Time-reversibility and integrability
of $p : -q$ resonant vector fields

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Abstract

We study the local analytical integrability in a neighborhood of $p : -q$ resonant singular point of a two-dimensional vector field and its connection to time-reversibility with respect to the non-smooth involution $\varphi(x, y) = (y^{p/q}, x^{q/p})$. Some generalizations of the theory developed by Sibirsky for $1 : -1$ resonant case to the $p : -q$ resonant case are presented.

1   Introduction

Consider an n-dimensional system of ordinary differential equations

$$\dot{x} = F(x),$$

where $F(x)$ is an n-dimensional vector functions defined on some domain D of $\mathbb{R}^n$ or $\mathbb{C}^n$. It is said (see e.g. [2, 15]) that system (1.2) is *time-reversible* on D if there exists an involution ψ defined on D such that

$$D_\psi^{-1} \cdot F \circ \psi = -F. \tag{1.1}$$ `inv_gen`

We say that system (1.2) is *completely analytically integrable* on D if it admits $n - 1$ functionally independent analytic first integrals on D.

A time-reversal symmetry is one of the fundamental symmetries that appears in nature, in particular, important both for classical and quantum mechanics. Various properties of systems exhibiting such symmetries have been studied by many authors, see e.g. [1, 2, 12, 13, 15, 27, 29] and the references therein.

Our paper is devoted to the investigation of the interconnection of time-reversibility and local integrability in a neighborhood of a singular point of systems of the form

$$\dot{\mathbf{x}} = A\mathbf{x} + X(\mathbf{x}), \qquad (1.2) \boxed{\texttt{sys\_X}}$$

where A is an $n \times n$ matrix with entries in $\mathbb{R}$ or $\mathbb{C}$ being $\mathbf{x} = (x_1, \ldots, x_n)$, $X(\mathbf{x})$ is a vector-function without constant and linear terms defined on some domain D of $\mathbb{R}^n$ or $\mathbb{C}^n$.

One of the first results in such studies is due to Poincaré. It follows from his results that if in the two-dimensional case the eigenvalues of A are pure imaginary and the system has an axis of symmetry passing through the origin, then it admits an analytic first integral in a neighborhood of the origin.

A generalization of this result is presented in [3], where it is shown that if system (1.2) is time-reversible with respect to a certain linear involution and two eigenvalues of the matrix A are pure imaginary, then under some assumptions, the system has at least one analytic first integral in a neighborhood of the origin.

A detailed study of the interconnection of time-reversibility and local integrability for systems (1.2) was presented in [17]. In [17] and [28] the notion of time-reversibility was generalized to the case when on the right-hand side of (1.1) "−1" is replaced by a primitive root of unity.

In this paper, we limit our consideration to the two-dimensional systems (1.2) with non-degenerate matrix A. In the case when a two-dimensional system (1.2) is real and the eigenvalues of A are pure imaginary, and the vector field is symmetric with respect to a curve passing through the origin, the origin of (1.2) is a center, and, therefore, has an analytic local integral in a neighborhood of the origin. This geometric argument was used in [30] in order to find some integrable systems in the family of real cubic systems (see also [2] for recent developments in this direction). The symmetry axis is, in the general case, an analytic curve passing through the origin. However, from the work of Montgomery and Zippin [18], any analytic involution ψ associated with a time-reversal symmetry can be linearized in such a way that the symmetry axis becomes a straight line. From this result in [1] the normal form theory is used to establish an algorithm to determine if a two-dimensional system (1.2) is orbitally reversible.

A detailed study of real polynomial systems which are time-reversible under reflection with respect to a line was performed by Sibirsky [24, 25]. In particular, he showed that in the polynomial case the set of such systems in the space of parameters is the variety of a binomial ideal defined by invariants of the rotation group of the system (some similar results were obtained also in [5, 16]). Later on, the results obtained by Sibirsky were generalized to the case of complex systems (1.2) with $1 : -1$ resonant singular point at the origin in [14, 19, 20].

In this paper, we consider system (1.2) having a $p : -q$ resonant singular point at the origin, which we write in the form

$$
\begin{aligned}
\dot{x} &= \quad px - \sum_{j+k\geq 1, j\geq -1}^{\infty} a_{jk} x^{j+1} y^k = px\Big(1 - \sum_{j+k\geq 1, j\geq -1}^{\infty} \frac{1}{p} a_{jk} x^j y^k\Big), \\
\dot{y} &= -qy + \sum_{j+k\geq 1, j\geq -1}^{\infty} b_{kj} x^k y^{j+1} = -qy\Big(1 - \sum_{j+k\geq 1, j\geq -1}^{\infty} \frac{1}{q} b_{kj} x^k y^j\Big),
\end{aligned}
\qquad (1.3) \boxed{\texttt{gspq}}
$$

where $p, q \in \mathbb{N}$, $\gcd(p, q) = 1$. Both vector field (1.3) and the associated differential operator are denoted by $\mathcal{X}$.

Unless $p = q = 1$ this system is not time-reversible under a linear transformation. Our study deals with the time-reversibility of system (1.3) with respect to the involution

$$\varphi(x, y) = (y^{p/q}, x^{q/p}). \tag{1.4} \boxed{\texttt{inv\_pq}}$$

We will prove that if a system (1.3) is time-reversible with respect to (1.4), then it admits an analytic first integral on a neighborhood of the origin. We will also give an extension of the results of [24, 25] and their generalizations obtained in [14, 19, 20], presenting an algorithm for finding subsets of polynomial systems (1.3) which are time-reversible with respect to (1.4) and generalizing the notion of Sibirsky ideal to polynomial systems (1.3). It will be shown that, in fact, the theory developed for the $1 : -1$ resonant case can be extended to systems (1.3), however not to the whole family, but only to a certain subfamily of (1.3).

2 Integrability of time-reversible systems

For a system (1.3) it is always possible to find a series of the form

$$\Psi(x, y) = x^q y^p + \sum_{\substack{j+k > p+q \\ j,k \in \mathbb{N}_0}} v_{j-q,k-p} x^j y^k \tag{2.1} \boxed{\texttt{pqInt}}$$

for which

$$\mathcal{X}\Psi = g_{q,p}(x^q y^p)^2 + g_{2q,2p}(x^q y^p)^3 + g_{3q,3p}(x^q y^p)^4 + \cdots,$$

where $g_{kq,kp}$, for $k = 1, 2, \ldots$, are polynomials in the parameters a_{jk}, b_{kj} of system (1.3). Polynomials $g_{kq,kp}$ are called the saddle quantities of system (1.3) (sometimes also the focus quantities). System (1.3) corresponding to some fixed values a_{jk}^*, b_{kj}^* of the parameters has a local analytical first integral in a neighborhood of the origin if and only if $g_{kq,kp}(a^*, b^*) = 0$ for all $k \in \mathbb{N}$ (see e.g. [20, 23]).

The following theorem shows that if system (1.3) is time-reversible with respect to (1.4) then it has an analytic first integral of the form (2.1). Observe, that if $p = q = 1$ then the map (1.4) is just a permutation of the variables, so the statement presents a generalization of known results of [3, 17, 19] to the case of $p : -q$ resonant systems.

⟨th1⟩ **Theorem 1.** *Assume that system* (1.3) *is time-reversible with respect to the involution* (1.4)*, that is,*

$$D_\varphi^{-1} \cdot \mathcal{X} \circ \varphi = -\mathcal{X}. \tag{2.2} \boxed{\texttt{inv\_yx}}$$

Then it admits an analytic first integral of the form (2.1) *in a neighborhood of the origin.*

To prove the above theorem we will need the following results.

⟨lem1⟩ **Lemma 2.** *System* (1.3) *with p or q different from 1 is time-reversible with respect to* (2.2) *if and only if*

$$b_{qv,up} = \frac{q}{p} a_{qu,pv}, \qquad a_{qu,pv} = \frac{p}{q} b_{qv,pu}, \tag{2.3} \boxed{\texttt{abpq}}$$

where $u, v = 0, 1, 2, \ldots$ and the other coefficients in (1.3) *are equal to zero.*

Proof. Using involution (1.4), that is, performing the substitution

$$x_1 = y^{p/q}, \qquad y_1 = x^{q/p}, \tag{2.4} \boxed{\texttt{subs\_x1y1}}$$

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after straightforward calculations we obtain

$$\dot{x}_1 = -px_1\left(1 - \sum_{j_1+k_1\geq 1, j_1\geq -1}^{\infty} \frac{1}{q}b_{k_1j_1}x_1^{\frac{qj_1}{p}}y_1^{\frac{pk_1}{q}}\right),$$

$$\dot{y}_1 = qy_1\left(1 - \sum_{j_1+k_1\geq 1, j_1\geq -1}^{\infty} \frac{1}{p}a_{j_1k_1}x_1^{\frac{qk_1}{p}}y_1^{\frac{pj_1}{q}}\right).$$

In view of (2.2) it should hold

$$\frac{1}{p}a_{jk}x^jy^k = \frac{1}{q}b_{k_1j_1}x^{\frac{qj_1}{p}}y^{\frac{pk_1}{q}}, \qquad (2.5)\;\boxed{\texttt{eq\_ab}}$$

where the exponents on the right-hand side should be non-negative integers or

$$\frac{qj_1}{p} = \frac{pj_1}{q} = -1.$$

However the latter equality is impossible unless $p = q = 1$. Thus, (2.5) can take place if we set $j_1 = pu$, $k_1 = qv$, where $u, v = 0, 1, 2, \ldots$. This yields formulas (2.3). $\qquad \square$

The following statement follows directly from (2.3).

Corollary 3. *If system* (1.3) *with* p *or* q *different from* 1 *is time-reversible with respect to* (2.2) *then* $x = 0$ *and* $y = 0$ *are the separatrixes of the system.*

Remark. Formulas (2.3) and the results obtained below remain valid also in the case of $1 : -1$ resonant singular points. But since the results in the $1 : -1$ resonant case are known, below we work under the assumption $p/q \neq 1$ taking advantage from the fact that in such case the subscripts of the parameters a_{jk}, b_{kj} of (1.3) are non-negative.

We augment the set of coefficients in (2.1) with the collection

$$J = \{v_{-q+s, q-s} : s = 0, \ldots, p+q\},$$

where, in agreement with formula (2.1), we set $v_{00} = 1$ and $v_{mn} = 0$ for all other elements of J, so that elements of J are the coefficients of the terms of degree $p + q$ in $\Psi(x, y)$ of the form (2.1). Let $\mathbb{N}_0$ be the set of nonnegative integers and for a given positive integer n we denote by $\mathbb{N}_{-n}$ the set $\{-n, -n+1, \ldots, -1\} \cup \mathbb{N}_0$.

By [22, p. 117] the coefficients v_{k_1,k_2} of the series (2.1) can be computed recursively using the formula

$$v_{k_1,k_2} = \begin{cases} \dfrac{1}{pk_1-qk_2} \displaystyle\sum_{\substack{s_1+s_2=0 \\ s_1\geq -q, s_2\geq -p}}^{k_1+k_2-1} [(s_1+q)a_{k_1-s_1,k_2-s_2} - (s_2+p)b_{k_1-s_1,k_2-s_2}]v_{s_1,s_2} & \text{if } pk_1 \neq qk_2, \\[4mm] 0 & \text{if } pk_1 = qk_2, \end{cases}$$

$$(2.6)\;\boxed{\texttt{vk1k2pq}}$$

(in [22] formula (2.6) was obtained for the case of polynomial system (1.3) but, obviously, it remains valid also in the case when the right-hand sides of (1.3) are series).

We order the index set of parameters a_{jk} in the first equation of (1.3) in some manner, say by degree lexicographic order from least to greatest, and write the ordered set as $S = \{(1,0), (0,1), (-1,2), (2,0), \ldots\}$. Consistently with this we then order the parameters

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as $(a_{10}, a_{01}, a_{-1,2}, a_{20}, \ldots, b_{02}, b_{2,-1}, b_{10}, b_{01})$ so that any monomial appearing in v_{ij} has the form $a_{10}^{\nu_1} a_{01}^{\nu_2} \cdots a_{s,t}^{\nu_\ell} b_{t,s}^{\nu_{\ell+1}} \cdots b_{10}^{\nu_{2\ell-1}} b_{01}^{\nu_{2\ell}}$ for some $\nu = (\nu_1, \ldots, \nu_{2\ell})$ and $\ell = 1, 2, \ldots$. To simplify the notation, for $\nu \in \mathbb{N}_+^{2\ell}$ we write

$$[\nu] \stackrel{\text{def}}{=} a_{10}^{\nu_1} a_{01}^{\nu_2} \cdots a_{s,t}^{\nu_\ell} b_{t,s}^{\nu_{\ell+1}} \cdots b_{10}^{\nu_{2\ell-1}} b_{01}^{\nu_{2\ell}}, \tag{2.7} \boxed{\texttt{nu\_gs}}$$

so if the k-th variable in the product is a_{pq}, then $2\ell - k + 1$-st variable is b_{qp}.

For each $m \in \mathbb{N}$ we consider the finite subset S_m of the set S which corresponds to the case when system (1.3) is a polynomial system of degree m, so

$$S_m = \{(1,0), (0,1), (-1,2), (2,0), \ldots, (-1,m)\}.$$

Denote by $\ell(m)$ the number of elements in S_m and let $L^m : \mathbb{N}_+^{2\ell(m)} \to \mathbb{Z}^2$ be the linear map defined by

$$L^m(\nu) = (L_1^m(\nu), L_2^m(\nu)) = \nu_1(1,0) + \nu_2(0,1) + \cdots + \nu_{\ell(m)}(-1, m) +$$
$$\nu_{\ell(m)+1}(m, -1) + \cdots + \nu_{2\ell(m)-1}(1,0) + \nu_{2\ell(m)}(0,1). \tag{2.8} \boxed{\texttt{Ldef}}$$

Let $\mathbb{K}[a, b]$ be the ring of polynomials in parameters a_{jk}, b_{jk} of system (1.3) over the field $\mathbb{K}$ and for $f \in \mathbb{K}[a, b]$ we write $f = \sum_{\nu \in \text{Supp}(f)} f^{(\nu)}[\nu]$, where $\text{Supp}(f)$ denotes those $\nu \in \mathbb{N}_+^{2\ell(m)}$, $m = 1, 2, \ldots$, that the coefficient of $[\nu]$ in the polynomial f is nonzero.

Definition 4. *For $(j, k) \in \mathbb{N}_{-1} \times \mathbb{N}_{-1}$, a polynomial*

$$f = \sum_{\nu \in \text{Supp}(f)} f^{(\nu)}[\nu]$$

in the polynomial ring $\mathbb{C}[a, b]$ is a (j, k)-polynomial if, for every $\nu \in \text{Supp}(f)$, $L^\ell(\nu) = (j, k)$ for all sufficiently large ℓ.

The reader can consult [22, Section 3.4] for more details about (j, k)-*polynomials* in the case of polynomial system (1.3).

From now on, we will limit our consideration to the systems of the form

$$\dot{x} = \quad x\Big(p - \sum_{t_1+t_2=1}^{\infty} a_{qt_1, pt_2} x^{qt_1} y^{pt_2}\Big),$$
$$\dot{y} = -y\Big(q - \sum_{t_1+t_2=1}^{\infty} b_{qt_2, pt_1} x^{qt_2} y^{pt_1}\Big). \tag{2.9} \boxed{\texttt{gs\_uv\_inf}}$$

By Lemma 2, in the case when $p/q \neq 1$ systems (1.3), which are time-reversible with respect to involution (1.4), form a subfamily of systems (2.9), so we do not lose generality working with family (2.9) if we are interesting in time-reversibility with respect to (1.4).

$\langle \texttt{lem2} \rangle$ **Lemma 5.** *For system (2.9) if v_{k_1, k_2} is a non-zero coefficient of series (2.1) computed by (2.6), then*

$$k_1 = t_1 q, \quad k_2 = t_2 p, \tag{2.10} \boxed{\texttt{t1t2}}$$

for some non-negative integers t_1, t_2. Moreover, under involution (2.2) the term

$$v_{qt_1, pt_2} x^{qt_1+q} y^{pt_2+p} \tag{2.11} \boxed{\texttt{vt1t2}}$$

of (2.1) is changed to the term

$$v_{qt_2, pt_1} x^{qt_2+q} y^{pt_1+p}, \tag{2.12} \boxed{\texttt{vt2t1}}$$

and vice versa.

Proof. For system (2.9) the linear map (2.8) can be written in the form

$$L^{qm}(\nu) = (qL_1^m(\nu), pL_2^m(\nu)),$$

where $(L_1^m(\nu), L_2^m(\nu))$ is defined by (2.8) and $L^{qm-q+1}(\nu) = \cdots = L^{qm-1}(\nu) = L^{qm}(\nu)$. By Theorem 4 of [21] $v_{k_1 k_2}$ is a (k_1, k_2)-polynomial. Therefore for each monomial $[\nu]$ of $v_{k_1 k_2}$ it holds that

$$(qL_1^m(\nu), pL_2^m(\nu)) = (k_1, k_2),$$

for all sufficiently large m. It means that q divides k_1 and p divides k_2, that is, (2.10) holds.

Performing in (2.11) substitution (2.4) we see that (2.12) holds. □

Remark. Theorem 4 of [21] mentioned above was formulated in [21] for the case of polynomial systems (1.3) but it remains correct also in the case when the right-hand sides of (1.3) are series.

⟨th2⟩ **Theorem 6.** *The formal series (2.1) computed according to (2.6) is unchanged under involution (2.2), that is, in view of Lemma 5,*

$$v_{qt_1, pt_2} = v_{qt_2, pt_1}. \tag{2.13} \boxed{\text{v\_cond}}$$

Proof. We prove the claim using induction on $t_1 + t_2$. When $t_1 = t_2 = 0$ we have by the definition $v_{00} = 1$, so the claim holds.

Using Lemma 5 we can write formula (2.6) as

$$v_{qt_1, pt_2} = \begin{cases} \dfrac{1}{pq(t_1 - t_2)} \displaystyle\sum_{\substack{s_1 + s_2 = 0 \\ s_1 \geq -q, s_2 \geq -p}}^{qt_1 + pt_2 - 1} [(s_1 + q)a_{qt_1 - s_1, pt_2 - s_2} - (s_2 + p)b_{qt_1 - s_1, pt_2 - s_2}]v_{s_1, s_2} & \text{if } t_1 \neq t_2, \\[4mm] 0 & \text{if } t_1 = t_2. \end{cases} \tag{2.14} \boxed{\text{vk1k2pq\_1}}$$

In view of (2.3) and taking into account that $v_{j,k}$ are (j, k)-polynomials for $t_1 \neq t_2$ we can change the rule of summation obtaining from (2.14)

$$v_{qt_1, pt_2} = \frac{1}{pq(t_1 - t_2)} \sum_{s_1 = 0}^{qt_1} \sum_{s_2 = 0}^{pt_2} [(s_1 + q)a_{qt_1 - s_1, pt_2 - s_2} - (s_2 + p)b_{qt_1 - s_1, pt_2 - s_2}]v_{s_1, s_2} = $$
$$\frac{1}{pq(t_1 - t_2)} \sum_{\tilde{s}_1 = 0}^{t_1} \sum_{\tilde{s}_2 = 0}^{t_2} [(\tilde{s}_1 + 1)qa_{q(t_1 - \tilde{s}_1), p(t_2 - \tilde{s}_2)} - (\tilde{s}_2 + 1)pb_{q(t_1 - \tilde{s}_1), p(t_2 - s_2)}]v_{q\tilde{s}_1, p\tilde{s}_2}, \tag{2.15} \boxed{\text{v\_1}}$$

where $s_1 = q\tilde{s}_1$, $s_2 = p\tilde{s}_2$.

Performing similar computations we have

$$v_{qt_2, pt_1} = \frac{1}{pq(t_2 - t_1)} \sum_{s_1 = 0}^{qt_2} \sum_{s_2 = 0}^{pt_1} [(s_1 + q)a_{qt_2 - s_1, pt_1 - s_2} - (s_2 + p)b_{qt_2 - s_1, pt_1 - s_2}]v_{s_1, s_2} = $$
$$\frac{1}{pq(t_2 - t_1)} \sum_{\tilde{s}_2 = 0}^{t_2} \sum_{\tilde{s}_1 = 0}^{t_1} [(\tilde{s}_2 + 1)qa_{q(t_2 - \tilde{s}_2), p(t_1 - \tilde{s}_1)} - (\tilde{s}_1 + 1)pb_{q(t_2 - \tilde{s}_2), p(t_1 - \tilde{s}_1)}]v_{q\tilde{s}_2, p\tilde{s}_1}, \tag{2.16} \boxed{\text{vk1k2pq\_2\_1}}$$

where $s_1 = q\tilde{s}_2$, $s_2 = p\tilde{s}_1$.

Using (2.3) we further obtain from (2.16)

$$v_{qt_2,pt_1} = \frac{1}{pq(t_2 - t_1)} \sum_{\tilde{s}_2=0}^{t_2} \sum_{\tilde{s}_1=0}^{t_1} [(\tilde{s}_2 + 1)pb_{q(t_1-\tilde{s}_1),p(t_2-\tilde{s}_2)} - (\tilde{s}_1 + 1)qa_{q(t_1-\tilde{s}_1),p(t_2-\tilde{s}_2)}]v_{q\tilde{s}_2,p\tilde{s}_1} =$$

$$\frac{1}{pq(t_1 - t_2)} \sum_{\tilde{s}_1=0}^{t_1} \sum_{\tilde{s}_2=0}^{t_2} [(\tilde{s}_1 + 1)qa_{q(t_1-\tilde{s}_1),p(t_2-\tilde{s}_2)} - (\tilde{s}_2 + 1)pb_{q(t_1-\tilde{s}_1),p(t_2-\tilde{s}_2)}]v_{q\tilde{s}_1,p\tilde{s}_2}, \quad (2.17)\;\boxed{\text{v\_2}}$$

where we have changed $v_{q\tilde{s}_2,p\tilde{s}_1}$ to $v_{q\tilde{s}_1,p\tilde{s}_2}$ using the induction hypothesis. Comparing the expressions for (2.15) and (2.17) we conclude that (2.13) holds, that is, the series $\Psi(x,y)$ computed by (2.6) is unchanged under the involution (2.2). $\qquad \square$

Using the obtained results we prove Theorem 1 as follows.

Proof of Theorem 1. Denote by $\mathfrak{X}$ the vector field of system (2.9). By Theorem 6 the series $\Psi(x,y)$ computed by (2.6) is unchanged under involution (1.4).

Assume that

$$\mathfrak{X}\Psi = \alpha(x,y).$$

Since by our assumption the system is time-reversible, it also holds that

$$\mathfrak{X}\Psi = -\alpha(x,y),$$

yielding $\alpha(x,y) \equiv 0$. That means, $\Psi(x,y)$ is a formal first integral of (1.3). But then also there exists an analytic first integral of the form (2.1) (see e.g. [21]). $\qquad \square$

Corollary 7. *If system* (1.3) *is time-reversible with respect to involution* (1.4), *then the system in the distinguished Poincaré–Dulac normal form is also time-reversible with respect to the same involution.*

Proof. Since by Theorem 1 any time-reversible system (1.3) is locally analytically integrable, its distinguished normal form can be written as

$$\dot{x} = px\left(1 + \sum_{k=1}^{\infty} g_k(x^q y^p)^k\right), \qquad \dot{y} = -qy\left(1 + \sum_{k=1}^{\infty} g_k(x^q y^p)^k\right). \qquad (2.18)\;\boxed{\text{normal}}$$

(see e.g. [21, 30]). Clearly, the latter system is time-reversible with respect to (1.4). $\qquad \square$

The normal form of any locally analytically integrable system (1.3) is given by (2.18). System (2.18) is time-reversible with respect to the involution (1.4). Therefore any locally analytically integrable system (1.3) is conjugate to a time-reversible system, in the sense that there exists a change of variables ϕ that transforms the original system to the normal form (2.18) and consequently the original system is time-reversible with respect to the involution $\tilde{\psi} = \phi^{-1} \circ \psi \circ \phi$, where ψ is the involution (1.4). Hence the analytical integrability of system (1.3) is always associated with a time-reversal symmetry. In fact, all nondegenerate centers are conjugate to a time-reversible system, and all the nilpotent centers are orbitally time-reversible. This does not happen for systems with null linear part, see [12].

The problem with the map $\tilde{\psi}$ is that we have no idea about the form of $\tilde{\psi}$ not even the leading terms of such involution. Therefore from the found results, we cannot deduce an algorithm based on the computation of the involution of the original system. However

several methods to compute the saddle or focus quantities are known, see for instance [8, 10, 11, 22] and references therein.

Nevertheless, always there exists a change ϕ such that any differential system (1.3) is transformed to its normal form

$$\dot{y}_1 = py_1(1 + Y_1(y_1^q y_2^p)), \qquad \dot{y}_2 = -qy_2(1 + Y_2(y_1^q y_2^p)), \qquad (2.19)\ \{?\}$$

and the results known for such resonance can be applied to the $p : -q$ resonance but only in the normal form. The change $z_1 = x^q$, $z_2 = y^p$ does not work for the original system (1.3), but it is possible to apply it to system (2.9) reducing the study to the $1 : -1$ resonant case. However we have chosen to work with system (2.9) directly obtaining the important property of series (2.1) given in Theorem 6, which is related to the results of the next section.

3 Conditions of time-reversibility and the Sibirsky ideal

In this section we propose an algorithmic approach which allows for a given polynomial family (2.9) to find the set of systems which are time-reversible with respect to (1.4). We also give a description of the set using the so-called Sibirsky ideal obtaining some generalizations of the results of [14, 19].

We will limit our consideration to polynomial systems of the form (2.9), that is, systems of the form

$$\begin{aligned}
\dot{x} = \ & x\Big(p - \sum_{u+v=1}^{n} a_{qu,pv} x^{qu} y^{pv}\Big), \\
\dot{y} = \ & -y\Big(q - \sum_{u+v=1}^{n} b_{qv,pu} x^{qv} y^{pu}\Big),
\end{aligned} \qquad (3.1)\ \boxed{\texttt{gs\_uv}}$$

assuming that $p/q \neq 1$.

Denote by ℓ the number of parameters in the first equation of (3.1). For $k = 1, \dots, \ell$ let

$$\zeta_k = u_k - v_k \qquad (3.2)\ \boxed{\texttt{zeta}}$$

and consider the ideal

$$H = \langle 1 - w\gamma,\ a_{qu_k,pv_k} - t_k,\ b_{qv_k,pu_k} - \tfrac{q}{p}\gamma^{\zeta_k} t_k\ :\quad k = 1, \dots, \ell\rangle. \qquad (3.3)\ \boxed{\texttt{idealH}}$$

$\langle\texttt{prTR}\rangle$ **Proposition 8.** *The following statements hold:*

(a) *The Zariski closure of the set of systems in family (3.1), which are time-reversible with respect to involution (1.4) after the transformation*

$$x \to \alpha x,\ y \to \alpha^{-1} y \qquad (3.4)\ \boxed{\texttt{trans\_ort}}$$

with $\alpha \in \mathbb{C} \setminus \{0\}$, is the variety $\mathbf{V}(\mathcal{I})$ of the ideal

$$\mathcal{I} = H \cap \mathbb{C}[a, b]. \qquad (3.5)\ \boxed{\texttt{Ir}}$$

(b) *If the parameters $a_{qu,pv}$, $b_{qv,pu}$ of system (3.1) belong to the variety $\mathbf{V}(\mathcal{I})$, then the system admits a local analytic first integral of the form (2.1).*

Proof. Performing in system (3.1) transformation (3.4) we obtain the system of the same shape with the parameters $a_{qu,pv}, b_{qv,pu}$ changed according to the rule

$$a_{qu,pv} \mapsto \alpha^{pv-qu} a_{qu,pv}, \qquad b_{qv,pu} \mapsto \alpha^{pu-qv} b_{qv,pu},$$

where $u + v = 1, \ldots, n$.

By Lemma 2 the system obtained after transformation (3.4) is time-reversible with respect to involution (1.4) if and only if for some $\alpha \neq 0$

$$\alpha^{pv-qu} a_{qu,pv} = \frac{p}{q} \alpha^{pu-qv} b_{qv,pu}, \tag{3.6} \boxed{\texttt{conpq}}$$

where $u + v = 1, \ldots, n$. Equivalently, we can rewrite (3.6) as

$$a_{qu_k,pv_k} = t_k, \quad b_{qv_k,pu_k} - \frac{q}{p} \gamma^{\zeta_k} t_k, \tag{3.7} \boxed{\texttt{conpq\_1}}$$

where $\gamma = \alpha^{-(p+q)}$, $k = 1, \ldots, \ell$ and ζ_k are defined by (3.2).

From (3.7) using the Implicitization Theorem (see e.g. [7]) we conclude that the first statement holds.

(b) By construction, $\mathbf{V}(\mathcal{I})$ is the Zariski closure of systems which are time-reversible with respect to (1.4) after a linear transformation (3.4), so, in view of Theorem 1 it is the Zariski closure of systems which admit a first integral of the form (2.1). However the set of systems in the space of parameters of (3.1) having an analytic first integral integral of the form (2.1) is an algebraic set (see e.g. Theorem 3.2.5 of [22]). Therefore all systems from $\mathbf{V}(\mathcal{I})$ admit an analytic first integral of the form (2.1). $\qquad\square$

Remark. Obviously, generically the set of time-reversible systems is a proper subset of $\mathbf{V}(\mathcal{I})$.

As an example we consider the $1 : -2$ resonant system of the form (3.1) of degree five:

$$\dot{x} = x\left(1 - \sum_{u+v=1}^{2} a_{2u,v} x^{2u} y^v\right) = x - a_{01}xy - a_{20}x^3 - a_{21}x^3y - a_{02}xy^2 - a_{40}x^5,$$

$$\dot{y} = -y\left(2 - \sum_{u+v=1}^{2} b_{2v,u} x^{2v} y^u\right) = -2y + b_{01}y^2 + b_{20}x^2y + b_{21}x^2y^2 + b_{02}y^3 + b_{40}x^4y. \tag{3.8} \boxed{\texttt{gs\_uv5}}$$

Proposition 9. *System (3.8) admits an analytic first integral of the form (2.1) if the 10-tuple $(a_{01}, \ldots, a_{40}, b_{40}, \ldots, b_{01})$ of its coefficients belong to the variety of the ideal*

$$\widetilde{\mathcal{I}} = \langle 2a_{21} - b_{21}, -a_{40}b_{01}^2 + 2a_{20}^2 b_{02}, 4a_{02}a_{40} - b_{02}b_{40}, 8a_{02}a_{20}^2 - b_{01}^2 b_{40}, -2a_{02}a_{20}b_{20} +$$
$$a_{01}b_{01}b_{40}, 2a_{01}a_{40}b_{01} - a_{20}b_{02}b_{20}, 4a_{01}a_{20} - b_{01}b_{20}, -a_{02}b_{20}^2 + 2a_{01}^2 b_{40}, 8a_{01}^2 a_{40} - b_{02}b_{20}^2 \rangle.$$

Proof. In the case of system (3.8) the ideal H of Proposition 8 is

$$\langle a_{01} - t_1, a_{20} - t_2, a_{02} - t_3, a_{40} - t_4, a_{21} - t_5,$$
$$b_{20} - 2t_1\gamma^{-1}, b_{01} - 2t_2\gamma, b_{40} - 2t_3\gamma^{-2}, b_{02} - 2t_4\gamma^2, b_{21} - 2t_5, 1 - w\gamma \rangle.$$

Computing the reduced Gröebner basis of this ideal with respect to the lexicographic ordering with $\gamma > w > t_1 > t_2 > t_3 > t_4 > a_{01} > a_{02} > a_{20} > a_{21} > a_{40} > b_{01} > b_{02} > b_{20} > b_{21} > b_{40}$ we obtain the set of polynomials

$$\{2a_{21} - b_{21}, 4a_{01}a_{20} - b_{01}b_{20}, a_{02}b_{20}^2 - 2a_{01}^2 b_{40}, 2a_{02}a_{20}b_{20} - a_{01}b_{01}b_{40}, 8a_{02}a_{20}^2 - b_{01}^2 b_{40},$$
$$a_{40}b_{01}^2 - 2a_{20}^2 b_{02}, 2a_{01}a_{40}b_{01} - a_{20}b_{02}b_{20}, 8a_{01}^2 a_{40} - b_{02}b_{20}^2, 4a_{02}a_{40} - b_{02}b_{40}, \ldots\},$$

where the dots stand for the polynomials which depend ob $\gamma, w, t_1, t_2, t_3, t_4, t_5$.

The polynomials of the Gröebner basis which do not depend on $\gamma, w, t_1, t_2, t_3, t_4$ form a basis of the ideal $\mathcal{I}$ of Proposition 8 and they are exactly the polynomials defining the ideal in the statement of the present proposition. $\qquad\square$

Remark. By (a) of Proposition 8 the variety $\mathbf{V}(\tilde{\mathcal{I}})$ is the Zariski closure of the set of time-reversible systems in family (3.8).

We denote by S the ordered set of subscripts of the coefficients of the nonlinear terms of the first equation in (3.1). Letting ℓ the number of elements of S, S can be written as

$$S = \{(qu_1, pv_1), \ldots, (qu_\ell, pv_\ell)\} = \{\bar{\imath}_1, \ldots, \bar{\imath}_\ell\}.$$

For $\bar{\imath}_s = (qu_s, pv_s)$, let $\bar{\jmath}_s = (qv_s, pu_s)$. We call $\bar{\imath}_s$ and $\bar{\jmath}_s$ *conjugate* vectors (or conjugate indices). Any monomial appearing in the coefficient $v_{k_1 k_2}$ of (2.1) has the form $a_{\bar{\imath}_1}^{\nu_1} \cdots a_{\bar{\imath}_\ell}^{\nu_\ell} b_{\bar{\jmath}_\ell}^{\nu_{\ell+1}} \cdots b_{\bar{\jmath}_1}^{\nu_{2\ell}}$ for some $\nu = (\nu_1, \ldots, \nu_{2\ell})$. We use notation (2.7) adapted to the case of system (3.1), so now

$$[\nu] \stackrel{\text{def}}{=} a_{\bar{\imath}_1}^{\nu_1} \cdots a_{\bar{\imath}_\ell}^{\nu_\ell} b_{\bar{\jmath}_\ell}^{\nu_{\ell+1}} \cdots b_{\bar{\jmath}_1}^{\nu_{2\ell}}.$$

For a given field $\mathbb{K}$ we will write just $\mathbb{K}[a, b]$ in place of $\mathbb{K}[a_{\bar{\imath}_1}, \ldots, a_{\bar{\imath}_\ell}, b_{\bar{\jmath}_\ell}, \ldots, b_{\bar{\jmath}_1}]$, and for $f \in \mathbb{K}[a, b]$ write $f = \sum_{\nu \in \text{Supp}(f)} f^{(\nu)}[\nu]$, where $\text{Supp}(f)$ denotes those $\nu \in \mathbb{N}_+^{2\ell}$ such that the coefficient of $[\nu]$ in the polynomial f is non-zero.

Definition 10. *Let*

$$f = \sum_{\nu \in Supp(f)} f^{(\nu)} a_{qu_1, pv_1}^{\nu_1} \cdots a_{qu_\ell, pv_\ell}^{\nu_\ell} b_{qv_\ell, pu_\ell}^{\nu_{\ell+1}} \cdots b_{qv_1, pu_1}^{\nu_{2\ell}} \in \mathbb{C}[a, b].$$

The conjugate $\widehat{f}$ *of* f *is the polynomial obtained from* f *by the involution*

$$f^{(\nu)} \to \bar{f}^{(\nu)} \qquad a_{qi, pj} \to b_{qj, pi} \qquad b_{qj, pi} \to a_{qi, pj};$$

that is,

$$\widehat{f} = \sum_{\nu \in Supp(f)} \bar{f}^{(\nu)} a_{qu_1, pv_1}^{\nu_{2\ell}} \cdots a_{qu_\ell, pv_\ell}^{\nu_{\ell+1}} b_{qv_\ell, pu_\ell}^{\nu_\ell} \cdots b_{qv_1, pu_1}^{\nu_1} \in \mathbb{C}[a, b].$$

Since $[\nu] = a_{qu_1, pv_1}^{\nu_1} \cdots a_{qu_\ell, pv_\ell}^{\nu_\ell} b_{qv_\ell, pu_\ell}^{\nu_{\ell+1}} \cdots b_{qv_1, pu_1}^{\nu_{2\ell}}$, we have

$$\widehat{[\nu]} = a_{qu_1, pv_1}^{\nu_{2\ell}} \cdots a_{qu_\ell, pv_\ell}^{\nu_{\ell+1}} b_{qv_\ell, pu_\ell}^{\nu_\ell} \cdots b_{qv_1, pu_1}^{\nu_1},$$

so that

$$[\widehat{(\nu_1, \ldots \nu_{2\ell})}] = [(\nu_{2\ell}, \ldots, \nu_1)].$$

For this reason we will also write, for $\nu = (\nu_1, \ldots, \nu_{2\ell})$, $\widehat{\nu} = (\nu_{2\ell}, \ldots, \nu_1)$.

Once the ℓ-element set S has been specified and ordered we let $L : \mathbb{N}_+^{2\ell} \to \mathbb{N}_+^2$ be the map defined by

$$L(\nu) = (L_1(\nu), L_2(\nu)) = \nu_1 \bar{\imath}_1 + \cdots + \nu_\ell \bar{\imath}_\ell + \nu_{\ell+1} \bar{\jmath}_\ell + \cdots + \nu_{2\ell} \bar{\jmath}_1, \qquad (3.9)\;\boxed{\text{L}}$$

which is similar to the map (2.8).

Let

$$\mathcal{M} = \{\nu \in \mathbb{N}_+^{2\ell} : \ L(\nu) = (qk, pk), \ k = 0, 1, 2, \ldots\}.$$

Clearly, $\mathcal{M}$ is an Abelian monoid.

Let $\mathfrak{X}$ be the vector field of system (3.1). The following result was obtained in [21] (where speaking about (s, t)-polynomials we mean (s, t)-polynomials with respect to map (3.9)).

Theorem 11. *Let family (3.1) be given. There exists a formal series $\Psi(x, y)$ of the form (2.1) and polynomials $g_{q,p}, g_{2q,2p}, \ldots$ in $\mathbb{Q}[a, b]$ such that*

(a)

$$\mathfrak{X}\Psi = \sum_{k=1}^{\infty} g_{qk,pk} x^{qk} y^{pk}; \tag{3.10} \boxed{\texttt{g\_qp}}$$

(b) *for every pair $(i, j) \in \mathbb{N}_{-q} \times \mathbb{N}_{-p}$, $i + j \geq 0$, $v_{ij} \in \mathbb{Q}[a, b]$, and v_{ij} is an (i, j)-polynomial;*

(c) *for every $k \geq 1$, $v_{qk,pk} = 0$; and*

(d) *for every $k \geq 1$, $g_{qk,pk} \in \mathbb{Q}[a, b]$, and $g_{qk,pk}$ is a (qk, pk)-polynomial.*

For a given family (3.1) and ordered set S of indices for any $\nu \in \mathbb{N}_+^{2\ell}$ define $V(\nu) \in \mathbb{Q}$ recursively, with respect to $|\nu| = \nu_1 + \cdots + \nu_{2\ell}$, as follows:

$$V(0, \ldots, 0) = 1;$$

for $\nu \neq (0, \ldots, 0)$
$$V(\nu) = 0 \qquad \text{if} \qquad pL_1(\nu) = qL_2(\nu);$$
and when $pL_1(\nu) \neq qL_2(\nu)$,

$$V(\nu) = \frac{1}{pL_1(\nu) - qL_2(\nu)} \times$$
$$\left[\sum_{j=1}^{\ell} V(\nu_1, \ldots, \nu_j - 1, \ldots, \nu_{2\ell})(L_1(\nu_1, \ldots, \nu_j - 1, \ldots, \nu_{2\ell}) + q) \right.$$
$$\left. - \sum_{j=\ell+1}^{2\ell} V(\nu_1, \ldots, \nu_j - 1, \ldots, \nu_{2\ell})(L_2(\nu_1, \ldots, \nu_j - 1, \ldots, \nu_{2\ell}) + p) \right],$$

where $L(\nu)$ is defined by (3.9).

⟨th_V⟩ **Theorem 12.** *For a family of systems of the form (3.1) let Ψ be the formal series of the form (2.1) computed by (2.6), $\{g_{qk,pk} : k \in \mathbb{N}\}$ be the polynomials in $\mathbb{C}[a, b]$ satisfying (3.10). Then*
(a) *for $\nu \in \mathrm{Supp}(v_{k_1,k_2})$, the coefficient $v_{k_1,k_2}^{(\nu)}$ of $[\nu]$ in v_{k_1,k_2} is $V(\nu)$,*
(b) *for $\nu \in \mathrm{Supp}(g_{qk,pk})$, the coefficient $g_{qk,pk}^{(\nu)}$ of $[\nu]$ in $g_{qk,pk}$ is*

$$g_{qk,pk}^{(\nu)} = - \left[\sum_{j=1}^{\ell} V(\nu_1, \ldots, \nu_j - 1, \ldots, \nu_{2\ell})(L_1(\nu_1, \ldots, \nu_j - 1, \ldots, \nu_{2\ell}) + q) \right.$$
$$\left. - \sum_{j=\ell+1}^{2\ell} V(\nu_1, \ldots, \nu_j - 1, \ldots, \nu_{2\ell})(L_2(\nu_1, \ldots, \nu_j - 1, \ldots, \nu_{2\ell}) + p) \right],$$

and

(c) the following identities hold:

$$\kappa V(\widehat{\nu}) = V(\nu) \quad and \quad \kappa g^{(\widehat{\nu})}_{qk,pk} = -g^{(\nu)}_{qk,pk} \qquad for\ all \quad \nu \in \mathbb{N}^{2\ell}_+$$

$$V(\nu) = g^{(\nu)}_{qk,pk} = 0 \quad if \quad \widehat{\nu} = \nu \neq (0,\dots,0),$$

where

$$\kappa = \left(\frac{q}{p}\right)^{(\nu_1 + \cdots + \nu_\ell) - (\nu_{l+1} + \cdots + \nu_{2\ell})}.$$

Statements (a) and (b) of the theorem are proved in [21], statement (c) can be proved similarly as statement 3) of Theorem 3.4.5 of [22].

The next statements follows immediately from c) of Theorem 12.

Corollary 13. *The saddle quantities $g_{qk,pk}$ of system* (3.1) *have the form*

$$g_{qk,pk} = \frac{1}{2} \sum_{v:L(\nu)=(qk,pk)} g^{(\nu)}_{qk,pk}(\kappa[\nu] - [\widehat{\nu}]). \qquad (3.12)\ \boxed{\texttt{g\_sq}}$$

By the analogy with $1:-1$ resonant case, we call the ideal

$$I_{Sib} = \langle \kappa[\nu] - [\widehat{\nu}] : \nu \in \mathcal{M} \rangle \qquad (3.13)\ \boxed{\texttt{id\_S\_def}}$$

the *Sibirsky ideal* of system (3.1). Obviously, transformations (3.4) form a group. It is easy to see that any $\nu \in \mathcal{M}\ [\nu]$ is an invariant of group (3.4). Sibirsky studied such invariants for the case of $1:-1$ resonant system (1.3) and used the ideal (3.13) to describe the basis of the invariants and the number of symmetry axis of the corresponding real systems [24, 25].

Theorem 14. *If the 2ℓ-tuple of parameters of* (3.1) *belong to* $\mathbf{V}(I_{Sib})$ *then the corresponding system admits an analytic first integral of the form* (2.1).

Proof. The conclusion follows from formula (3.12). $\qquad\square$

The following statement shows that the variety of the Sibirsky ideal is the Zariski closure of the set of systems, which are time-reversible with respect to (1.4). The proof is based on an adaption of the ideas of [6, 26].

Theorem 15. *Let $\mathcal{I}$ be the ideal defined by* (3.5). *Then*

$$I_{Sib} = \mathcal{I}. \qquad (3.14)\ \boxed{\texttt{t.isym}}$$

Proof. For $k = 1,\dots,\ell$ let, as above, $\zeta_k = u_k - v_k$ and consider the ring homomorphism

$$\theta : \mathbb{Q}(a,b,t_1,\dots,t_\ell,\gamma,w) \longrightarrow \mathbb{Q}(\gamma,t_1,\dots,t_\ell)$$

defined by

$$a_{qu_k,pv_k} \mapsto t_k,\ b_{qv_k,pu_k} \mapsto \frac{q}{p}\gamma^{\zeta_k}t_k,\ w \mapsto 1/\gamma, \quad (k = 1,\dots,\ell). \qquad (3.15)\ \boxed{\texttt{theta}}$$

Let H be the ideal (3.3). Clearly,

$$H = \ker(\theta).$$

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A reduced Gröebner basis G of $\mathbb{Q}[a,b] \cap H$ can be found computing a reduced Gröebner basis of H using an elimination ordering with $\{a_{qu_j,pv_j}, b_{qv_j,pu_j}\} < \{w, \gamma, t_j\}$ for all $j = 1, \ldots, \ell$, and then intersecting it with $\mathbb{Q}[a,b]$. Since H is binomial, any reduced Gröebner basis G of H also consists of binomials. This means that $\mathcal{I} = H \cap \mathbb{Q}[a,b]$ is a binomial ideal.

We show that $I_{Sib} \subset \mathcal{I}$. Taking into account that $\zeta_k = -\zeta_{2\ell-k}$ by (3.15) for any $\alpha = (\alpha_1, \ldots, \alpha_{2\ell}) \in \mathcal{M}$ we have

$$\theta([\alpha]) = t_1^{\alpha_1} \cdots t_\ell^{\alpha_\ell} t_\ell^{\alpha_{\ell+1}} (\frac{q}{p})^{\alpha_{\ell+1}} \gamma^{\zeta_\ell \alpha_{\ell+1}} \cdots t_1^{\alpha_{2\ell}} (\frac{q}{p})^{\alpha_{2\ell}} \gamma^{\zeta_1 \alpha_{2\ell}} =$$

$$(\frac{q}{p})^{\alpha_{\ell+1}+\cdots+\alpha_{2\ell}} t_1^{\alpha_1} \cdots t_\ell^{\alpha_\ell} t_\ell^{\alpha_{\ell+1}} \cdots t_1^{\alpha_{2\ell}} \gamma^{-(\zeta_{\ell+1}\alpha_{\ell+1}+\cdots+\zeta_{2\ell}\alpha_{2\ell})}$$

and

$$\theta([\hat{\alpha}]) = t_1^{\alpha_{2\ell}} \cdots t_\ell^{\alpha_{\ell+1}} t_\ell^{\alpha_\ell} (\frac{q}{p})^{\alpha_\ell} \gamma^{\zeta_\ell \alpha_\ell} \cdots t_1^{\alpha_1} (\frac{q}{p})^{\alpha_1} \gamma^{\zeta_1 \alpha_1} =$$

$$(\frac{q}{p})^{\alpha_1+\cdots+\alpha_\ell} t_1^{\alpha_1} \cdots t_\ell^{\alpha_\ell} t_\ell^{\alpha_{\ell+1}} \cdots t_1^{\alpha_{2\ell}} \gamma^{\zeta_1 \alpha_1 + \cdots + \zeta_\ell \alpha_\ell}.$$

Since

$$-(\zeta_{\ell+1}\alpha_{\ell+1} + \cdots + \zeta_{2\ell}\alpha_{2\ell}) = \zeta_1 \alpha_1 + \cdots + \zeta_\ell \alpha_\ell,$$

we obtain $\theta(\kappa[\alpha] - [\hat{\alpha}]) = 0$. Thus, $\kappa[\alpha] - [\hat{\alpha}] \in \ker(\theta)$ yielding $\kappa[\alpha] - [\hat{\alpha}] \in \mathcal{I}$.

The proof of the inclusion $\mathcal{I} \subset I_{Sib}$ is similar as the proof in Theorem 5.2.2 of [22]. □

As an immediate consequence of Theorem 3.14 and Proposition 8 we have the next result.

Theorem 16. *The variety of the Sibirsky ideal I_{Sib} is the Zariski closure of the set $\mathcal{R}$ of all time-reversible systems in family* (3.1).

The above studies show that the theory regarding the computation and the structure of the saddle quantities for $1 : -1$ resonant systems has a counterpart also in the family of $p : -q$ resonant systems, however not in the whole family, but just in subfamilies of the form (2.9) and (3.1). It is in agreement with the known fact that the study of local integrability of $p : -q$ resonant systems is much more difficult that the studies in $1 : -1$ case, which can be observed already in the quadratic and the cubic case [4, 9].

Acknowledgements

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