A CHARACTERIZATION OF THE GENERALIZED LIÉNARD POLYNOMIAL DIFFERENTIAL SYSTEMS HAVING INVARIANT ALGEBRAIC CURVES

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Abstract. The generalized Liénard polynomial differential systems are the differential systems of the form $x' = y$, $y' = -f(x)y - g(x)$, where f and g are polynomials.

We characterize all the generalized Liénard polynomial differential systems having an invariant algebraic curve. We show that the first four higher coefficients of the polynomial in the variable y, defining the invariant algebraic curve, determine completely the generalized Liénard polynomial differential system. This fact does not hold for arbitrary polynomial differential systems.

1. Introduction and statement of the main results

In this work we study the generalized Liénard polynomial differential systems of the form

$$x' = y, \qquad y' = -f(x)y - g(x), \tag{1}$$

where the degrees of the polynomials f and g are m and n respectively.

Let $F(x, y)$ be a polynomial such that

$$\frac{\partial F}{\partial x} y + \frac{\partial F}{\partial y}(-f(x)y - g(x)) = KF, \tag{2}$$

for some polynomial $K = K(x, y)$. Then $F(x, y) = 0$ is an *invariant algebraic curve* of the differential system (1), i.e. if an orbit of system (1) has a point on the curve $F(x, y) = 0$, the whole orbit is contained in this curve. The polynomial K is called as/or to be the *cofactor* of the invariant algebraic curve $F(x, y) = 0$.

The knowledge of the algebraic curves of system (1) allows to study the Darboux and Liouvillian theories of integrability, see [4, 9, 22, 24] and references therein. In fact the existence of invariant algebraic curves is a measure of the integrability in such theories. Another problem is finding a bound for the degree of the irreducible invariant algebraic curves of system (1). This problem goes back to Poincaré for any polynomial differential system and it is known as the *Poincaré problem* for the invariant algebraic curves. The invariant algebraic curves of generalized Liénard systems (1) have been studied by several authors in function of degrees of f and g, see for instance [1, 3, 10, 11, 12, 13, 18, 25] and references therein. The determination of invariant algebraic curves is also important when we study the algebraic

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limit cycles of such systems, see [14, 20, 21]. Several works are also devoted to the Liouville integrability of such systems, see [2, 16, 17, 19, 23]. Finally we remark that a new method to determine the invariant algebraic curves have been developed on [5, 6, 7, 8] based on the solutions of the differential system expressed in Puiseux series. In this note we study the reciprocal problem. This problem consists in given an invariant algebraic curve characterize the generalized Liénard poynomial differential systems having such an invariant algebraic curve.

In the following we use the notation $a_j^k(x)$ to denote $(a_j(x))^k$.

Theorem 1. *Assume that a generalized Liénard polynomial differential system* (1) *with the polynomials f and g non-identically zero has an invariant algebraic curve that we write as*

$$F(x,y) = \sum_{j=0}^{s} a_j(x) y^{s-j} = 0 \text{ with } a_0(x) \neq 0 \text{ and } s \geq 2. \tag{3}$$

Then the polynomials f and g are

$$f(x) = \frac{s a_3'(x) - (s-1)a_1(x)a_2'(x) + (s-1)a_1^2(x)a_1'(x) - s a_2(x)a_1'(x)}{(s-1)a_1^2(x) - 2\,s\,a_2(x)},$$

$$g(x) = \frac{a_1(x)a_2(x)a_1'(x) + a_1(x)a_3'(x) - 2a_2(x)a_2'(x)}{(s-1)a_1^2(x) - 2sa_2(x)},$$

$a_0(x)$ is a constant and the cofactor of $F(x,y)=0$ only depends on x.

Theorem 1 is proved in section 2.

Note that the common denominator in the expressions of $f(x)$ and $g(x)$ given in the statement of Theorem 1 must divide their numerators, otherwise $f(x)$ and $g(x)$ would not be polynomials.

We remark that if a generalized Liénard polynomial differential system (1) has an invariant algebraic curve (3) the coefficients $a_1(x)$, $a_2(x)$ and $a_3(x)$ determine completely such differential system. Of course this is not true for general polynomial differential systems, for instance the polynomial differential system

$$\dot{x} = a(x,y)F - c(x,y)\frac{\partial F}{\partial y}, \qquad \dot{y} = b(x,y)F + c(x,y)\frac{\partial F}{\partial x},$$

where a, b and c are arbitrary polynomials, has $F = F(x,y) = 0$ as an invariant algebraic curve.

Corollary 2. *Under the assumptions of Theorem* 1 *if the common denominator of the expressions of $f(x)$ and $g(x)$ are zero, then the polynomials $f(x)$ and $g(x)$ become*

$$f(x) = \frac{a_1'(x)}{2}, \qquad g(x) = \frac{a_2'(x)}{s} - \frac{a_1(x)a_1'(x)}{2s}.$$

Proposition 3. *Under the assumptions of Theorem* 1 *if $a_i(x) = \sum_{j=1}^{3} a_{ij}x^j$ for $j = 1, 2, 3$ are arbitrary polynomials such that the maximum degree of all them is* 3, *then the generalized Liénard polynomial differential systems having an irreducible invariant algebraic curve of degree* 3 *in the variable y are the following ones:*

(I) $f(x) = \dfrac{4a_{11}}{9}, \quad g(x) = -\dfrac{a_{11}}{27}(a_{10} + a_{11}x),$
$F(x,y) = \dfrac{1}{27}((a_{10} + a_{11}x)(3a_{20} + a_{11}x(2a_{10} + a_{11}x))$
$+9(3a_{20} + a_{11}x(2a_{10} + a_{11}x))y + 27(a_{10} + a_{11}x)y^2 + 27y^3).$

(II) $f(x) = \dfrac{5a_{11}}{9}, \quad g(x) = -\dfrac{2a_{11}}{27}(3a_{10} + a_{11}x),$
$F(x,y) = \dfrac{1}{81}(3a_{10} + a_{11}x)^2(a_{10} + 3a_{11}x)$
$+ \dfrac{1}{9}(3a_{10} + a_{11}x)(a_{10} + 3a_{11}x)y + (a_{10} + a_{11}x)y^2 + y^3.$

(III) $f(x) = \dfrac{a_{11}}{3}, \quad g(x) = -\dfrac{2a_{11}^2}{9}x,$
$F(x,y) = \dfrac{1}{27}(27a_{30} - 4a_{11}^3x^3 + 27a_{11}xy^2 + 27y^3).$

(IV) $f(x) = \dfrac{2a_{11}^3 - k - 6a_{11}a_{22}}{3(a_{11}^2 - 3a_{22})}, \quad g(x) = \dfrac{a_{11}^4 + ka_{11} - 9a_{11}^2a_{22} + 18a_{22}^2}{9(a_{11}^2 - 3a_{22})},$
$F(x,y) = \dfrac{1}{27}(-2a_{11}^3x^3 - 2kx^3 + 9a_{11}a_{22}x^3 + 27a_{22}x^2y + 27a_{11}xy^2 + 27y^3),$
where $k = (a_{11}^2 - 3a_{22})^{3/2}$.

Taking into account that the degrees of the polynomials f and g are m and n respectively, from Corollary 2 it follows that in this case the $\deg a_1 = m + 1$ and $\deg a_2 \leq \max\{n, m(m+1)\} + 1$.

Under the assumptions of Corollary 2 we note that the generalized Liénard polynomial differential system (1) having an invariant algebraic curve (3) already the coefficients $a_1(x)$ and $a_2(x)$ determine completely such differential system.

Proposition 4. *Under the assumptions of Corollary 2 if* $a_1(x) = \sum_{i=1}^4 a_{1i}x^i$ *and* $a_2(x) = \sum_{i=1}^4 a_{2i}x^i$ *are arbitrary polynomials such that the maximum degree of both is 4, then the generalized Liénard polynomial differential systems having an irreducible invariant algebraic curve of degree 4 in the variable y are the following ones:*

(I) $f(x) = \dfrac{a_{11}}{2}, \quad g(x) = -\dfrac{a_{11}^2}{8}x,$
$F(x,y) = -\dfrac{a_{11}^4}{128}x^4 + \dfrac{a_{11}^3}{16}x^3y + a_{11}xy^3 + y^4;$

(II) $f(x) = \dfrac{a_{11}}{2}, \quad g(x) = -\dfrac{a_{11}}{8}(a_{10} + a_{11}x),$
$F(x,y) = -\dfrac{a_{10}^4}{128} - \dfrac{a_{11}}{128}x(4a_{10}^3 + 6a_{10}^2a_{11}x + 4a_{10}a_{11}^2x^2 + a_{11}^3x^3) + \dfrac{1}{16}(a_{10} +$
$a_{11}x)^3y + (a_{10} + a_{11}x)y^3 + y^4;$

(III) $f(x) = \dfrac{1}{2}(a_{11} + 2a_{12}x + 3a_{13}x^2 + 4a_{14}x^3),$
$g(x) = -\dfrac{1}{4}f(x)(a_{10} + a_{11}x + a_{12}x^2 + a_{13}x^3 + a_{14}x^4),$

$$F(x,y) = -\frac{1}{128}\{a_{10}+x(a_{11}+x(a_{12}+x(a_{13}+a_{14}x)))\}^4+\frac{1}{16}\{a_{10}+x(a_{11}+x(a_{12}+x(a_{13}+a_{14}x)))\}^3 y+\{a_{10}+x(a_{11}+x(a_{12}+x(a_{13}+a_{14}x)))\}y^3+y^4.$$

Corollary 2 and Propositions 3 and 4 are proved in section 2.

We remark that the next polynomial Liénard differential system shows that there are invariant algebraic curves of arbitrary degree in the variable y. In [15] the authors proved that the linear polynomial Liénard differential system

$$\dot{x} = y, \qquad \dot{y} = x - \frac{p-1}{\sqrt{p}}y,$$

with $p > 1$ has the first integral $H = (\sqrt{p}\,y - x)(\sqrt{p}\,y + x)^p$. Therefore such differential system has invariant algebraic curves of degree $p+1$ for all p.

2. Proofs

Proof of Theorem 1. If $F(x,y) = 0$ is an invariant algebraic curve of system (1) it must satisfy (2) with a cofactor of the form $K = \sum_{j=0}^{r} K_j(x)y^j$. From equation (2) it follows easily that $r \leq 1$. If $r = 1$ then the coefficient of y^{s+1} is $a_0'(x) - a_0(x)K_1(x) = 0$. Since $a_0(x)$ must be a polynomial it follows that $K_1(x) = 0$ and $a_0(x)$ is a constant. Then without loss of generality we can take $a_0(x) = 1$, because we can divide the invariant algebraic curve by the non-zero constant $a_0(x)$. In summary the cofactor of $F(x,y) = 0$ only depends on the variable x.

From the coefficient of y^s in equation (2) we obtain that the cofactor $K_0(x) = a_1'(x) - sf(x)$. And from the coefficient of y^{s-1} in equation (2) we get that

$$g(x) = \frac{1}{s}\left(a_2'(x) - (s-1)a_1(x)f(x) - K_0(x)a_1(x)\right).$$

Substituting $K_0(x)$ is the above expression we have

$$g(x) = \frac{1}{s}\left(a_2'(x) + a_1(x)f(x) - a_1(x)a_1'(x)\right). \tag{4}$$

The coefficient of y^{s-2} in equation (2) is

$$\begin{aligned}&a_3'(x) - (s-2)a_2(x)f(x) - \frac{s-1}{s}a_1(x)(a_2'(x) + a_1(x)f(x) - a_1(x)a_1'(x))\\ &= a_1'(x)a_2(x) - s\,a_2(x)f(x).\end{aligned}$$

Consequently we obtain the function $f(x)$ stated in Theorem 1, and substituting $f(x)$ in (4) we get the expression of $g(x)$ stated in the theorem. □

Example 1. Consider the polynomial Liénard differential system

$$\dot{x} = y, \qquad \dot{y} = x - \frac{1}{16}a^2(x^2-4)x^3 - a(x^2-1)y, \tag{5}$$

having the invariant algebraic curve

$$\begin{aligned}F(x,y) = {}& y^2 + a\left(\frac{1}{2}x^3 - 2x\right)y - 4 + (a^2+1)x^2 - \frac{a^2x^4}{2} + \frac{a^2x^6}{16}\\ = {}& y^2 + a_1(x)y + a_2(x).\end{aligned}$$

Substituting $a_1(x)$ and $a_2(x)$ in the expressions of the polynomials f and g given in the statement of Theorem 1 we obtain

$$f(x) = a(x^2-1), \qquad g(x) = -x + \frac{1}{16}a^2(x^2-4)x^3.$$

Proof of Corollary 2. Since the denominator of the polynomial $f(x)$ divides its numerator, then $f(x)$ must vanish when $a_2(x) = (s-1)a_1^2(x)/(2s)$. This implies that

$$s^2a_1^2(x)a_1'(x) - 3sa_1^2(x)a_1'(x) + 2a_1^2(x)a_1'(x) - 2s^2a_3'(x) = 0.$$

From this equation we get

$$a_3(x) = \frac{(s-2)(s-1)a_1^3(x)}{6s^2} + c_1,$$

where c_1 is an arbitrary constant. Substituting $a_3(x)$ in $f(x)$ we obtain

$$f(x) = \frac{N(x)}{D(x)} \tag{6}$$

where $N(x) = -2s^2a_2(x)a_1'(x) + 3s^2a_1^2(x)a_1'(x) - 5sa_1^2(x)a_1'(x) + 2a_1^2(x)a_1'(x) - 2s^2a_1(x)a_2'(x) + 2sa_1(x)a_2'(x)$ and $D(x) = 2s[(s-1)a_1^2(x) - 2sa_2(x)]$.

Dividing the numerator $N(x)$ by the denominator $D(x)$ the quotient is $a_1'(x)/2$, and the remainder $R(x)$ is equal to $2(s-1)a_1(x)(s\,a_1(x)a_1'(x) - a_1(x)a_1'(x) - s\,a_2'(x))$, which is zero when the denominator $D(x)$ is zero. Consequently the expression (6) reduces to

$$f(x) = \frac{a_1'(x)}{2}. \tag{7}$$

In this case $g(x)$ takes the form

$$g(x) = \frac{a_1(x)a_1'(x)\left((s-2)(s-1)a_1^2(x) + 2s^2a_2(x)\right) - 4s^2a_2(x)a_2'(x)}{2s^2\left((s-1)a_1^2(x) - 2sa_2(x)\right)}. \tag{8}$$

In the above expression of $g(x)$ since the denominator divides the numerator, working in the same way as we did for the polynomial $f(x)$ we obtain that

$$g(x) = \frac{a_2'(x)}{s} - \frac{a_1(x)a_1'(x)}{2s},$$

and this completes the proof of the corollary. □

We observe that the generalized Liénard polynomial differential system of Example 1 does not satisfy the assumption that the common denominator of the expressions of f and g stated in Theorem 1 vanishes. So we cannot apply Corollary 2 to differential system (5).

Proof of Proposition 3. Under the assumptions of the proposition we arrive to an algebraic system between the parameters of $a_i(x)$ for $i = 1, 2, 3$. The equation for the highest power in x which corresponds to x^{11} is $3a_{13}^2(4a_{13}a_{33} - a_{23}^2) = 0$.

First we consider $a_{33} = a_{23}^2/(4a_{13})$ with $a_{13} \neq 0$. Then vanishing the coefficients associated to the next powers of x we obtain $a_{32} = (2a_{13}a_{22}a_{23} - a_{12}a_{23}^2)/(4a_{13}^2)$,

$a_{31} = (a_{13}^2 a_{22}^2 + 2a_{13}^2 a_{21} a_{23} - 2a_{12} a_{13} a_{22} a_{23} + a_{12}^2 a_{23}^2 - a_{11} a_{13} a_{23}^2)/(4a_{13}^3)$, and

$$a_{30} = \frac{1}{8a_{13}^4}(4a_{13}^3 a_{21} a_{22} - 2a_{12} a_{13}^2 a_{22}^2 + 4a_{13}^3 a_{20} a_{23} - 4a_{12} a_{13}^2 a_{21} a_{23} + 4a_{12}^2 a_{13} a_{22} a_{23} \\ - 4a_{11} a_{13}^2 a_{22} a_{23} - 2a_{12}^3 a_{23}^2 + 4a_{11} a_{12} a_{13} a_{23}^2 - 2a_{10} a_{13}^2 a_{23}^2 + a_{13} a_{23}^3).$$

But the solutions obtained vanishing the coefficients of next powers of x provides a reducible invariant algebraic curve $F = 0$.

Second we consider $a_{13} = 0$. The next coefficient is $2a_{12}^2 a_{23}^2 = 0$. The case $a_{23} = 0$ with $a_{12} \neq 0$ implies $a_{23} = a_{33} = 0$ and $a_{32} = a_{22}^2/(4a_{12})$, and all the invariant curves $F = 0$ are reducible. The case $a_{12} = 0$ implies $a_{23} = 0$. Imposing that the rest of equations vanish we obtain the polynomials f, g and F of the statement of the proposition. □

Proof of Proposition 4. This proof follows in a similar way to the proof of Proposition 3, but firstly imposing that f and g satisfy (7) and (8). We omit the details that the reader can compute easily. □

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