the functions f_i $(i = 0, 1, \ldots, n)$ has at most $n + 1$ zeros and for any set of $m \leq n + 1$ zeros there exists a linear combination of $m + 1$ functions f_i realizing it.

Theorem 7 ([36, Theorem 1.2])**.** *Let $\mathcal{F} = [f_0, f_1, \ldots, f_n]$ be an ordered set of $\mathcal{C}^\infty$ functions defined in the interval $[a, b]$. Assume that all the Wronskians do not vanish except $W(f_0, f_1, \ldots, f_{n-1})(t)$ and $W(f_0, f_1, \ldots, f_n)(t)$, which have respectively k and l zeros in the interval (a, b) and these zeros are simple. Then there exists a linear combination of $n + 1$ functions f_i $(i = 0, 1, \ldots, n)$ having exactly $n + k + l$ simple zeros.*

Theorem 8 ([36, Theorem 1.1])**.** *Let $\mathcal{F} = [f_0, f_1, \ldots, f_n]$ be an ordered set of analytic functions defined in the interval $[a, b]$. Assume that all the ν_k zeros of the Wronskian $W(f_0, f_1, \ldots, f_k)(t)$ are simple for $k = 0, \ldots, n$. Then the number of isolated zeros for any linear combination of $n+1$ functions f_i $(i = 0, 1, \ldots, n)$ does not exceed*

$$n + \nu_n + \nu_{n-1} + 2(\nu_{n-2} + \cdots + \nu_0) + \mu_{n-1} + \cdots + \mu_3,$$

where $\mu_i = \min(2\nu_i, \nu_{i-3} + \cdots \nu_0)$, for $i = 3, \ldots, n - 1$.

3. Proofs of the main results

Proof of Theorem 1. In the polar coordinates $x = r\cos\theta$ and $x = r\sin\theta$ system (2) takes the form

$$(9)\qquad \begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} = \begin{cases} \begin{pmatrix} \varepsilon\big(A_1(x, y)\cos\theta + B_1(x, y)\sin\theta\big) \\ +\varepsilon^2\big(A_2(x, y)\cos\theta + B_2(x, y)\sin\theta\big), \\ -1 + \varepsilon\big(B_1(x, y)\cos\theta - A_1(x, y)\sin\theta\big)/r \\ +\varepsilon^2\big(B_2(x, y)\cos\theta - A_2(x, y)\sin\theta\big)/r \end{pmatrix} & \text{if } \tilde{h}_n(\theta, r) \geq 0, \\ \begin{pmatrix} \varepsilon\big(C_1(x, y)\cos\theta + D_1(x, y)\sin\theta\big) \\ +\varepsilon^2\big(C_2(x, y)\cos\theta + D_2(x, y)\sin\theta\big), \\ -1 + \varepsilon\big(D_1(x, y)\cos\theta - C_1(x, y)\sin\theta\big)/r \\ +\varepsilon^2\big(D_2(x, y)\cos\theta - C_2(x, y)\sin\theta\big)/r \end{pmatrix} & \text{if } \tilde{h}_n(\theta, r) \leq 0, \end{cases}$$

where $\tilde{h}_n(\theta, r) := r\sin\theta - r^n\cos^n\theta$ with $n \geq 4$. Taking θ as the new time variable system (9) is equivalent to the following discontinuous piecewise differential system

$$(10)\qquad \dot{r} = \begin{cases} P(\theta, r) := \varepsilon P_1(\theta, r) + \varepsilon^2 P_2(\theta, r) + \mathcal{O}(\varepsilon^3) \text{ if } \tilde{h}_n(\theta, r) \geq 0, \\ Q(\theta, r) := \varepsilon Q_1(\theta, r) + \varepsilon^2 Q_2(\theta, r) + \mathcal{O}(\varepsilon^3) \text{ if } \tilde{h}_n(\theta, r) \leq 0, \end{cases}$$

where $Q_i(\theta, r) := P_i(\theta, r)|_{a_{ijk}=c_{ijk},\, b_{ijk}=d_{ijk}}$ for $i = 1, 2$, $j, k = 0, 1, 2$ and $0 \leq j+k \leq 2$, and $P_i(\theta, r)$ $(i = 1, 2)$ are given in the Data.pdf.

Note that each point (θ, r) on $\tilde{h}_n(\theta, r) = 0$ is determined by

$$r^2 = x^2 + x^{2n}, \quad \sin\theta = \frac{x^n}{r}, \quad \cos\theta = \frac{x}{r},$$

implying that $\tan\theta = x^{n-1}$. Thus there exists a unique $\theta_1 := \arctan(x^{n-1}) \in (0, \pi/2)$ such that $\tilde{h}_n(\theta_1, r) = 0$. Let $\theta_2 := \pi - (-1)^n\theta_1$. Then θ_2 satisfies $\tilde{h}_n(\theta_2, r) = 0$ and $\theta_2 \in (\pi/2, \pi)$ (resp. $\in (\pi, 3\pi/2)$) if n is even (resp. odd). Furthermore one can check that $\tilde{h}_n(\theta, r) > 0$ (resp. < 0) if $\theta \in (\theta_1, \theta_2)$ (resp. $\in (0, \theta_1) \cup (\theta_2, 2\pi)$). Hence the switching curve of system (10) is given by $\widetilde{\Sigma} := \{(\theta_1, r) : r > 0\} \cup \{(\theta_2, r) : r > 0\}$. Computations show that

$$\begin{aligned}
&\left\langle \nabla\tilde{h}_n(\theta_1, r), (1, P(\theta_1, r)) \right\rangle \left\langle \nabla\tilde{h}_n(\theta_1, r), (1, Q(\theta_1, r)) \right\rangle \\
&\quad = \left(x + nx^{2n-1}\right)^2 + \frac{(n-1)^2 x^{2n}}{x^2 + x^{2n}} + \mathcal{O}(\varepsilon) > 0, \\
&\left\langle \nabla\tilde{h}_n(\theta_2, r), (1, P(\theta_2, r)) \right\rangle \left\langle \nabla\tilde{h}_n(\theta_2, r), (1, Q(\theta_2, r)) \right\rangle \\
&\quad = \left(x + (-1)^{n-1} n x^{2n-1}\right)^2 + \frac{\left(x^n + n(-x)^n\right)^2}{x^2 + x^{2n}} + \mathcal{O}(\varepsilon) > 0.
\end{aligned}$$

So the hypothesis (CH) holds for system (10). From (8) the averaged function of order one is

$$f_1(r) = \int_0^{2\pi} \frac{d\dot{r}}{d\varepsilon}\Big|_{\varepsilon=0} d\theta = \int_0^{\theta_1} Q_1(\theta, r)d\theta + \int_{\theta_1}^{\theta_2} P_1(\theta, r)d\theta + \int_{\theta_2}^{2\pi} Q_1(\theta, r)d\theta,$$

whose computation gives

$$f_1^e(r) := f_1(r) = \frac{k_1^e g_1^e + k_2^e g_2^e + k_3^e g_3^e + k_4^e g_4^e + k_5^e g_5^e + k_6^e g_6^e}{6\sqrt{x^2 + x^{2n}}}$$

for $n \geq 4$ even, and

$$f_1^o(r) := f_1(r) = \frac{k_1^e g_1^o + k_2^e g_2^o + k_3^e g_3^o + k_4^e g_4^o + k_5^e g_5^o + k_6^e g_6^o + k_7^e g_7^o}{6\sqrt{x^2 + x^{2n}}}$$

for $n \geq 5$ odd, where

$$\begin{aligned}
&g_1^e := x, \quad g_2^e := x^3, \quad g_3^e := x^{n+1}, \quad g_4^e := x^{2n+1}, \quad g_5^e := x^2 + x^{2n}, \\
&g_6^e := (x^2 + x^{2n})\arctan(x^{n-1}), \quad g_1^o := x, \quad g_2^o := x^3, \quad g_3^o := x^n, \\
&g_4^o := x^{n+2}, \quad g_5^o := x^{2n+1}, \quad g_6^o := x^{3n}, \quad g_7^o := x^2 + x^{2n},
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
&k_1^e := -12(b_{100} - d_{100}), \quad k_2^e := -4(a_{111} + 2b_{102} + b_{120} - c_{111} - 2d_{102} - d_{120}), \\
&k_3^e := 6(a_{110} + d_{101} - b_{101} - c_{110}), \quad k_4^e := -12(b_{102} - d_{102}), \\
&k_5^e := -3\pi(a_{110} + b_{101} + c_{110} + d_{101}), \quad k_6^e := 6(a_{110} + b_{101} - c_{110} - d_{101}), \\
&k_1^o := -12(b_{100} - d_{100}), \quad k_2^o := 4(-2b_{102} + 2d_{102} + c_{111} + d_{120} - a_{111} - b_{120}), \\
&k_3^o := 12(a_{100} - c_{100}), \quad k_4^o := 12(a_{120} - c_{120}), \quad k_5^o := -12(b_{102} - d_{102}), \\
&k_6^o := 4(a_{102} + 2a_{120} + b_{111} - c_{102} - 2c_{120} - d_{111}), \\
&k_7^o := -3\pi(a_{110} + b_{101} + c_{110} + d_{101}).
\end{aligned}$$

From Theorem 4 every simple zero of the function $f_1(r)$ corresponds to a crossing limit cycle of system (2). Compute the Wronskians

$$\begin{aligned}
W_1(g_1^e) &:= x, \\
W_2(g_1^e, g_2^e) &:= 2x^3, \\
W_3(g_1^e, g_2^e, g_3^e) &:= 2n(n-2)x^{n+2},
\end{aligned}$$

$$
\begin{aligned}
W_4(g_1^e,\dots,g_4^e) &:= 8n^3(n-1)(n-2)x^{3n},\\
W_5(g_1^e,\dots,g_5^e) &:= -8n^3(n-2)(n-1)^2(2n-1)((2n-3)x^{2n-2}+1)x^{3n-2},\\
W_6(g_1^e,\dots,g_6^e) &:= \frac{32n^3(n-2)(n-1)^4(2n-1)x^{6n}\Phi_1^e(\varsigma)}{(x^{2n}+x^2)^4},\\
W_1(g_1^o) &:= x,\\
W_2(g_1^o,g_2^o) &:= 2x^3,\\
W_3(g_1^o,g_2^o,g_3^o) &:= 2(n-1)(n-3)x^{n+1},\\
W_4(g_1^o,\dots,g_4^o) &:= 4(n+1)(n-1)^2(n-3)x^{2n},\\
W_5(g_1^o,\dots,g_5^o) &:= 16n(n+1)^2(n-3)(n-1)^4x^{4n-3},\\
W_6(g_1^o,\dots,g_6^o) &:= 192n^2(n-3)(n-1)^7(n+1)^2(3n-1)x^{7n-8},\\
W_7(g_1^o,\dots,g_7^o) &:= 192n^3(n-2)(n-3)(n-1)^7(n+1)^2(2n-1)(3n-1)x^{7n-12}\Phi_1^o(\varsigma),
\end{aligned}
$$

where $\varsigma = x^{n-1}$ and

$$
\begin{aligned}
\Phi_1^e(\varsigma) :=& -n(n-2)(2n-3)(3n-2)\varsigma^6 + (3n-2)(12n^3-45n^2+62n-32)\varsigma^4\\
&-(3n-4)(2n^3-13n^2+12n-4)\varsigma^2-(n-2)(3n-4)(3n-2),\\
\Phi_1^o(\varsigma) :=& \, n(2n-3)\varsigma^2-(3n-2).
\end{aligned}
$$

Note that neither $W_i(g_1^e,\dots,g_i^e)$ $(i=1,\dots,5)$ with $n\ge 4$ even, nor $W_j(g_1^o,\dots,g_j^o)$ $(j=1,\dots,6)$ with $n\ge 5$ odd has zeros in the interval $(0,+\infty)$. Then we only need to discuss the zeros of $W_6(g_1^e,\dots,g_6^e)$ and $W_7(g_1^o,\dots,g_7^o)$, i.e. to discuss the zeros of $\Phi_1^e(\varsigma)$ with $n\ge 4$ even and the zeros of $\Phi_1^o(\varsigma)$ with $n\ge 5$ odd, respectively. Obviously $\Phi_1^o(\varsigma)$ has only one zero in the interval $(0,+\infty)$, i.e.

$$
\varsigma = \sqrt{\frac{3n-2}{n(2n-3)}},
$$

which is simple. From Theorem 6 the set of functions $\{g_1^o,\dots,g_7^o\}$ is an ECT-system in the interval $(0,+\infty)$. Hence $f_1^o(r)$ has at most 7 simple zeros in the interval $(0,+\infty)$ and thereby system (2) with $n\ge 5$ odd has at most 7 crossing limit cycles.

Let $\widetilde{\Phi}_1^e(\nu) := \Phi_1^e(\varsigma)$ with $\nu=\varsigma^2$. Then $\widetilde{\Phi}_1^e(\nu)$ is a polynomial of degree 3 in ν. The discriminant of $\widetilde{\Phi}_1^e(\nu)$ with respect to ν is

$$
\Delta_{\widetilde{\Phi}_1^e} := 512(n-1)^6(3n-4)(3n-2)\Upsilon_1,
$$

where $\Upsilon_1 := \big((3n^2-16n+26)n^2+8n(n-4)+10\big)\big(3n^4+2n(37n^2-142n+204)+96(n-4)\big)$. One can check that $\Upsilon_1>0$ for $n\ge 4$ even. Actually we have $3n^2-16n+26>0$ and $37n^2-142n+204>0$ for $n\in\mathbb{N}$ because their discriminants with respect to the variable n are -56 and -10028, respectively. Then $\Upsilon_1>0$ for $n\ge 4$ even, implying that $\Delta_{\widetilde{\Phi}_1^e}>0$ for $n\ge 4$ even. Thus $\widetilde{\Phi}_1^e(\nu)$ as well as $\Phi_1^e(\varsigma)$ has only one zero in the interval $(0,+\infty)$. Furthermore the resultant of $\Phi_1^e(\varsigma)$ and $(\Phi_1^e(\varsigma))'$ is

$$
\begin{aligned}
&512n(n-2)(2n-3)(3n-4)(n-1)^6(3n-2)^2(3n^4-16n^3\\
&+34n^2-32n+10)(3n^4+74n^3-284n^2+504n-384).
\end{aligned}
$$

One can show that the resultant has no zeros for $n \geq 4$ even, indicating that $\Phi_1^e(\varsigma)$ and $(\Phi_1^e(\varsigma))'$ have no common zeros with respect to the variable ς for $n \geq 4$ even. Thus the uniquely positive zero of $\Phi_1^e(\varsigma)$ is simple. From Theorem 6 the set of functions $\{g_1^e, \dots, g_6^e\}$ is an ECT-system in the interval $(0, +\infty)$, implying that $f_1^e(r)$ has at most 6 simple zeros in the interval $(0, +\infty)$. Hence system (2) has at most 6 crossing limit cycles for $n \geq 4$ even.

Furthermore one can compute the determinants

$$\det \frac{\partial(k_1^e, k_2^e, k_3^e, k_4^e, k_5^e, k_6^e)}{\partial(a_{110}, a_{111}, b_{100}, b_{101}, b_{102}, c_{110})} = \det \begin{pmatrix} 0 & 0 & -12 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & -8 & 0 \\ 6 & 0 & 0 & -6 & 0 & -6 \\ 0 & 0 & 0 & 0 & -12 & 0 \\ -3\pi & 0 & 0 & -3\pi & 0 & -3\pi \\ 6 & 0 & 0 & 6 & 0 & -6 \end{pmatrix}$$
$$= -248832\pi,$$

and

$$\det \frac{\partial(k_1^o, k_2^o, k_3^o, k_4^o, k_5^o, k_6^o, k_7^o)}{\partial(a_{100}, a_{110}, a_{120}, a_{111}, a_{102}, b_{100}, b_{102})}$$
$$= \det \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -12 & 0 \\ 0 & 0 & 0 & -4 & 0 & 0 & -8 \\ 12 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 12 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -12 \\ 0 & 0 & 8 & 0 & 4 & 0 & 0 \\ 0 & -3\pi & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = 995328\pi,$$

which are different from zero, implying that the coefficients $\{k_1^e, k_2^e, k_3^e, k_4^e, k_5^e, k_6^e\}$ and $\{k_1^o, k_2^o, k_3^o, k_4^o, k_5^o, k_6^o, k_7^o\}$ are linearly independent, respectively. Hence we can choose parameters $a_{ijk}, b_{ijk}, c_{ijk}$ and d_{ijk} ($i = 1, 2$ and $0 \leq j + k \leq 2$) such that system (2) with $n \geq 4$ even (resp. $n \geq 5$ odd) has exactly $0, 1, 2, 3, 4, 5$ or 6 (resp. $0, 1, 2, 3, 4, 5, 6$ or 7) crossing limit cycles. This proves Theorem 1. □

Proof of Theorem 2. The averaged function of order two is computed when $f_1(r) = 0$. Solving $f_1(r) = 0$, i.e. $f_1^e(r) = 0$ for $n \geq 4$ even and $f_1^o(r) = 0$ for $n \geq 5$ odd, which are equivalent to $k_1^e = k_2^e = k_3^e = k_4^e = k_5^e = k_6^e = 0$ and $k_1^o = k_2^o = k_3^o = k_4^o = k_5^o = k_6^o = k_7^o = 0$ respectively, yields the conditions $\mathcal{Q}_1^e$ given in (3) for $n \geq 4$ even and the conditions $\mathcal{Q}_1^o$ given in (4) for $n \geq 5$ odd. Let

$$\Gamma_i(r) := \left\langle \nabla h\big(\theta_i, r\big), \left(s, \int_0^{\theta_i} \frac{d\dot{r}}{d\varepsilon}\Big|_{(\varepsilon,\theta)=(0,\varphi)} d\varphi\right)\right\rangle, \quad i = 1, 2,$$

where $\theta_1 = \arctan(x^{n-1})$, $\theta_2 = \pi - (-1)^n\theta_1$ and x satisfies $r^2 = x^2 + x^{2n}$. Computations show that

$$\Gamma_1(r) = \left\langle \nabla h\big(\theta_1, r\big), \left(s, \int_0^{\theta_1} Q_1(t, r)dt\right)\right\rangle$$
$$= s\big(nr^n \tan\theta_1 \cos^n\theta_1 + r\cos\theta_1\big) + \frac{\big(\sin\theta_1 - nr^{n-1}\cos^n\theta_1\big)\Psi_1(r)}{12},$$

$$\Gamma_2(r) = \Big\langle \nabla h\big(\theta_2, r\big), \Big(s, \int_0^{\theta_1} Q_1(t,r)dt + \int_{\theta_1}^{\theta_2} P_1(t,r)dt\Big)\Big\rangle$$
$$= s\big(nr^n \tan\theta_2 \cos^n\theta_2 + r\cos\theta_2\big) + \frac{\big(\sin\theta_2 - nr^{n-1}\cos^n\theta_2\big)\big(\Psi_1(r) + \Psi_2(r)\big)}{12},$$

where

$$\begin{aligned}
\Psi_1(r) :=& -12c_{100}\sin\theta_1 - 6r(c_{101}+d_{110})\sin^2\theta_1 - 4r^2(c_{102}+d_{111})\sin^3\theta_1\\
&- 6c_{110}r(\theta_1 + \sin\theta_1\cos\theta_1) + 4r^2(c_{111}+d_{120})(\cos\theta_1 - 1)(\cos^2\theta_1\\
&+ \cos\theta_1 + 1) - c_{120}r^2(9\sin\theta_1 + \sin(3\theta_1)) + 12d_{100}(\cos\theta_1 - 1)\\
&+ 3d_{101}r(\sin(2\theta_1) - 2\theta_1) - 16d_{102}r^2(\cos\theta_1 + 2)\sin^4(\theta_1/2),\\
\Psi_2(r) :=& -3((-1)^n - 1)(4a_{100} + r^2(a_{102} + 3a_{120} + b_{111}))\sin(\theta_1)\\
&+ r^2((-1)^n - 1)(a_{102} - a_{120} + b_{111})\sin(3\theta_1) - 6r(a_{110} + b_{101})(\pi\\
&- \theta_1((-1)^n + 1)) + 3r((-1)^n + 1)(a_{110} - b_{101})\sin(2\theta_1) - (24b_{100}\\
&+ 6r^2(a_{111} + 3b_{102} + b_{120}))\cos(\theta_1) - 2r^2(a_{111} - b_{102} + b_{120})\cos(3\theta_1).
\end{aligned}$$

In order to ensure that $\Gamma_i(r) = 0$ if and only if $s = 0$, we need to eliminate Ψ_i for $i = 1, 2$, equivalently all coefficients of Ψ_i are equal to zero, i.e.

$$\begin{aligned}
&b_{100} = c_{100} = c_{110} = c_{120} = d_{100} = d_{101} = d_{102} = 0,\\
&c_{101} + d_{110} = c_{102} + d_{111} = c_{111} + d_{120} = a_{110} + b_{101} = 0,\\
&a_{110} - b_{101} = a_{111} + 3b_{102} + b_{120} = a_{111} - b_{102} + b_{120} = 0
\end{aligned} \tag{12}$$

for $n \geq 4$ even, and

$$\begin{aligned}
&a_{100} = b_{100} = c_{100} = c_{110} = c_{120} = d_{100} = d_{101} = d_{102} = c_{101} + d_{110} = 0,\\
&c_{102} + d_{111} = c_{111} + d_{120} = a_{102} + 3a_{120} + b_{111} = a_{102} - a_{120} + b_{111} = 0,\\
&a_{110} + b_{101} = a_{111} + 3b_{102} + b_{120} = a_{111} - b_{102} + b_{120} = 0
\end{aligned} \tag{13}$$

for $n \geq 5$ odd. From (12) and $f_1^e(r) = 0$ we obtain the conditions $\mathcal{Q}_2^e$ given in (5) for $n \geq 4$ even. Similarly from (13) and $f_1^o(r) = 0$ we obtain the conditions $\mathcal{Q}_2^o$ given in (6) for $n \geq 5$ odd.

From (8) the averaged function of order two is

$$\begin{aligned}
f_2(r) =& \int_0^{2\pi}\left\{\left(\frac{\partial^2\dot r}{\partial r\partial\varepsilon}\Big|_{\varepsilon=0}\int_0^{\theta}\frac{\partial\dot r}{\partial\varepsilon}\Big|_{(\varepsilon,\theta)=(0,\varphi)}d\varphi\right) + \frac{1}{2}\frac{\partial^2\dot r}{\partial\varepsilon^2}\Big|_{\varepsilon=0}\right\}d\theta\\
=& \int_0^{\theta_1}\left\{\frac{\partial Q_1(\theta,r)}{\partial r}\int_0^{\theta} Q_1(\varphi,r)d\varphi + Q_2(\theta,r)\right\}d\theta\\
&+ \int_{\theta_1}^{\theta_2}\left\{\frac{\partial P_1(\theta,r)}{\partial r}\left(\int_0^{\theta_1} Q_1(\varphi,r)d\varphi + \int_{\theta_1}^{\theta} P_1(\varphi,r)d\varphi\right) + P_2(\theta,r)\right\}d\theta\\
&+ \int_{\theta_2}^{2\pi}\left\{\frac{\partial Q_1(\theta,r)}{\partial r}\left(\int_0^{\theta_1} Q_1(\varphi,r)d\varphi + \int_{\theta_1}^{\theta_2} P_1(\varphi,r)d\varphi + \int_{\theta_2}^{\theta} Q_1(\varphi,r)d\varphi\right)\right.\\
&\qquad\left. + Q_2(\theta,r)\right\}d\theta.
\end{aligned}$$

Computations show that

$$f_2^e(r) := f_2(r) = \frac{\tilde{k}_1^e \tilde{g}_1^e + \tilde{k}_2^e \tilde{g}_2^e + \tilde{k}_3^e \tilde{g}_3^e + \tilde{k}_4^e \tilde{g}_4^e + \tilde{k}_5^e \tilde{g}_5^e + \tilde{k}_6^e \tilde{g}_6^e + \tilde{k}_7^e \tilde{g}_7^e + \tilde{k}_8^e \tilde{g}_8^e}{24\sqrt{x^2 + x^{2n}}}$$

for $n \geq 4$ even, and

$$f_2^o(r) := f_2(r) = \frac{\tilde{k}_1^o g_1^o + \tilde{k}_2^o g_2^o + \tilde{k}_3^o g_3^o + \tilde{k}_4^o g_4^o + \tilde{k}_5^o g_5^o + \tilde{k}_6^o g_6^o + \tilde{k}_7^o g_7^o}{6\sqrt{x^2 + x^{2n}}}$$

for $n \geq 5$ odd, where g_i^o $(i = 1, \dots, 6)$ are given in (11) and

$$\tilde{g}_1^e := x, \quad \tilde{g}_2^e := x^3, \quad \tilde{g}_3^e := x^{n+1}, \quad \tilde{g}_4^e := x^{n+3}, \quad \tilde{g}_5^e := x^{2n+1}, \quad \tilde{g}_6^e := x^2 + x^{2n},$$
$$\tilde{g}_7^e := (x^2 + x^{2n})\arctan(x^{n-1}), \quad \tilde{g}_8^e := x^{3n+1} - (x^2 + x^{2n})^2 \left(\frac{\pi}{2} - \arctan(x^{n-1})\right),$$
$$\tilde{k}_1^e := -48(b_{200} - d_{200}),$$
$$\tilde{k}_2^e := -16(a_{101}b_{120} + a_{211} + b_{110}b_{120} + 2b_{202} + b_{220} - c_{211} - 2d_{202} - d_{220}),$$
$$\tilde{k}_3^e := 24(a_{100}b_{120} + a_{210} - b_{201} - c_{210} + d_{201}), \quad \tilde{k}_4^e := -6b_{120}(a_{102} - 5a_{120} + b_{111}),$$
$$\tilde{k}_5^e := -48(b_{202} - d_{202}), \quad \tilde{k}_6^e := -12\pi(a_{100}b_{120} + a_{210} + b_{201} + c_{210} + d_{201}),$$
$$\tilde{k}_7^e := 24(a_{100}b_{120} + a_{210} + b_{201} - c_{210} - d_{201}), \quad \tilde{k}_8^e := 6b_{120}(a_{102} + 3a_{120} + b_{111}),$$
$$\tilde{k}_1^o := -12(b_{200} - d_{200}),$$
$$\tilde{k}_2^o := -4(a_{101}b_{120} + a_{211} + b_{101}b_{111} + b_{110}b_{120} + 2b_{202} + b_{220} - c_{211} - 2d_{202} - d_{220}),$$
$$\tilde{k}_3^o := 12(a_{200} - c_{200}), \quad \tilde{k}_4^o := 12(a_{220} - b_{101}b_{120} - c_{220}),$$
$$\tilde{k}_5^o := -12(b_{101}b_{111} + b_{202} - d_{202}),$$
$$\tilde{k}_6^o := 4(a_{101}b_{111} + a_{202} + 2a_{220} - b_{101}b_{120} + b_{110}b_{111} + b_{211} - c_{202} - 2c_{220} - d_{211}),$$
$$\tilde{k}_7^o := -3\pi(a_{210} + b_{201} + c_{210} + d_{201}).$$

Note that $f_2^o(r)$ has the same form as $f_1^o(r)$. Then $f_2^o(r)$ has at most 6 simple zeros. Moreover one can check that

$$\det \frac{\partial(\tilde{k}_1^o, \tilde{k}_2^o, \tilde{k}_3^o, \tilde{k}_4^o, \tilde{k}_5^o, \tilde{k}_6^o, \tilde{k}_7^o)}{\partial(a_{200}, a_{210}, a_{211}, a_{202}, b_{200}, b_{202}, c_{220})}$$
$$= \det \begin{pmatrix} 0 & 0 & 0 & 0 & -12 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 & -8 & 0 \\ 12 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -12 \\ 0 & 0 & 0 & 0 & 0 & -12 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & -8 \\ 0 & -3\pi & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = -995328\pi \neq 0,$$

implying that the coefficients $\{\tilde{k}_1^o, \tilde{k}_2^o, \tilde{k}_3^o, \tilde{k}_4^o, \tilde{k}_5^o, \tilde{k}_6^o, \tilde{k}_7^o\}$ are linearly independent. Hence the averaged function of order two produces the same result as the ones of order 1 for $n \geq 5$ odd.

For $n \geq 4$ even we compute the Wronskians

$$\begin{aligned}
W_1(\tilde{g}_1^e) &:= x,\\
W_2(\tilde{g}_1^e, \tilde{g}_2^e) &:= 2x^3,\\
W_3(\tilde{g}_1^e, \tilde{g}_2^e, \tilde{g}_3^e) &:= 2n(n-2)x^{n+2},\\
W_4(\tilde{g}_1^e, \ldots, \tilde{g}_4^e) &:= 4n^2(n^2-4)x^{2n+2},\\
W_5(\tilde{g}_1^e, \ldots, \tilde{g}_5^e) &:= 16n^4(n-2)^2(n^2+n-2)x^{4n-1},\\
W_6(\tilde{g}_1^e, \ldots, \tilde{g}_6^e) &:= -16n^4(n+2)(2n-1)(n-1)^2(n-2)^2x^{4n-4}\Phi_2^e(\varsigma),\\
W_7(\tilde{g}_1^e, \ldots, \tilde{g}_7^e) &:= \frac{128n^4(n+2)(2n-1)(n-1)^4(n-2)^2x^{7n-1}\Phi_3^e(\varsigma)}{(x^2+x^{2n})^5},\\
W_8(\tilde{g}_1^e, \ldots, \tilde{g}_8^e) &:= \frac{128n^4(n+2)(2n-1)(n-1)^5(n-2)^2x^{7n+6}\Phi_4^e(\varsigma)}{(x^2+x^{2n})^{10}},
\end{aligned}$$

where $\varsigma = x^{n-1}$ and

$$\begin{aligned}
\Phi_2^e(\varsigma) :=& (n-3)(2n-3)\varsigma^2 - (n+1),\\
\Phi_3^e(\varsigma) :=& n^2(n-3)(n-2)(2n-3)(3n-2)\varsigma^8 - 2(3n-2)(15n^5 - 110n^4 + 302n^3\\
&- 395n^2 + 240n - 48)\varsigma^6 + 2(3n-4)(3n-2)(5n^4 - 33n^3 + 83n^2 - 87n\\
&+ 38)\varsigma^4 - 2(3n-4)(n^5 + 18n^3 - 61n^2 + 54n - 16)\varsigma^2\\
&+ (n-2)^2(n+1)(3n-4)(3n-2),\\
\Phi_4^e(\varsigma) :=& \Phi_{41}^e(\varsigma) + 2\Phi_{42}^e(\varsigma)\left(1+\varsigma^2\right)^4 \arctan(\varsigma),
\end{aligned}$$

and $\Phi_{4i}^e(\varsigma)$ $(i = 1, 2)$, given in the Data.pdf, are polynomials of degrees 22 and 14 in ς, respectively. For $n \geq 4$ even $\Phi_2^e(\varsigma)$ has exactly one positive zero

$$\varsigma = \sqrt{\frac{n+1}{(n-3)(2n-3)}},$$

which is simple. Let $\widetilde{\Phi}_3^e(\nu) := \Phi_3^e(\varsigma)$ with $\nu = \varsigma^2$. Then $\widetilde{\Phi}_3^e(\nu)$ is a polynomial of degree 4 in ν. We can prove that $\widetilde{\Phi}_3^e(\nu)$ has exactly two positive zeros for $4 \leq n \leq 74$ even, where the two positive zeros are simple. We present the isolation intervals for the two positive zeros of $\widetilde{\Phi}_3^e(\nu)$ with $4 \leq n \leq 74$ even in Table 1. In order to determine if $\widetilde{\Phi}_3^e(\nu)$ has multiple zeros we compute the resultant between the polynomials $\widetilde{\Phi}_3^e(\nu)$ and $(\widetilde{\Phi}_3^e(\nu))'$ with respect to the variable ν, which is

$$\begin{aligned}
\mathcal{R} =& 49152n^2(n-2)(n-3)(2n-3)(n-1)^{12}(3n-2)^3(3n-4)^2(3n^8 - 49n^7\\
&+ 315n^6 - 1084n^5 + 2257n^4 - 2973n^3 + 2445n^2 - 1146n + 228)(108n^{12}\\
&- 9972n^{11} + 161301n^{10} - 1155787n^9 + 4608695n^8 - 11264874n^7 + 17422780n^6\\
&- 16171208n^5 + 6010272n^4 + 5255936n^3 - 9284608n^2 + 5511168n - 860160).
\end{aligned}$$

We obtain that $\mathcal{R}$ has no zeros for $n \geq 4$ even, implying that $\widetilde{\Phi}_3^e(\nu)$ and $(\widetilde{\Phi}_3^e(\nu))'$ has no common zeros. Thus the two zeros of $\widetilde{\Phi}_3^e(\nu)$ with $4 \leq n \leq 74$ even are simple.

$4 \le n \le 74$ even	The isolation intervals of positive zeros of $\widetilde{\Phi}_3^e(\nu)$	
$n = 4$	[81/1024, 41/512],	[25831/2048, 51663/4096]
$n = 6$	[609/4096, 305/2048],	[51565/4096, 25783/2048]
$n = 8$	[49/64, 785/1024],	[52347/4096, 13087/1024]
$n = 10$	[215/256, 861/1024],	[13259/1024, 53037/4096]
$n = 12$	[449/512, 899/1024],	[26787/2048, 53575/4096]
$n = 14$	[921/1024, 461/512],	[26997/2048, 53995/4096]
$n = 16$	[937/1024, 469/512],	[54327/4096, 6791/512]
$n = 18$	[237/256, 949/1024],	[54597/4096, 27299/2048]
$n = 20$	[957/1024, 479/512],	[54819/4096, 13705/1024]
$n = 22$	[241/256, 965/1024],	[55005/4096, 27503/2048]
$n = 24$	[485/512, 971/1024],	[27581/2048, 55163/4096]
$n = 26$	[975/1024, 61/64],	[27649/2048, 55299/4096]
$n = 28$	[489/512, 979/1024],	[55415/4096, 6927/512]
$n = 30$	[491/512, 983/1024],	[27759/2048, 55519/4096]
$n = 32$	[985/1024, 493/512],	[6951/512, 55609/4096]
$n = 34$	[987/1024, 247/256],	[55689/4096, 27845/2048]
$n = 36$	[989/1024, 495/512],	[55761/4096, 27881/2048]
$n = 38$	[991/1024, 31/32],	[27913/2048, 55827/4096]
$n = 40$	[993/1024, 497/512],	[13971/1024, 55885/4096]
$n = 42$	[995/1024, 249/256],	[27969/2048, 55939/4096]
$n = 44$	[249/256, 997/1024],	[27993/2048, 55987/4096]
$n = 46$	[997/1024, 499/512],	[56031/4096, 1751/128]
$n = 48$	[499/512, 999/1024],	[7009/512, 56073/4096]
$n = 50$	[999/1024, 125/128],	[28055/2048, 56111/4096]
$n = 52$	[125/128, 501/512],	[56145/4096, 28073/2048]
$n = 54$	[125/128, 501/512],	[56177/4096, 28089/2048]
$n = 56$	[501/512, 251/256],	[56207/4096, 3513/256]
$n = 58$	[501/512, 251/256],	[14059/1024, 56237/4096]
$n = 60$	[251/256, 503/512],	[28131/2048, 56263/4096]
$n = 62$	[251/256, 503/512],	[56287/4096, 1759/128]
$n = 64$	[251/256, 503/512],	[28155/2048, 56311/4096]
$n = 66$	[503/512, 63/64],	[14083/1024, 56333/4096]
$n = 68$	[503/512, 63/64],	[56353/4096, 28177/2048]
$n = 70$	[503/512, 63/64],	[14093/1024, 56373/4096]
$n = 72$	[503/512, 63/64],	[28195/2048, 56391/4096]
$n = 74$	[63/64, 505/512],	[7051/512, 56409/4096]

TABLE 1. The isolation intervals for two positive zeros of $\widetilde{\Phi}_3^e(\nu)$ with $4 \le n \le 74$ even.

For $n \ge 76$ even we take four closed intervals $I_i \subset (0, +\infty)$ $(i = 1, \dots, 4)$, where

$$I_1 := \left[0, \frac{1}{25}\right], \quad I_2 := \left[\frac{1}{25}, \frac{1}{2}\right], \quad I_3 := \left[\frac{1}{2}, 1\right], \quad I_4 := [1, 15]\,.$$

Note that $\widetilde{\Phi}_3^e(\nu)$ is a polynomial of degree 4 in ν. Then $\widetilde{\Phi}_3^e(\nu)$ has at most 4 zeros with respect to ν. We claim that $\widetilde{\Phi}_3^e(\nu)$ has exactly one positive zero in each interval

I_i $(i = 1, \dots, 4)$ for $n \geq 76$ even. We prove that $\widetilde{\Phi}_3^e(\nu)$ has the opposite signs at the endpoints of I_i $(i = 1, \dots, 4)$ for $n \geq 76$ even. More precisely we show that for $n \geq 76$ even

$$\widetilde{\Phi}_3^e(0) > 0, \quad \widetilde{\Phi}_3^e(1/25) < 0, \quad \widetilde{\Phi}_3^e(1/2) > 0, \quad \widetilde{\Phi}_3^e(1) < 0, \quad \widetilde{\Phi}_3^e(15) > 0,$$

Obviously $\widetilde{\Phi}_3^e(0) = (n-2)^2(n+1)(3n-4)(3n-2) > 0$ for $n \geq 76$ even. Computations show that

$$\begin{aligned}\widetilde{\Phi}_3^e(1/25) =& -16(2484n^6 - 198427n^5 + 1099723n^4 - 1818787n^3 \\ &+ 303779n^2 + 1348050n - 679700)/390625.\end{aligned}$$

We obtain that $\widetilde{\Phi}_3^e(1/25)$ has exactly one zero with respect to the variable n in each interval J_{1j} $(j = 1, \dots, 6)$, where

$$\begin{aligned} J_{11} &:= \left[-\frac{841}{1024}, -\frac{105}{128}\right], \quad J_{12} := \left[\frac{167}{256}, \frac{669}{1024}\right], \quad J_{13} := \left[\frac{5461}{4096}, \frac{2731}{2048}\right], \\ J_{14} &:= \left[\frac{3661}{2048}, \frac{7323}{4096}\right], \quad J_{15} := \left[\frac{5927}{2048}, \frac{741}{256}\right], \quad J_{16} := \left[\frac{18953}{256}, \frac{75813}{1024}\right]. \end{aligned}$$

One can check that $J_{1j} \cap [76, +\infty) = \emptyset$ for $i = 1, \dots, 6$. Then $\widetilde{\Phi}_3^e(1/25)$ has no zeros in the interval $[76, +\infty)$, implying that $\widetilde{\Phi}_3^e(1/25)$ does not change sign in the interval $[76, +\infty)$. Note that

$$\widetilde{\Phi}_3^e(1/25)|_{n=76} = -\frac{183080538143552}{390625} < 0.$$

Thus $\widetilde{\Phi}_3^e(1/25) < 0$ for $n \geq 76$ even. Similarly we have

$$\begin{aligned} \widetilde{\Phi}_3^e(1/2) &= 3(46n^6 - 497n^5 + 1689n^4 - 2736n^3 + 2756n^2 - 1824n + 512)/16, \\ \widetilde{\Phi}_3^e(1) &= -16(5n^5 - 29n^4 + 65n^3 - 85n^2 + 66n - 20), \\ \widetilde{\Phi}_3^e(15) &= 64(315n^6 + 1236n^5 - 24435n^4 + 85508n^3 - 112887n^2 + 56232n - 8017). \end{aligned}$$

We obtain that $\widetilde{\Phi}_3^e(1/2)$, $\widetilde{\Phi}_3^e(1)$ and $\widetilde{\Phi}_3^e(15)$ have exactly one zero in each interval J_{2j} $(j = 1, \dots, 4)$, J_{3j} $(j = 1, 2, 3)$ and J_{4j} $(j = 1, \dots, 4)$ respectively, where

$$\begin{aligned} J_{21} &:= \left[\frac{309}{512}, \frac{619}{1024}\right], \quad J_{22} := \left[\frac{2759}{2048}, \frac{5519}{4096}\right], \quad J_{23} := \left[\frac{1071}{512}, \frac{8569}{4096}\right], \\ J_{24} &:= \left[\frac{6357}{1024}, \frac{3179}{512}\right], \quad J_{31} := \left[\frac{321}{512}, \frac{643}{1024}\right], \quad J_{32} := \left[\frac{701}{512}, \frac{5609}{4096}\right], \\ J_{33} &:= \left[\frac{1417}{512}, \frac{5669}{2048}\right], \quad J_{41} := \left[-\frac{6277}{512}, -\frac{50215}{4096}\right], \quad J_{42} := \left[\frac{481}{2048}, \frac{963}{4096}\right], \\ J_{43} &:= \left[\frac{341}{512}, \frac{683}{1024}\right], \quad J_{44} := \left[\frac{3033}{2048}, \frac{6067}{4096}\right]. \end{aligned}$$

Note that $J_{ij} \cap [76, +\infty) = \emptyset$ for $i = 2, 3, 4$ and $j = 1, 2, 3$ or 4. Then $\widetilde{\Phi}_3^e(1/2)$, $\widetilde{\Phi}_3^e(1)$ and $\widetilde{\Phi}_3^e(15)$ have no zeros with respect to the variable n in the interval $[76, +\infty)$, indicating that $\widetilde{\Phi}_3^e(1/2)$, $\widetilde{\Phi}_3^e(1)$ and $\widetilde{\Phi}_3^e(15)$ do not change signs in the interval $[76, +\infty)$ separately. Furthermore one can check that

$$\begin{aligned} \widetilde{\Phi}_3^e(1/2)|_{n=76} &= 1436100761268 > 0, \\ \widetilde{\Phi}_3^e(1)|_{n=76} &= -187810740032 < 0, \end{aligned}$$

$$\widetilde{\Phi}_3^e(15)|_{n=76} = 4035588498286528 > 0.$$

Thus $\widetilde{\Phi}_3^e(1/2) > 0$, $\widetilde{\Phi}_3^e(1) < 0$ and $\widetilde{\Phi}_3^e(15) > 0$ for $n \geq 76$ even, which together with $\widetilde{\Phi}_3^e(0) > 0$ and $\widetilde{\Phi}_3^e(1/25) < 0$ implies that $\widetilde{\Phi}_3^e(\nu)$ has exactly one positive zero in each interval I_i $(i = 1, \dots, 4)$ for $n \geq 76$ even. This proves the claim that $\widetilde{\Phi}_3^e(\nu)$ has exactly one positive zero in each interval I_i $(i = 1, \dots, 4)$ for $n \geq 76$ even.

From Theorem 7 there exist two linear combinations of functions $\{\tilde{g}_1^e, \dots, \tilde{g}_7^e\}$ such that they have 9 positive zeros for $4 \leq n \leq 74$ even and 11 positive zeros for $n \geq 76$ even, respectively.

Solving $\tilde{k}_8^e = 0$ we obtain that either $b_{120} = 0$ or $b_{111} = -a_{102} - 3a_{120}$. Choosing $b_{111} = -a_{102} - 3a_{120}$, where we do not take $b_{120} = 0$ because it results in $\tilde{k}_4^e = 0$, we compute the determinant

$$\begin{aligned}
&\det \frac{\partial(\tilde{k}_1^e, \tilde{k}_2^e, \tilde{k}_3^e, \tilde{k}_4^e, \tilde{k}_5^e, \tilde{k}_6^e, \tilde{k}_7^e)}{\partial(a_{211}, b_{200}, b_{201}, b_{120}, b_{202}, c_{210}, d_{201})} \\
&= \det \begin{pmatrix}
0 & -48 & 0 & 0 & 0 & 0 & 0 \\
-16 & 0 & 0 & -16(a_{101} + b_{110}) & -32 & 0 & 0 \\
0 & 0 & -24 & 24a_{100} & 0 & -24 & 24 \\
0 & 0 & 0 & 48a_{120} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -48 & 0 & 0 \\
0 & 0 & -12\pi & -12\pi a_{100} & 0 & -12\pi & -12\pi \\
0 & 0 & 24 & 24a_{100} & 0 & -24 & -24
\end{pmatrix} \\
&= 48922361856\pi a_{120},
\end{aligned}$$

which is different from zero for fixed $a_{120} \neq 0$. It implies that the coefficients $\{\tilde{k}_1^e, \tilde{k}_2^e, \tilde{k}_3^e, \tilde{k}_4^e, \tilde{k}_5^e, \tilde{k}_6^e, \tilde{k}_7^e\}$ are linearly independent for fixed $a_{120} \neq 0$. Hence in the case $b_{111} = -a_{102} - 3a_{120}$ we obtain that $f_2^e(r)$ has at most 9 positive zeros for $4 \leq n \leq 74$ even and at most 11 positive zeros for $n \geq 76$ even, where the bounds 9 and 11 are reachable from Theorem 7 but we do not know whether the bounds $0, 1, 2, 3, 4, 5, 6, 7, 8$ for $4 \leq n \leq 74$ even and the bounds $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ for $n \geq 76$ even are reachable from Theorem 7.

Next we give more information for the above unknown bounds. From $\tilde{k}_3^e = \tilde{k}_8^e = 0$ we obtain

$$b_{111} = -a_{102} - 3a_{120}, \qquad d_{201} = -a_{210} + b_{201} + c_{210} - a_{100}b_{120}. \tag{14}$$

One can compute the Wronskians

$$\begin{aligned}
W_1(\tilde{g}_1^e) &:= x, \\
W_2(\tilde{g}_1^e, \tilde{g}_2^e) &:= 2x^3, \\
W_3(\tilde{g}_1^e, \tilde{g}_2^e, \tilde{g}_4^e) &:= 2n(n+2)x^{n+4}, \\
W_4(\tilde{g}_1^e, \tilde{g}_2^e, \tilde{g}_4^e, \tilde{g}_5^e) &:= 8n^2(n-2)(n-1)(n+2)x^{3n+2}, \\
W_5(\tilde{g}_1^e, \tilde{g}_2^e, \tilde{g}_4^e, \tilde{g}_5^e, \tilde{g}_6^e) &:= -8n^2(n-2)(n-1)(n+2)(2n-1)x^{3n}\Theta_1(\nu), \\
W_5(\tilde{g}_1^e, \tilde{g}_2^e, \tilde{g}_4^e, \tilde{g}_5^e, \tilde{g}_6^e, \tilde{g}_7^e) &:= -\frac{16n^2(n-2)(n-1)^2(n+2)(2n-1)x^{4n-6}\Theta_2(\nu)}{(x^{2n}+x^2)^4},
\end{aligned}$$

where $\nu = x^{2n-2}$ and

$$\Theta_1(\nu) := (n-3)(2n-3)\nu + (n+1),$$

$$\begin{aligned}\Theta_2(\nu) := & n^2(n-3)(n-2)(2n-3)\nu^5 + n^2(n-2)(12n^3-54n^2+53n-1)\nu^4 \\ & -2(36n^6-303n^5+1009n^4-1708n^3+1504n^2-624n+96)\nu^3 \\ & +2(6n^6-81n^5+307n^4-578n^3+664n^2-456n+128)\nu^2 \\ & +(n-2)(16n^4-41n^3+11n^2+56n-32)\nu + n^2(n-2)(n+1).\end{aligned}$$

Obviously $\Theta_1(x)$ has no positive zeros with respect to the variable ν for $n \geq 4$ even. We claim that $\Theta_2(\nu)$ has five zeros with respect to the variable ν for $n \geq 4$ even. Note that $\Theta_2(\nu)$ has at most five zeros because it is a polynomial of degree 5 in ν. For $n \geq 4$ even we take the five intervals

$$\tilde{I}_1 := (-\infty, -2], \quad \tilde{I}_2 := \left[-2, -\frac{1}{n}\right], \quad \tilde{I}_3 := \left[-\frac{1}{n}, 0\right], \quad \tilde{I}_4 := [0,1], \quad \tilde{I}_5 := [1,6].$$

Since the leading coefficient of $\Theta_2(\nu)$ with respect to the variable ν is $n^2(n-3)(n-2)(2n-3)$, which is positive for $n \geq 4$ even, we have

$$\lim_{\nu \to -\infty} \Theta_2(\nu) = -\infty \tag{15}$$

for each fixed $n \geq 4$ even. We can prove that

$$\Theta_2(-2) > 0, \quad \Theta_2(-1/n) < 0, \quad \Theta_2(0) > 0, \quad \Theta_2(1) < 0, \quad \Theta_2(6) > 0$$

for $n \geq 4$ even, which together with (15) implies that $\Theta_2(\nu)$ has a unique zero with respect to the variable ν in each $\tilde{I}_i$ $(i = 1, \ldots, 5)$ for $n \geq 4$ even and, moreover, the five zeros are simple because $\Theta_2(\nu)$ is a polynomial of degree 5 in ν. This proves the claim. Hence $\Theta_2(\nu)$ has two positive zeros with respect to the variable ν for $n \geq 4$ even, where the two positive zeros are simple. Under (14) compute the determinant

$$\begin{aligned}&\det \frac{\partial(\tilde{k}_1^e, \tilde{k}_2^e, \tilde{k}_4^e, \tilde{k}_5^e, \tilde{k}_6^e, \tilde{k}_7^e)}{\partial(a_{211}, b_{200}, b_{201}, b_{120}, b_{202}, c_{210})} \\ &= \det \begin{pmatrix} 0 & -48 & 0 & 0 & 0 & 0 \\ -16 & 0 & 0 & -16(a_{101}+b_{110}) & -32 & 0 \\ 0 & 0 & 0 & 48a_{120} & 0 & 0 \\ 0 & 0 & 0 & 0 & -48 & 0 \\ 0 & 0 & -24\pi & 0 & 0 & -24\pi \\ 0 & 0 & 0 & 48a_{100} & 0 & -48 \end{pmatrix} \\ &= 2038431744\pi a_{120},\end{aligned}$$

which is different from zero for fixed $a_{120} \neq 0$, implying that the coefficients $\{\tilde{k}_1^e, \tilde{k}_2^e, \tilde{k}_4^e, \tilde{k}_5^e, \tilde{k}_6^e, \tilde{k}_7^e\}$ are linearly independent. From Theorem 6 we obtain that $f_2^e(r)$ has at most 7 positive zeros for $n \geq 4$ even under (14), where the bounds $0, 1, 2, 3, 4, 5, 6, 7$ are reachable.

From $\tilde{k}_5^e = \tilde{k}_8^e = 0$ we obtain

$$b_{111} = -a_{102} - 3a_{120}, \qquad d_{202} = b_{202}. \tag{16}$$

Under (16) using the similar method we can prove that $f_2^e(r)$ has at most 9 positive zeros for $n \geq 76$ even. In fact one can compute the Wronskians

$$\begin{aligned} W_1(\tilde{g}_1^e) &:= x, \\ W_2(\tilde{g}_1^e, \tilde{g}_2^e) &:= 2x^3, \\ W_3(\tilde{g}_1^e, \tilde{g}_2^e, \tilde{g}_3^e) &:= 2n(n-2)x^{n+2}, \end{aligned}$$

$$\begin{aligned} W_4(\tilde{g}_1^e, \tilde{g}_2^e, \tilde{g}_3^e, \tilde{g}_4^e) &:= 4n^2(n^2-4)x^{2n+2},\\ W_5(\tilde{g}_1^e, \tilde{g}_2^e, \tilde{g}_3^e, \tilde{g}_4^e, \tilde{g}_6^e) &:= 4n^2(n-2)(n-1)(n+2)x^{2n}\Xi_1(\nu),\\ W_6(\tilde{g}_1^e, \tilde{g}_2^e, \tilde{g}_3^e, \tilde{g}_4^e, \tilde{g}_6^e, \tilde{g}_7^e) &:= \frac{32n^2(n-2)(n-1)^3(n+2)x^{5n+2}\Xi_2(\nu)}{(x^{2n}+x^2)^4}, \end{aligned}$$

where $\nu = x^{2n-2}$ and

$$\begin{aligned} \Xi_1(\nu) :=& (n-3)(2n-3)(2n-1)\nu - (n+1),\\ \Xi_2(\nu) :=& n^2(n-3)(n-2)(2n-3)(2n-1)\nu^3 + (-32n^6 + 248n^5 - 747n^4\\ &+ 1141n^3 - 916n^2 + 348n - 48)\nu^2 + (12n^6 - 108n^5 + 377n^4\\ &- 641n^3 + 594n^2 - 304n + 64)\nu - (n-2)(n+1)(3n-4)(3n-2). \end{aligned}$$

Clearly $\Xi_1(\nu)$ has a unique positive zero with respect to the variable ν for $n \geq 76$ even. Note that $\Xi_2(\nu)$ is a polynomial of degree 3 in ν. Then $\Xi_2(\nu)$ has at most three zeros. We can prove that $\Xi_2(\nu)$ has three positive zeros for $n \geq 76$ even, which lie in the intervals

$$\left[0, \frac{1}{4}\right], \quad \left[\frac{1}{4}, \frac{1}{2}\right], \quad \left[\frac{1}{2}, 8\right],$$

respectively, where the three zeros are simple. Under (16) we compute the determinant

$$\begin{aligned} &\det \frac{\partial(\tilde{k}_1^e, \tilde{k}_2^e, \tilde{k}_3^e, \tilde{k}_4^e, \tilde{k}_6^e, \tilde{k}_7^e)}{\partial(a_{211}, b_{200}, b_{201}, b_{120}, c_{210}, d_{201})}\\ &= \det \begin{pmatrix} 0 & -48 & 0 & 0 & 0 & 0\\ -16 & 0 & 0 & -16(a_{101}+b_{110}) & 0 & 0\\ 0 & 0 & -24 & 24a_{100} & -24 & 24\\ 0 & 0 & 0 & 48a_{120} & 0 & 0\\ 0 & 0 & -12\pi & -12\pi a_{100} & -12\pi & -12\pi\\ 0 & 0 & 24 & 24a_{100} & -24 & -24 \end{pmatrix}\\ &= -1019215872\pi a_{120}, \end{aligned}$$

which is different from zero for fixed $a_{120} \neq 0$, implying that the coefficients $\{\tilde{k}_1^e, \tilde{k}_2^e, \tilde{k}_3^e, \tilde{k}_4^e, \tilde{k}_6^e, \tilde{k}_7^e\}$ are linearly independent. From Theorem 6 we obtain that $f_2^e(r)$ has at most 9 positive zeros for $n \geq 76$ even under (16), where the bound 9 is reachable. Here we cannot determine whether the bound 8 for $n \geq 4$ even and the bound 10 for $n \geq 76$ even are reachable. This proves Theorem 2. □

Theorem 2 with $n \geq 4$ even is obtained based on $\tilde{k}_8^e = 0$. When $\tilde{k}_8^e \neq 0$, we need to discuss the positive zeros of $W_8(\tilde{g}_1^e, \ldots, \tilde{g}_8^e)$, equivalently to discuss the positive zeros of $\Phi_4^e(\varsigma)$. However it is difficult because the transcendental function $\arctan(\varsigma)$ is involved in $\Phi_4^e(\varsigma)$. Once the maximum number of the positive zeros of $\Phi_4^e(\varsigma)$ is obtained, where all zeros are simple, we can give an upper bound on the number of the positive zeros of $f_2^e(r)$ as well as an upper bound for the number of the crossing limit cycles for system (2) with $n \geq 4$ even. In fact, if Φ_4^e has at most ℓ positive zeros and they are simple, then from Theorem 8 and our Theorem 2 we obtain that $f_2^e(r)$ has at most $11+\ell$ positive zeros for $4 \leq n \leq 74$ even and at most $13+\ell$ positive zeros for $n \geq 76$ even, implying that system (2) has at most $11+\ell$ crossing limit cycles for $4 \leq n \leq 74$ even and at most $13+\ell$ crossing limit cycles for $n \geq 76$

even, where we do not determine whether the bounds $11+\ell$ and $13+\ell$ are reachable from Theorem 8.

Although it is difficult to give the exact number for the positive zeros of $\Phi_4^e(\varsigma)$, we have the following.

Proposition 9. *$\Phi_4^e(\varsigma)$ has at least two positive zeros for $n \geq 4$ even and their multiplicities are not greater than 2.*

Proof. One can check that

$$\Phi_4^e(0) = -3(n-3)(n-2)^2(n+1)(2n-3)(3n-5)(3n-4)(3n-2)\pi < 0 \tag{17}$$

for $n \geq 4$ even. We claim that

$$\Phi_4^e(1/2) > 0, \qquad \lim_{\varsigma \to +\infty} \Phi_4^e(\varsigma) = -\infty, \tag{18}$$

for $n \geq 4$ even. In fact computations show that

$$\Phi_4^e(1/2) = \frac{\mathcal{P}(n)}{4194304}, \tag{19}$$

where $\mathcal{P}(n) := \mathcal{P}_1(n, \pi, \arctan(1/2)) - \mathcal{P}_2(n, \pi, \arctan(1/2))$ and

$$\begin{aligned}
\mathcal{P}_1(n,u,v) := {} & 1134000000n^{12}v + 1441352448n^{12} + 3412800000n^{11}v \\
& + 121218048n^{11} + 60970837500n^{10}u + 583290525000n^9v \\
& + 509788601040n^9 + 663611153125n^8u + 1999679006250n^7v \\
& + 889035231524n^7 + 1396262796875n^6u + 294867027364n^6 \\
& + 4021862818750n^5v + 2409126075000n^4u + 2540209625760n^4 \\
& + 4139680025000n^3v + 1225019550000n^2u + 1192575320128n^2 \\
& + 926299200000nv + 82454400000u + 76723261440, \\
\mathcal{P}_2(n,u,v) := {} & 567000000n^{12}u + 1706400000n^{11}u + 121941675000n^{10}v \\
& + 104802757040n^{10} + 291645262500n^9u + 1327222306250n^8v \\
& + 1025258589540n^8 + 999839503125n^7u + 2792525593750n^6v \\
& + 2010931409375n^5u + 1821531770900n^5 + 4818252150000n^4v \\
& + 2069840012500n^3u + 2117692526960n^3 + 2450039100000n^2v \\
& + 463149600000nu + 436788493312n + 164908800000v.
\end{aligned}$$

Since each term in $\mathcal{P}_i(n,u,v)$ $(i=1,2)$ is positive for $u>0$, $v>0$ and $n \geq 4$ even, we obtain that $\mathcal{P}_i(n,u,v)$ $(i=1,2)$ are monotonically increasing with respect to u and v, i.e.

$$\mathcal{P}_i(n,u_1,v_1) < \mathcal{P}_i(n,u_2,v_2), \quad i=1,2, \tag{20}$$

for $0 < u_1 < u_2$ and $0 < v_1 < v_2$. It is not difficult to obtain that $\pi \in (\alpha_l, \alpha_r)$ and $\arctan(1/2) \in (\beta_l, \beta_r)$, where

$$\alpha_l := \frac{31415926535}{10000000000}, \quad \alpha_r := \frac{31415926536}{10000000000},$$
$$\beta_l := \frac{4636476090}{10000000000}, \quad \beta_r := \frac{4636476091}{10000000000}.$$

From (20) we have

$$\mathcal{P}_1(n,\pi,\arctan(1/2)) > \mathcal{P}_1(n,\alpha_l,\beta_l), \qquad \mathcal{P}_2(n,\pi,\arctan(1/2)) < \mathcal{P}_2(n,\alpha_r,\beta_r),$$

which together with (19) implies that

$$\mathcal{P}(n) > \mathcal{P}_1(n,\alpha_l,\beta_l) - \mathcal{P}_2(n,\alpha_r,\beta_r) = \frac{\mathcal{P}_3(n)}{3200000}, \tag{21}$$

where

$$\begin{aligned}\mathcal{P}_3(n) :=\, & 594706566447360n^{12} - 11703229107545088n^{11} + 96655398605897804n^{10} \\ & - 435202418619644304n^9 + 1421356431555721857n^8 \\ & - 4239701812023250196n^7 + 10837146103965175535n^6 \\ & - 20077862508308980156n^5 + 25199140207829937432n^4 \\ & - 21442988667971270224n^3 + 12496425497233152368n^2 \\ & - 4679486287651730432n + 829765849245441024.\end{aligned}$$

We find that $\mathcal{P}_3(n)$ has no zeros greater than 4 with respect to the variable n, indicating that $\mathcal{P}_3(n)$ does not change sign for $n \geq 4$ even. Computations show that

$$\mathcal{P}_3(4) = \frac{728196387383804091}{2500} > 0,$$

indicating that $\mathcal{P}_3(n) > 0$ for $n \geq 4$ even. It together with (21) implies that $\mathcal{P}(n) > 0$ for $n \geq 4$ even, i.e. $\Phi_4^e(1/2) > 0$ for $n \geq 4$ even.

Next we prove the second equality in (18). Since $\Phi_{42}^e(\varsigma)$ is a polynomial in ς and its leading coefficient with respect to ς does not vanish for $n \geq 4$ even, we have $\Phi_{42}^e(\varsigma) \neq 0$ for $\varsigma > 0$ sufficiently large. Then we write

$$\Phi_4^e(\varsigma) = \frac{\mathcal{Q}_2(\varsigma)}{\mathcal{Q}_1(\varsigma)},$$

where

$$\mathcal{Q}_1(\varsigma) := \frac{1}{(1+\varsigma^2)^4\Phi_{42}^e(\varsigma)}, \qquad \mathcal{Q}_2(\varsigma) := \frac{\Phi_{41}^e(\varsigma)}{(1+\varsigma^2)^4\Phi_{42}^e(\varsigma)} + 2\arctan\varsigma. \tag{22}$$

Obviously $\lim_{\varsigma\to+\infty}\mathcal{Q}_1(\varsigma) = 0$. On the other hand, since both $\Phi_{41}^e(\varsigma)$ and $(1+\varsigma^2)^4\Phi_{42}^e(\varsigma)$ are polynomials of degree 22 in ς, we have

$$\lim_{\varsigma\to+\infty}\frac{\Phi_{41}^e(\varsigma)}{(1+\varsigma^2)^4\Phi_{42}^e(\varsigma)} = \frac{\operatorname{lcoeff}(\Phi_{41}^e(\varsigma),\varsigma)}{\operatorname{lcoeff}\big((1+\varsigma^2)^4\Phi_{42}^e(\varsigma),\varsigma\big)} = -\pi,$$

where the notation $\operatorname{lcoeff}(f(x),x)$ denotes the leading coefficient of a polynomial $f(x)$ with respect to the variable x. It together with $\lim_{\varsigma\to+\infty}\arctan\varsigma = \pi/2$ means that $\lim_{\varsigma\to+\infty}\mathcal{Q}_2(\varsigma) = 0$. From the l'Hospital rule we have

$$\lim_{\varsigma\to+\infty}\frac{\mathcal{Q}_2(\varsigma)}{\mathcal{Q}_1(\varsigma)} = \lim_{\varsigma\to+\infty}\frac{\mathcal{Q}_2'(\varsigma)}{\mathcal{Q}_1'(\varsigma)} = \lim_{\varsigma\to+\infty}\frac{\mathcal{Q}_4(\varsigma)}{\mathcal{Q}_3(\varsigma)},$$

where $\mathcal{Q}_4(\varsigma) := -32\varsigma\mathcal{Q}_{41}(\varsigma)\mathcal{Q}_{42}(\varsigma)$ and $\mathcal{Q}_3(\varsigma)$, $\mathcal{Q}_{41}(\varsigma)$, $\mathcal{Q}_{42}(\varsigma)$ are given in the Data.pdf. Note that

$$\operatorname{lcoeff}(\mathcal{Q}_3(\varsigma),\varsigma) = 53909856000, \qquad \operatorname{lcoeff}(\mathcal{Q}_4(\varsigma),\varsigma) = -2922784594329600000$$

for $n = 4$, and that

$$\begin{aligned}\mathrm{lcoeff}(\mathcal{Q}_3(\varsigma),\varsigma) &= 33n^3(n-2)(n-3)(5n-3)(4n-3)(3n\\&\quad -1)(2n-1)(2n-3)(3n-2)(4n-1),\\ \mathrm{lcoeff}(\mathcal{Q}_4(\varsigma),\varsigma) &= -32n^5(n-4)(4n-3)(3n-1)(3n-4)(2n-1)(5n\\&\quad -3)(3n-5)(4n-1)(n-3)^2(2n-3)^2(3n-2)^2(n-2)^4,\end{aligned}$$

for $n \geq 6$ even. Moreover one can check that $\deg(\mathcal{Q}_4(\varsigma),\varsigma) = 29$ for $n = 4$ and $\deg(\mathcal{Q}_4(\varsigma),\varsigma) = 31$ for $n \geq 6$ even, which together with $\deg(\mathcal{Q}_3(\varsigma),\varsigma) = 14$ indicates that

$$\lim_{\varsigma\to+\infty}\frac{\mathcal{Q}_4(\varsigma)}{\mathcal{Q}_3(\varsigma)} = \begin{cases}\dfrac{\mathrm{lcoeff}(\mathcal{Q}_4(\varsigma),\varsigma)}{\mathrm{lcoeff}(\mathcal{Q}_3(\varsigma),\varsigma)}\lim\limits_{\varsigma\to+\infty}\varsigma^{15} = -\dfrac{596377600}{11}\lim\limits_{\varsigma\to+\infty}\varsigma^{15} & \text{if } n=4,\\ \dfrac{\mathrm{lcoeff}(\mathcal{Q}_4(\varsigma),\varsigma)}{\mathrm{lcoeff}(\mathcal{Q}_3(\varsigma),\varsigma)}\lim\limits_{\varsigma\to+\infty}\varsigma^{17} = \dfrac{32\Upsilon_2(n)}{33}\lim\limits_{\varsigma\to+\infty}\varsigma^{17} & \text{if } n\geq 6 \text{ even,}\end{cases}$$

where $\Upsilon_2(n) := -n^2(n-3)(n-4)(3n-4)(2n-3)(3n-2)(3n-5)(n-2)^3$. Clearly $\Upsilon_2(n) < 0$ for $n \geq 6$ even. Thus we have

$$\lim_{\varsigma\to+\infty}\frac{\mathcal{Q}_4(\varsigma)}{\mathcal{Q}_3(\varsigma)} = -\infty, \quad \text{i.e.} \quad \lim_{\varsigma\to+\infty}\frac{\mathcal{Q}_2(\varsigma)}{\mathcal{Q}_1(\varsigma)} = -\infty,$$

for $n \geq 4$ even. This proves the claim. From (17) and (18) together with the continuity of $\Phi_4^e(\varsigma)$ we obtain that $\Phi_4^e(\varsigma)$ has at least two positive zeros with respect to the variable ς for $n \geq 4$ even, one in the interval $(0, 1/2)$ and the other in the interval $(1/2, +\infty)$.

Finally we prove the second part of the proposition. Let ς_0 be a positive zero of $\Phi_4^e(\varsigma)$. First we show that $\Phi_{42}^e(\varsigma_0) \neq 0$. In fact, if $\Phi_{42}^e(\varsigma_0) = 0$ then we have $\Phi_{41}^e(\varsigma_0) = 0$ from $\Phi_4^e(\varsigma_0) = 0$. Note that

$$\begin{aligned}\mathrm{lcoeff}\big(\Phi_{42}^e(\varsigma),\varsigma\big) &= 3n^3(n-2)(n-3)(5n-3)(4n-3)(3n-1)(2n\\&\quad -1)(2n-3)(3n-2)(4n-1),\end{aligned}$$

which is different from zero for $n \geq 4$ even. The resultant of the polynomials $\Phi_{41}^e(\varsigma)$ and $\Phi_{42}^e(\varsigma)$ with respect to the variable ς is

$$\mathrm{res}\big(\Phi_{41}^e(\varsigma),\Phi_{42}^e(\varsigma),\varsigma\big) = \mathcal{R}_1(n)\mathcal{R}_2^2(n)\mathcal{R}_3^2(n),$$

where

$$\begin{aligned}\mathcal{R}_1(n) &:= n^{23}(3n-5)^3(4n-1)^5(3n-1)^5(5n-3)^5(2n-1)^5(4n-3)^5(n\\&\quad +1)^5(3n-4)^9(2n-3)^{10}(n-3)^{10}(3n-2)^{18}(n-2)^{21}(n-1)^{108},\\ \mathcal{R}_2(n) &:= 108n^{12} - 9972n^{11} + 161301n^{10} - 1155787n^9 + 4608695n^8\\&\quad - 11264874n^7 + 17422780n^6 - 16171208n^5 + 6010272n^4\\&\quad + 5255936n^3 - 9284608n^2 + 5511168n - 860160,\end{aligned}$$

and we omit the long expression of $\mathcal{R}_3(n)$. Using the command "IsolatingInterval" together with "Root" in the software MATHEMATICA we can prove that $\mathcal{R}_2(n)$ and $\mathcal{R}_3(n)$ have no zeros for $n \geq 4$ even, which together with $\mathcal{R}_1(n) \neq 0$ for $n \geq 4$ even implies that $\Phi_{41}^e(\varsigma)$ and $\Phi_{42}^e(\varsigma)$ have no common zeros for $n \geq 4$ even. Thus the positive zero ς_0 of $\Phi_4^e(\varsigma)$ satisfies $\Phi_{42}^e(\varsigma_0) \neq 0$.

Assume that the positive zero ς_0 of $\Phi_4^e(\varsigma)$ is of multiplicity 3. Then ς_0 is a positive zero of $\mathcal{Q}_2(\varsigma)$ with multiplicity 3, where $\mathcal{Q}_2(\varsigma)$ is given in (22). Therefore $\mathcal{Q}_2'(\varsigma)$ and $\mathcal{Q}_2''(\varsigma)$ have positive common zeros with respect to the variable ς for $n \geq 4$ even. It is not difficult to check that the zeros of $\mathcal{Q}_2'(\varsigma)$ and $\mathcal{Q}_2''(\varsigma)$ are determined by two polynomials $\mathcal{Q}_5(\varsigma)$ and $\mathcal{Q}_6(\varsigma)$ respectively, where we omit the long expressions of $\mathcal{Q}_i(\varsigma)$ for $i = 5, 6$. Using the command "Resultant" together with the commands "IsolatingInterval" and "Root" in the software MATHEMATICA we can prove that $\mathcal{Q}_5(\varsigma)$ and $\mathcal{Q}_6(\varsigma)$ have no positive common zeros with respect to the variable ς for $n \geq 4$ even, a contradiction with the assumption. Hence the multiplicity of ς_0 is at most 2. This proves the proposition. □

Numerically we do not find that $\Phi_4^e(\varsigma)$ has three positive zeros with respect to the variable ς for $n \geq 4$ even. Moreover it is difficult to determine whether the two positive zeros of $\Phi_4^e(\varsigma)$ obtained in Proposition 9 are simple because the transcendental function $\arctan(\varsigma)$ is involved. For some fixed $n \geq 4$ even we plot the graphs of $\Phi_4^e(\varsigma)$ and $(\Phi_4^e(\varsigma))'$ and find that they have no common intersection points at the positive ς-axis.

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