A NEW SUFFICIENT CONDITION IN ORDER THAT THE REAL JACOBIAN CONJECTURE IN $\mathbb{R}^2$ HOLDS

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Abstract. Let $F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial map such that $\det(DF(x, y))$ is nowhere zero and $F(0, 0) = (0, 0)$. In this work we give a new sufficient condition for the injectivity of F. We also state a conjecture when $\det(DF(x, y)) = \text{constant} \neq 0$ and $F(0, 0) = (0, 0)$ equivalent to the Jacobian conjecture.

1. Introduction and statement of the main result

Let $F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth map such that the determinant of the Jacobian matrix $\det(DF)$ is nowhere zero. By the Inverse Theorem such a map F is a local diffeomorphism. However this map is not always an injective map. But with some additional conditions it holds that F is a global diffeomorfphism, see for instance [9, 12, 20].

The *real Jacobian conjecture*, stated by Keller [18] in 1939, says that when F is a polynomial map, then F is injective. However in 1994 Pinchuk [19] gaves a counterexample to this conjecture providing a non injective polynomial map with nonvanishing Jacobian determinant. Nevertheless with additional conditions the conjecture holds, for instance in [3, 5] it was shown that the conjecture is true if the degree of f is at most 4. In [4] the following result provides two independent sufficient conditions for the validity of the real Jacobian conjecture.

Theorem 1. *Let $F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial map such that $\det(DF)$ is nowhere zero and $F(0, 0) = (0, 0)$. If the higher homogeneous terms of the polynomials $ff_x + gg_x$ and $ff_y + gg_y$ do not have real linear factors in common, then F is injective.*

Theorem 1 improves a preliminary result in [6] which said: if $\deg f = \deg g$ and that the homogeneous terms of higher degree of f and g do not have real linear factors in common, then F is injective. A similar

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result to this also works for the real Jacobian conjecture in $\mathbb{R}^n$ see [7]. Moreover in [8], and based in the structure of polynomial maps, another sufficient condition was given in order that the real Jacobian conjecture in $\mathbb{R}^n$ holds.

In this work we restrict to the case when $\det(DF(x,y)) =$ constant $\neq 0$. This case is known as the *Jacobian conjecture*. Many authors have work in this conjecture, see for instance the surveys [2] and [11] on the Jacobian conjecture and related problems. Our main result is the following one.

Theorem 2. *Let $F = (f,g) : \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial map such that $\det(DF(x,y))$ is nowhere zero and $F(0,0) = (0,0)$. If the differential equation $y'(x) = -(ff_x + gg_x)/(ff_y + gg_y)$ does not have any formal solution at infinity except $y = \pm ix$, then F is injective.*

This result is proved using the qualitative theory of ordinary differential equations, following ideas started by Gavrilov [13] and Sabatini [21], while the approach of the previous studies (with the exceptions of Theorem 1 and its preliminary version in [6]) are based in the structure of the polynomial maps. Moreover we state the following conjecture.

Conjecture 3. *Let $F = (f,g)$ be a polynomial map with $\det(DF(x,y))$ $=$ constant $\neq 0$ and $F(0,0) = (0,0)$. Then the following statements are equivalent.*

(a) *The differential equation $y'(x) = -(ff_x + gg_x)/(ff_y + gg_y)$ does not have any formal solution at infinity except $y = \pm ix$.*
(b) *F is a global diffeomorphism of the plane onto itself.*

In section 2 we summarize some preliminary results that we shall use in the proof of Theorem 2 given in section 3.

2. Preliminary results

A singular point p of a vector field defined in $\mathbb{R}^2$ is a *center* if p has a neighborhood U such that $U \setminus \{p\}$ is filled of periodic orbits. The *period annulus* of the center p is the maximal neighborhood $\mathcal{P}$ of p such that all the orbits contained in $\mathcal{P} \setminus \{p\}$ are periodic. A center is *global* if its period annulus is the whole $\mathbb{R}^2$.

Gavrilov [13] and Sabatini [21] gave the following result connecting the global invertibility of a local invertible map with the globality of the period annulus of an associated differential system.

Theorem 4. *Let $F = (f, g)$ be a real polynomial map with nowhere zero Jacobian determinant such that $F(0, 0) = (0, 0)$. Then the following statements are equivalent.*

(a) *The origin is a global center for the polynomial vector field $\mathcal{X} = (-ff_y - gg_y, ff_x + gg_x)$ in $\mathbb{R}^2$.*
(b) *F is a global diffeomorphism of the plane $\mathbb{R}^2$ onto itself.*

Let $\mathcal{X}$ be a planar polynomial vector field of degree n and $\mathbb{S}^2 = \{y = (y_1, y_2, y_3) \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\}$ (the *Poincaré sphere*). The *Poincaré compactification* of $\mathcal{X}$, denoted by $p(\mathcal{X})$, is an induced vector field on $\mathbb{S}^2$ defined as follows. For more details see Chapter 5 of [10].

Denote by $T_y\mathbb{S}^2$ the tangent space to $\mathbb{S}^2$ at the point y. Assume that $\mathcal{X}$ is defined in the plane $T_{(0,0,1)}\mathbb{S}^2 \equiv \mathbb{R}^2$. Consider the central projection $f : T_{(0,0,1)}\mathbb{S}^2 \to \mathbb{S}^2$. This map defines two copies of $\mathcal{X}$ on $\mathbb{S}^2$, one in the open northern hemisphere $\mathbb{H}^+$ and the other in the open southern hemisphere $\mathbb{H}^-$. Denote by $\mathcal{X}'$ the vector field $Df \circ \mathcal{X}$ defined on $\mathbb{S}^2$ except on its equator $\mathbb{S}^1 = \{y \in \mathbb{S}^2 : y_3 = 0\}$. Clearly $\mathbb{S}^1$ is identified with the *infinity* of $\mathbb{R}^2$. In order to extend $\mathcal{X}'$ to a vector field on $\mathbb{S}^2$ (including $\mathbb{S}^1$) it is necessary that $\mathcal{X}$ satisfies suitable conditions. In the case that $\mathcal{X}$ is a planar polynomial vector field of degree n then $p(\mathcal{X})$ is the only analytic extension of $y_3^{n-1}\mathcal{X}'$ to $\mathbb{S}^2$. On $\mathbb{S}^2 \setminus \mathbb{S}^1 = \mathbb{H}^+ \cup \mathbb{H}^-$ there are two symmetric copies of $\mathcal{X}$, one in $\mathbb{H}^+$ and other in $\mathbb{H}^-$, and knowing the behavior of $p(\mathcal{X})$ around $\mathbb{S}^1$, we know the behavior of $\mathcal{X}$ at infinity. The Poincaré compactification has the property that $\mathbb{S}^1$ is invariant under the flow of $p(\mathcal{X})$.

The singular points of $\mathcal{X}$ are called the *finite singular points* of $\mathcal{X}$ or of $p(\mathcal{X})$, while the singular points of $p(\mathcal{X})$ contained in $\mathbb{S}^1$, i.e. at infinity, are called the *infinite singular points* of $\mathcal{X}$ or of $p(\mathcal{X})$. It is known that the infinity singular points appear in pairs diametrically opposed.

Given a polynomial $p : \mathbb{R}^2 \to \mathbb{R}$, we denote by p_k the homogeneous term of degree k of p.

Let q be an infinite singular point and let h be a hyperbolic sector of q. We say that h is *degenerated* if its two separatrices are contained in the equator of $\mathbb{S}^2$ (i.e. in $\mathbb{S}^1$), otherwise h is called *non–degenerated*.

The next result is the Poincaré–Hopf Theorem for the Poincaré compactification of a polynomial vector field. See for instance Theorem 6.30 of [10] for a proof.

Theorem 5. *Let $\mathcal{X}$ be a polynomial vector field. If $p(\mathcal{X})$ defined on the Poincaré sphere $\mathbb{S}^2$ has finitely many singular points, then the sum of their topological indices is two.*

3. Proof of Theorem 2

Assume that $F = (f, g)$ is a polynomial map such that $\det(DF(x, y))$ is nowhere zero and $F(0, 0) = (0, 0)$.

Consider the function $H : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$H(x, y) = f(x, y)^2 + g(x, y)^2 \tag{1}$$

and its associated Hamiltonian vector field $\mathcal{X} = (P, Q)$ given by

$$\dot{x} = P = -H_y = -2ff_y - 2gg_y, \qquad \dot{y} = Q = H_x = 2ff_x + 2gg_x, \tag{2}$$

whose first integral is the function H. First we prove that each finite singular point of $\mathcal{X}$ is a center, and consequently has topological index 1, see for more details chapter 6 of [10]. Indeed, $(a, b) \in \mathbb{R}^2$ is a singular point of $\mathcal{X}$ if and only if

$$\begin{pmatrix} f_x(a,b) & g_x(a,b) \\ f_y(a,b) & g_y(a,b) \end{pmatrix} \begin{pmatrix} f(a,b) \\ g(a,b) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which implies that $f(a, b) = g(a, b) = 0$, because $\det(DF(x, y))$ never vanishes. Moreover there exists a neighborhood U of (a, b) in which the map F is injective. Taking into account that the Hamiltonian H is positive in all the points of U except at (a, b) where $H(a, b) = 0$. This proves that (a, b) is an isolated minimum of H. Therefore the orbits of $\mathcal{X}$ in $U \setminus \{(a, b)\}$ are closed, consequently the singular point (a, b) is a center. Hence, from Theorem 4, in order to prove Theorem 2 it is enough to show that $(0, 0)$ is a global center of the vector field $\mathcal{X}$. First we recall the following corollary from the main result proved in [4].

Corollary 6. *Let $F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial map such that $\det(DF)$ is nowhere zero and $F(0, 0) = (0, 0)$. Then F is injective if and only if the vector field $\mathcal{X} = (-2ff_y - 2gg_y, 2ff_x + 2gg_x)$ has no infinite singular points or each of them is formed by two degenerated hyperbolic sectors.*

By this corollary if F is not injective, then $\mathcal{X}$ has infinite singular points and each of them is not formed by two degenerated hyperbolic sectors. In this case the vector field $\mathcal{X}$ has some separatrices not contained in the equator of $\mathbb{S}^2$ (i.e. in $\mathbb{S}^1$) and since all the finite singular points are centers, these separatrices must go from one infinite singular point to another infinite singular point. We can see this behavior in the in the phase portrait of the counterexample to the real conjecture given by Pinchuk, see [1]. In this example the Jacobian $\det(DF)$ is nowhere zero but not constant.

Now we consider the Bendixon compactification of system (2), for more details on this compactification see for instance chapter 5 of [10]. We do the change of variables

$$u = \frac{x}{x^2+y^2}, \qquad v = \frac{y}{x^2+y^2}. \tag{3}$$

Hence the infinity of system (2) is transformed into the origin of the system

$$\dot{u} = \frac{1}{(u^2+v^2)^d}\tilde{P}(u,v), \qquad \dot{v} = \frac{1}{(u^2+v^2)^d}\tilde{Q}(u,v), \tag{4}$$

where $d = \max\{\deg P, \deg Q\}$ and $\tilde{P} = P(x(u,v), y(u,v))$ and $\tilde{Q} = Q(x(u,v), y(u,v))$ whose terms of lowest order are of degree at least $d+2$. Introducing the change of time scale

$$\frac{dt}{d\tau} = (u^2+v^2)^d, \tag{5}$$

system (4) becomes

$$u' = \frac{du}{d\tau} = \tilde{P}(u,v), \qquad v' = \frac{dv}{d\tau} = \tilde{Q}(u,v). \tag{6}$$

Since the lowest homogeneous parts of $\tilde{P}$ and $\tilde{Q}$ have minimum degree $d+2$ it is clear that the origin is a singular point of system (6).

More precisely, system (2) after the change of variables (3) becomes

$$\begin{aligned} \dot{u} &= 2\Big[\tilde{f}((u^2-v^2)\tilde{f}_v - 2uv\tilde{f}_u) + \tilde{g}\left((u^2-v^2)\tilde{g}_v - 2uv\tilde{g}_u\right)\Big], \\ \dot{v} &= 2\Big[\tilde{f}(2uv\tilde{f}_v + (u^2-v^2)\tilde{f}_u) + \tilde{g}\left(2uv\tilde{g}_v + (u^2-v^2)\tilde{g}_u\right)\Big], \end{aligned} \tag{7}$$

where

$$\tilde{f} = f\left(\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2}\right) \quad \text{and} \quad \tilde{g} = g\left(\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2}\right).$$

System (7) is not well-defined at the origin but doing the change of time (5) we get the system

$$u' = (u^2+v^2)^d\dot{u}, \qquad v' = (u^2+v^2)^d\dot{v}, \tag{8}$$

which is well-defined at the origin. Moreover as the terms of lowest order of u' and v' after the scaling of time have minimum degree $d+2$, the origin of system (8) is a degenerate singular point. The first integral (1) is transformed to

$$H(x,y) = f^2\left(\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2}\right) + g^2\left(\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2}\right). \tag{9}$$

This first integral is not well-defined at the origin of system (8), hence we cannot deduce the analytic integrability of system (8) around the origin. Indeed, system (8) has the polynomial inverse integrating factor $V = (u^2+v^2)^2$, but also from its existence we cannot deduce the local integrability at the origin of system (8) because $V(0,0) = 0$, see [15]. In fact for proving Theorem 2 it would be sufficient to prove the monodromy of the degenerate singular point located at the origin of system (8) and the existence of a center at this point. The degenerate center problem is a hard open problem, see for instance [14] and references therein. Hence the origin of system (8) only has two possibilities, either is a monodromic point and consequently is a center (recall that cannot be a focus because system (8) is the transformation of a Hamiltonian system),b or is not monodromic and then has homoclinic orbits with defined tangent that enter or escape from the origin.

Now we consider the differential equation associate to system (8) given by

$$\frac{dv}{du} = \frac{\tilde{f}\left(2uv\tilde{f}_v + (u^2-v^2)\tilde{f}_u\right) + \tilde{g}\left(2uv\tilde{g}_v + (u^2-v^2)\tilde{g}_u\right)}{\tilde{f}\left((u^2-v^2)\tilde{f}_v - 2uv\tilde{f}_u\right) + \tilde{g}\left((u^2-v^2)\tilde{g}_v - 2uv\tilde{g}_u\right)}. \tag{10}$$

We know that the solutions $v = v(u)$ with initial condition at the origin are always a formal solution (in fact analytic) by the classical theorem of the analytic dependence respect to the initial conditions and parameters. Hence if we construct these solutions and we only get the solutions $v = \pm iu$, this implies that there are not homoclinic orbits with defined tangent that enter or escape from the origin, and consequently the origin is a center. The solutions $v = \pm iu$ correspond to the invariant curve $u^2 + v^2 = 0$ that only contains the origin. Hence Theorem 2 is proved.

The technique to construct formal solutions of the associated differential equation (10) is described in the works [16, 17].

It seems that Theorem 2 is true for the case that $\det(DF(x,y)) = \text{constant} \neq 0$ and $F(0,0) = (0,0)$. Almost all the examples that we have analyzed satisfy the condition of Theorem 2, as the following example.

Example. Consider the following map $F = (f, g) = (x, y + x + x^2 + x^4)$. This map has $\det(DF(x, y)) = 1$. System (2) associate is

$$\dot{x} = -2(x+x^2+x^4+y), \quad \dot{y} = 2x+2(1+2x+4x^3)(x+x^2+x^4+y). \tag{11}$$

Now to study the infinity we do the change of variables (3) and we get system (8) **Jo afegiria l'expressió explícita d'aquest sistema i del segúent, així facilitem que si algu vol refer els càlculs ho tingui més fàcil**. Next we construct the associated differential equation (10) and we compute the formal solutions of the form

$$v(u) = \sum_{i=1}^{\infty} a_i u^i. \tag{12}$$

Substituting this power series into equation (10) we obtain the condition $-8(1 + a_1^2) = 0$. The cases $a_1 = \pm i$ give the formal solution $v(u) = \pm iu$ which correspond to the invariant algebraic curve $u^2 + v^2 = 0$ that only contains the origin.

We can also study the other differential equation du/dv and the results obtained are the same. In this case we compute the formal solutions of the form

$$u(v) = \sum_{i=1}^{\infty} b_i v^i. \tag{13}$$

Substituting this power series into equation du/dv we obtain the condition $8b_1^8(1 + b_1^2) = 0$. The cases $b_1 = \pm i$ give the formal solution $v(u) = \pm iu$ which correspond to the invariant algebraic curve $u^2 + v^2 = 0$. For the case $b_1 = 0$ does not exist a formal solution. Consequently the system cannot have any homoclinic orbit with defined tangent that enter or escape from the origin and system (11) has a global center. Consequently by Theorem 4 the map F is injective.

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