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A characterization of the generalized Liénard polynomial differential systems having invariant algebraic curves

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ABSTRACT

The generalized Liénard polynomial differential systems are the differential systems of the form $x' = y, y' = -f(x)y - g(x)$, where f and g are polynomials.
We characterize all the generalized Liénard polynomial differential systems having an invariant algebraic curve. We show that the first four higher coefficients of the polynomial in the variable y, defining the invariant algebraic curve, determine completely the generalized Liénard polynomial differential system. This fact does not hold for arbitrary polynomial differential systems.
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1. Introduction and statement of the main results

In this work we study the generalized Liénard polynomial differential systems of the form

$$x' = y, \qquad y' = -f(x)y - g(x), \tag{1}$$

where the degrees of the polynomials f and g are m and n respectively.

Let $F(x, y)$ be a polynomial such that

$$\frac{\partial F}{\partial x}y + \frac{\partial F}{\partial y}(-f(x)y - g(x)) = KF, \tag{2}$$

for some polynomial $K = K(x, y)$. Then $F(x, y) = 0$ is an *invariant algebraic curve* of the differential system (1), i.e. if an orbit of system (1) has a point on the curve $F(x, y) = 0$, the whole orbit is contained in this curve. The polynomial K is called as/or to be the *cofactor* of the invariant algebraic curve $F(x, y) = 0$.

The knowledge of the algebraic curves of system (1) allows to study the Darboux and Liouvillian theories of integrability, see [1–4] and references therein. In fact the existence of invariant algebraic curves is a measure of the integrability in such theories. Another problem is finding a bound for the degree of the irreducible invariant algebraic curves of system (1). This problem goes back to Poincaré for any polynomial differential system and it is known as the *Poincaré problem* for the invariant algebraic curves. The invariant algebraic curves of generalized Liénard systems (1) have been studied by several authors in function of degrees of f and g, see for instance [5–12] and references therein. The determination of invariant algebraic curves is also important when we study the algebraic limit cycles of such systems, see [13–15]. Several works are also devoted to the Liouville integrability of such systems, see [16–20]. Finally we remark that a new method to determine the invariant algebraic curves have been developed on [21–24] based on the solutions of the differential system expressed in Puiseux series. In this note we study the reciprocal problem. This problem consists in given an invariant algebraic curve characterize the generalized Liénard poynomial differential systems having such an invariant algebraic curve.

In the following we use the notation $a_j^k(x)$ to denote $(a_j(x))^k$.

Theorem 1. *Assume that a generalized Liénard polynomial differential system (1) with the polynomials f and g non-identically zero has an invariant algebraic curve that we write as*

$$F(x, y) = \sum_{j=0}^{s} a_j(x)y^{s-j} = 0 \text{ with } a_0(x) \neq 0 \text{ and } s \geq 2. \tag{3}$$

Then the polynomials f and g are

$$f(x) = \frac{sa_{3'}(x) - (s-1)a_1(x)a_{2'}(x) + (s-1)a_1^2(x)a_{1'}(x) - sa_2(x)a_{1'}(x)}{(s-1)a_1^2(x) - 2sa_2(x)},$$

$$g(x) = \frac{a_1(x)a_2(x)a_{1'}(x) + a_1(x)a_{3'}(x) - 2a_2(x)a_{2'}(x)}{(s-1)a_1^2(x) - 2sa_2(x)},$$

$a_0(x)$ is a constant and the cofactor of $F(x, y) = 0$ only depends on x.

Theorem 1 is proved in Section 2.

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