mathematics

MDPI

Article

Invariant Algebraic Curves of Generalized Liénard Polynomial Differential Systems

Jaume Giné [1,*] and Jaume Llibre [2]

[1] Departament de Matemàtica, Universitat de Lleida, Avda. Jaume II, 69, 25001 Lleida, Catalonia, Spain
[2] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Catalonia, Spain; jllibre@mat.uab.cat
* Correspondence: jaume.gine@udl.cat

Abstract: In this study, we focus on invariant algebraic curves of generalized Liénard polynomial differential systems $x' = y$, $y' = -f_m(x)y - g_n(x)$, where the degrees of the polynomials f and g are m and n, respectively, and we correct some results previously stated.

Keywords: Liénard differential systems; invariant algebraic curve; first integrals

MSC: Primary 34A05; Secondary 34C05; 37C10

check for updates

Citation: Giné, J.; Llibre, J. Invariant Algebraic Curves of Generalized Liénard Polynomial Differential Systems. *Mathematics* **2022**, *10*, 209. https://doi.org/10.3390/math10020209

Academic Editor: Ioannis G. Stratis

Received: 16 December 2021
Accepted: 5 January 2022
Published: 10 January 2022

1. Introduction and Statement of the Main Results

In this work, we study the generalized Liénard polynomial differential systems of the following form:

$$x' = y, \qquad y' = -f_m(x)y - g_n(x), \tag{1}$$

where the degrees of the polynomials f and g are given by the subscripts m and n, respectively. These generalized Liénard systems are used to model different problems in numerous areas of knowledge and have been intensively studied in the last decades (see for instance [1,2] and references therein).

Consider $F(x,y) = 0$ an *invariant algebraic curve* of the differential system (1) where $F(x,y)$ is a polynomial, then there exists a polynomial $K(x,y)$ such that the following is the case.

$$\frac{\partial F}{\partial x}y + \frac{\partial F}{\partial y}(-f_m(x)y - g_n(x)) = KF \tag{2}$$

The knowledge of the algebraic curves of system (1) allows studying modern Darboux and Liouvillian theories of integrability (see [3] and references therein). In fact the existence of invariant algebraic curves is a measure of integrability in such theories. Another problem is finding a bound on the degree of irreducible invariant algebraic curves of system (1). This problem goes back to Poincaré for any differential system and is known as *Poincaré problem*.

In 1996, Hayashi [4] stated the following result.

Theorem 1. *The generalized Liénard polynomial differential system (1) with $f_m \not\equiv 0$ and $m+1 \geq n$ has an invariant algebraic curve if and only if there is an invariant curve $y - P(x) = 0$ satisfying $g_n(x) = -(f_m(x) + P'(x))P(x)$, where $P(x)$ or $P(x) + F(x)$ is a polynomial with a degree of at most one, such that $F(x) = \int_0^x f(s)ds$.*

Given P and Q polynomials, an algebraic curve of the form $(y + P(x))^2 - Q(x) = 0$ is called *hyperelliptic curve* (see for instance [5–8]). In such works, hyperelliptic curves are used to determine the algebraic limit cycles of generalized Liénard systems (1).

Theorem 1 is also announced in [9], where the author seems to not be aware that the theorem is false. Theorem 1 is not correct as the following proposition shows. More