LOWER BOUNDS FOR THE LOCAL CYCLICITY FOR FAMILIES OF CENTERS

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ABSTRACT. In this paper we are interested on how the local cyclicity of a family of centers depends on the parameters. This fact, was pointed out in [21], to prove that there exists a family of cubic centers, labeled by CD_{31}^{12} in [25], with more local cyclicity than expected. In this family there is a special center such that at least twelve limit cycles of small amplitude bifurcate from the origin when we perturb it in the cubic polynomial general class. The original proof has some important gaps that we correct here. We take the advantage of better understanding of the bifurcation phenomenon in non generic cases to show two new cubic systems exhibiting 11 limit cycles and another exhibiting 12. Finally, using the same techniques, we study the local cyclicity of holomorfic quartic centers, proving that 21 limit cycles of small amplitude bifurcate from the origin, when we perturb in the class of quartic polynomial vector fields.

1. INTRODUCTION

The study of limit cycles began at the end of the 19th century with Poincaré. Years later, in 1900, Hilbert presents a list of unsolved problems. From the original 23 problems of this list, the 16th is still open. The second part of this problem consists in determining a uniform bound of the maximal number of limit cycles (named $H(n)$), and their relative positions, of a planar polynomial systems of degree n. However, there are also weak versions of 16th Hilbert's problem. Arnold in [1] proposed a version focused on studying the number of limit cycles bifurcating from the period annulus of Hamiltonians systems. In this paper, we are interested in provide the maximal number $M(n)$ of small amplitude limit cycles bifurcating from an elementary center or an elementary focus, in special for degrees 3 and 4. The main idea is to study the local cyclicity of families of centers depending on a finite number of parameters.

As it is well known, for $n = 2$, Bautin proved in [2] that $M(2) = 3$. The case $n = 3$ but without quadratic terms (homogeneous cubic perturbation) was studied in [3, 19] and solved in [23], then $M_h(3) = 5$. In [24, 26] Zoladek shown that $M(3) \geq 11$. Christopher, in [5], gave a simple proof of Zoladek's result perturbing another cubic center with a rational first integral, using only the linear parts of the Lyapunov constants. The interest of this result is that we can compute these linear parts [5], in a parallelized way [14, 17], near a center without having the complete expressions of the Lyapunov constants. Basically the used technique consists in to choose a point on the center variety and at this point consider the linear term of the Lyapunov constants, if the point is chosen on a component of the center variety of codimension k, then the first k linear terms of the Lyapunov constants are

independent. This is a direct application of the Implicit Function Theorem to prove that $M(n) \geq k$. Usually we use this technique to provide lower bounds for the local cyclicity problem in the class of polynomial vector fields of degree n. In [10, 11], Giné presents a conjecture that the number $M(n)$ is bounded below by $n^2 + 3n - 7$ and study the cyclicity of different families of centers presented in [9]. In [13, 14] new lower bounds for $M(n)$ and n small has been obtained. The new values are $M(4) \geq 20$, $M(5) \geq 33$, $M(6) \geq 44$, $M(7) \geq 61$, $M(8) \geq 76$, and $M(9) \geq 88$.

In [21], Yu and Tian point out that the 1-parameter family of centers labeled by CD_{31}^{12} in [25] is quite special because it can exhibit one more limit cycle than expected in Giné's conjecture. This family has the next rational first integral

$$H(x,y) = \frac{(xy^2 + x + 1)^5}{x^3(xy^5 + 5xy^3/2 + 5y^3/2 + 15xy/8 + 15y/4 + a)^2} \tag{1}$$

and it has, following Żołądek computations, codimension 12. The original proof has some important gaps that we correct here, proving effectively that there exist some special values of the parameter a in (1) such that 12 limit cycles of small amplitude bifurcate from the origin, when we perturb in the class of complete cubic polynomial vector fields. This family was also studied by Christopher in [5] and it was the first clear proof about the existence of at least 11 limit cycles of small amplitude bifurcating from an equilibrium in polynomial vector fields of degree three.

The main result of this paper is the following.

Theorem 1.1. *The number of limit cycles of small amplitude bifurcating from an equilibrium of monodromic type in the classes of polynomial vector fields of degrees 3 and 4 are $M(3) \geq 12$ and $M(4) \geq 21$, respectively.*

After the above result, clearly, the commented general lower bound for $M(n)$ should be updated to be $n^2 + 3n - 6$. We remark that the total number of parameters for polynomial vector fields of degree n is $n^2 + 3n + 2$. Then, the new conjecture removes 8 to this total number of parameters. Six corresponding to an affine change of variables that writes the linear part in its normal form, one corresponding to a rotation and another to a rescaling. The previous conjecture took into account that the number of limit cycles in a center component do not change. But this is only generically. In this work, we will provide examples where this property fails. Hence we establish the following conjecture.

Conjecture 1.2. *The number of limit cycles of small amplitude bifurcating from an equilibrium of monodromic type in the class of polynomial vector fields of degree n is $M(n) = n^2 + 3n - 6$.*

The proof of the above theorem is based on an extension of Christopher results ([5]) for linear and higher order studies when the considered center has parameters. This new result, Theorem 3.1, is proved in Section 3. For completeness we also include here the Christopher results, see Theorems 2.2 and 2.4 in the next section. We remark that, the parallelization algorithm introduced in [14, 17] is crucial to get the results because facilitates all the needed computations. In Section 4 we do the proof of the statement of Theorem 1.1 corresponding to degree 3 vector fields. Moreover, we study also the bifurcation diagrams of limit cycles of small amplitude bifurcating from three families of centers. The first is 1-parametric and it is the rational reversible center family labeled by CR_{12}^{17} in [25]. The second is a

2-parameter holomorphic cubic center family. The third is labeled by $CD_{10}^{(11)}$ in [25], it has also 2 parameters and no maximal codimension 12 but, from it, also 12 limit cycles bifurcate from the center inside the cubic polynomial class. In Section 5 we study the bifurcation diagram for a 2-parameter center family of degree 4 that allow us to prove the statement of Theorem 1.1 corresponding to vector fields of degree 4. Finally, we also study partially the bifurcation diagram for a 4-parameter quartic holomorphic family of centers.

We have used a cluster of computers with 128 processors simultaneously with 725 GB of ram memory. All the computations have been made with `Maple` [18].

2. Lyapunov constants and parallelization

Let us consider the system

$$\begin{cases} \dot{x} &= -y + P_n(x,y), \\ \dot{y} &= x + Q_n(x,y), \end{cases}$$

with P_n and Q_n polynomials of degree n in variables x, y. Writing the system in complex coordinates, we have

$$R(z, \bar{z}) = iz + R_n(z, \bar{z}), \tag{2}$$

where $R_n(z, \bar{z})$ are polynomials of degree n in variables $(z, \bar{z})$. We seek for a first integral in the form $H(z, \bar{z})$ in a neighborhood of the origin such as

$$\mathcal{X}(H) = \sum_{k=0}^{\infty} v_k (z\bar{z})^{k+1},$$

where $\mathcal{X}$ is the vector field associated to (2) and v_k the k-Lyapunov constant. Clearly, the origin will be a center if, and only if $v_n(2\pi) = 0$ for all n. If $v_n(2\pi) \neq 0$, for some n, so we have a focus of order n.

In most cases, the process of calculating Lyapunov constants is very hard, being impossible to calculate them manually. Therefore, the use of an algebraic manipulator system is necessary. Moreover the Parallelization process will be so important in our work, as it offers us a reduction in the time of very large computations. The first result about Parallelization is given by Liang and Torregrosa in [17] and using it we can obtain the linear part of Lyapunov constants of the more easy way.

Theorem 2.1 ([17]). *Let $p(z, \bar{z})$ and $Q_j(z, \bar{z})$, $j = 1, \ldots, s$ be polynomials with monomials of degree higher or equal than two such that the origin of $\dot{z} = iz + p(z, \bar{z})$ is a center. If $L_{k,j}^{(1)}$ denotes the linear part, with respect to $\lambda_j \in \mathbb{R}$, of the k-Lyapunov constant of equation*

$$\dot{z} = iz + p(z, \bar{z}) + \lambda_j Q_j(z, \bar{z}), \quad j = 1, \ldots, s,$$

then the linear part of the k-Lyapunov constants, with respect to $\Lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{R}^N$, of equation

$$\dot{z} = iz + p(z, \bar{z}) + \sum_{j=1}^{N} \lambda_j Q_j(z, \bar{z}),$$

are $L_k^{(1)} = \sum_{j=1}^{N} L_{k,j}^{(1)}$.

The next theorem was developed by Christopher and show how we can use Lyapunov constants to obtain lower bounds of limit cycles.

Theorem 2.2 ([5])**.** *Suppose that s is a point on the center variety and that the first k Lyapunov constants, $L_1, \ldots, L_k$, have independent linear parts (with respect to the expansion of L_i about s), then s lies on a component of the center variety of codimension at least k and there are bifurcations which produce k limit cycles locally from the center corresponding to the parameter value s. If, furthermore, we know that s lies on a component of the center variety of codimension k, then s is smooth point of the variety, and the cyclicity of the center for the parameter value s is exactly k. In the latter case, k is also the cyclicity of a generic point on this component of the center variety.*

We notice that, to perform higher order parallelization, that is, to calculate high order of Lyapunov constants we need to decompose the global problem in simpler problems having exactly ℓ parameters or monomials. However, as many parameters appear in some of the simple perturbation problems we need to correct the obtained coefficients of the developments of order ℓ.

Theorem 2.3 ([14])**.** *Let $p(z, \bar{z})$ and $Q_j(z, \bar{z})$, $j = 1, \ldots, N$ be polynomials with monomials of degree higher or equal than two such that the origin of $\dot{z} = iz + p(z, \bar{z})$ is a center. For $\ell \leq N$, we denote by $L_k^{(\ell)}$ the k-Lyapunov constant of order ℓ of equation*

$$\dot{z} = iz + p(z, \bar{z}) + \sum_{j=1}^{N} \lambda_j Q_j(z, \bar{z}),$$

with $\Lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{R}^N$. Let S be the set of all combinations of the components of Λ taken ℓ at a time. That is, $S = \{(\lambda_1, \ldots, \lambda_\ell), (\lambda_2, \ldots, \lambda_{\ell+1}), \ldots, (\lambda_{\ell-N}, \ldots, \lambda_N)\}$ and it has $\binom{N}{\ell}$ elements. For each element $S_j = (\lambda_{\sigma(j,1)}, \ldots, \lambda_{\sigma(j,\ell)})$ in S, $j = 1, \ldots, \binom{N}{\ell}$, we denote by $L_{k,j}^{(\ell)}$ the k-Lyapunov constant of order ℓ with respect to S_j of equation

$$\dot{z} = iz + p(z, \bar{z}) + \sum_{l=1}^{\ell} \lambda_{\sigma(j,l)} Q_{\sigma(j,l)}(z, \bar{z}).$$

Then

$$L_k^{(\ell)} = \sum_{l=1}^{N} \hat{L}_{k,j}^{(\ell)},$$

where $\hat{L}_{k,j}^{(\ell)} = \sum_p \dfrac{\mu_{k,j,p}}{\binom{N-s(p)}{\ell-s(p)}} \Lambda_j^p$, for $\Lambda_j^p = \lambda_{\sigma(j,1)}^{p_1} \lambda_{\sigma(j,2)}^{p_2} \cdots \lambda_{\sigma(j,\ell)}^{p_\ell}$ and $p = (p_1, \ldots, p_\ell)$

writing $L_{k,j}^{(\ell)} = \sum_p \mu_{k,j,p} \Lambda_j^p$ with $s(p) = \sum_{l=1}^{\ell} \text{sgn}(p_l)$ where $\text{sgn}(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x = 0. \end{cases}$

The next result developed by Christopher in [5] show how we can obtain limit cycles using high order of Lyapunov constants. A new proof of this result, using blow-up, has been given in [14].

Theorem 2.4 ([5])**.** *Suppose that, after a change of variables if necessary, $L_0 = L_1 = \cdots = L_k = 0$ and the next Lyapunov constants $L_i = h_i(u) + O_{m+1}(u)$, $i = k+1, \ldots, k+l$, where h_i are homogeneous polynomials of degree $m \geq 2$ and $u =$*

$(u_{k+1}, \ldots, u_{k+l})$. *If there exists a line ℓ, in the parameter space, such that $h_i(\ell) = 0$, $i = k+1, \ldots, k+l-1$, the hypersurfaces $h_i = 0$ intersect transversally along ℓ for $i = k+1, \ldots, k+l-1$, and $h_{k+l}(\ell) \neq 0$, then there are perturbations of the center which can produce $k + l$ limit cycles.*

We remark that from the proof of the above theorem it is clear that there exists a pertubation that produces the total number of limit cycles.

As we will see in the proofs of the results of the next sections, sometimes the application of Theorem 2.4 is not so simple. Because it depends on finding explicitly the intersection of some manifolds and if it is transversal. Although this intersection point can be obtained numerically, we use a computer assisted proof to prove analytically the existence of such point. This is done using Poincaré–Miranda's Theorem together with the results of the appendix. For the transversality property we can use the Circles of Gershgorin Theorem. For completeness we add them here.

Theorem 2.5 ([16], Poincaré–Miranda)**.** *Let c be a positive real number and $S = [-c, c]^n$ a n-dimensional cube. Consider $f = (f_1, \ldots, f_n) : S \to \mathbb{R}^n$ a continuous function such $f_i(S_i^-) < 0$ and $f_i(S_i^+) > 0$ for each $i \leq n$, where $S_i^{\pm} = \{(x_1, \ldots, x_n) \in S : x_i = \pm b\}$. So, there exists a point $d \in S$ such that $f(d) = 0$.*

Theorem 2.6 ([8], Circles of Gershgorin)**.** *Let $A = (a_{i,j}) \in \mathbb{C}^{n \times n}$ and α_k your eigenvalues. Consider for each $i = 1, \ldots, n$*

$$D_i = \{z \in \mathbb{C} : |z - a_{i,i}| \leq r_i,$$

where $r_i = \displaystyle\sum_{i \neq j} |a_{i,j}|$. So, for all k, each $\alpha_k \in D_i$ for some i.

The Poincaré–Miranda's Theorem was conjectured by Poincaré in the 19th century and proved by Miranda in last century. Note that this result is a generalization of the Bolzano's Theorem for higher dimensions. The reader can get more details on Gershgorin Circles Theorem in [12].

3. Local cyclicity depending on parameters

This section is devoted to extend Theorems 2.2 and 2.4 to families of centers that depend on some parameters. Let $(\dot{x}, \dot{y}) = (P_c(x, y, \mu), Q_c(x, y, \mu))$ be a family of polynomial centers of degree n depending on a parameter $\mu \in \mathbb{R}^\ell$, having a center equilibrium point at the origin. We consider the perturbed polynomial system

$$\begin{cases} \dot{x} &= P_c(x, y, \mu) + \alpha y + P(x, y, \lambda), \\ \dot{y} &= Q_c(x, y, \mu) + \alpha x + Q(x, y, \lambda), \end{cases} \tag{3}$$

with P, Q polynomials of degree n having no constant nor linear terms. More concretely,

$$P(x, y, \lambda) = \sum_{k+l=2}^{n} a_{k,l} x^k y^l, \quad Q(x, y, \lambda) = \sum_{k+l=2}^{n} b_{k,l} x^k y^l,$$

with $\lambda = (a_{20}, a_{11}, a_{02}, \ldots, b_{20}, b_{11}, b_{02}) \in \mathbb{R}^{n^2+3n-4}$. The trace parameter α sometimes is also denoted by L_0.

Theorem 3.1. *When $a = 0$, we denote by $L_j^{(1)}(\lambda, b)$ the first order development, with respect to $\lambda \in \mathbb{R}^k$, of the $j-$Lyapunov constant of system (3). We assume that, after a change of variables in the parameter space if necessary, we can write*

$$L_j = \begin{cases} \lambda_j + O_2(\lambda), \text{ for } j = 1, \ldots, k-1, \\ \sum_{l=1}^{k-1} g_{j,l}(\mu)\lambda_l + f_{j-k}(\mu)\lambda_k + O_2(\lambda), \text{ for } j = k, \ldots, k+\ell. \end{cases} \tag{4}$$

Where with $O_2(\lambda)$ we denote all the monomials of degree higher or equal than 2 in λ with coefficients analytic functions in μ. If there exists a point μ^* such that $f_0(\mu^*) = \cdots = f_{\ell-1}(\mu^*) = 0$, $f_\ell(\mu^*) \neq 0$, and the Jacobian matrix of $(f_0, \ldots, f_{\ell-1})$ with respect to μ has rank ℓ at μ^*, then system (3) has $k + \ell$ hyperbolic limit cycles of small amplitude bifurcating from the origin.

Remark 3.2. *We remark the importance, in the above result, of the number of components in parameters λ and μ. Because, if there are more parameters than the relevant k in λ, in $O_2(\lambda)$ term can appear monomials of degree 2 that can affect the monomials of degree 1 and the result could be not valid.*

Proof of Theorem (3.1). We assume first that the trace parameter α is zero. Then, using scheme mentioned in [14], we can remove the sums in (4) and consider a simpler list

$$L_j = \begin{cases} \lambda_j + O_2(\lambda), \text{ for } j = 1, \ldots, k-1, \\ f_{j-k}(\mu)\lambda_k + O_2(\lambda), \text{ for } j = k, \ldots, k+\ell. \end{cases}$$

With the Implicit Function Theorem in the first $k - 1$ components and writing $\lambda_k = u_k$ the above expression writes as

$$L_j = \begin{cases} u_j, \text{ for } j = 1, \ldots, k-1, \\ f_{j-k}(\mu)u_k + O_2(u), \text{ for } j = k, \ldots, k+\ell. \end{cases} \tag{5}$$

From the hypothesis on the functions f_j at $\mu = \mu^*$, using again the Implicit Function Theorem, we can write, close to $\mu = \mu^*$, $f_{j-k}(\mu) = v_{j-k} + O_2(v)$, with $v_{j-k} = \mu_{j-k} - \mu^*_{j-k}$, for $j = k, \ldots, k+\ell-1$, and $v = (v_0, \ldots, v_{\ell-1})$.

Now, we consider the change of variables, like a partial blow-up, $u_j = zw_j$ for $j = 1, \ldots, k-1$, $u_k = z$, and $v_{j-k} = w_j$ for $j = k, \ldots, k+\ell-1$. Then (5) write as

$$L_j = \begin{cases} zw_j, \text{ for } j = 1, \ldots, k-1, \\ z(w_j + A_j z + O_2(z, w_1, \ldots, w_{k+\ell-1})), \text{ for } j = k, \ldots, k+\ell-1, \end{cases} \tag{6}$$

for some real numbers A_j. They, as the higher order terms in u, come from the terms $O_2(u)$ and the terms $O_2(v)$, after the change to $(z, w_1, \ldots, w_{k+\ell-1})$ coordinates. Moreover, the last Lyapunov constant writes as

$$L_{k+\ell} = z(B + O_1(z, w_1, \ldots, w_{k+\ell-1})).$$

Finally, in (6) we can use again the Implicit Function Theorem to write $z_j = w_j$ for $j = 1, \ldots, k-1$ and $z_j = w_j + A_j z + O_2(z, w_1, \ldots, w_{k+\ell-1})$, for $j = k, \ldots, k+\ell-1$. We notice that z is small enough and we have, near the origin of the parameter space, a curve (parametrized) by z of weak-foci of order $k + \ell$ that unfolds exactly, using the Weierstrass Preparation Theorem, $k + \ell - 1$ hyperbolic limit cycles of small amplitude bifurcating from the equilibrium point located at the origin. The last limit cycle appears using the trace parameter α in a classical Hopf bifurcation. $\qquad\square$

Christopher in [5] comments the generic unfolding of k limit cycles in families of polynomial vector fields when we consider centers on a component of the center variety of codimension k. This is the aim of Theorems 2.2 and 2.4. The above result shows that on some special points on such component the cyclicity can increase. This is the mechanism that we have used in the following sections to improve the known lower bounds for the local cyclicity $M(n)$ for some low values of n. In particular for $n = 3$ and $n = 4$. We think that the Giné's conjecture in [10, 11] about the lower bound for $M(n) = n^2 + 3n - 7$ can be thought in the sense of generic centers. We remark that, for providing higher values for $M(n)$ for higher degree n, we need to know better center families. Because the known families or have low codimension or they have too many parameters and the computational difficulties, as we will see in the following examples, increase so fast.

The fact that the cyclicity of Hamiltonian families depends on the parameters was previously studied by Han and Yu in [15], but not applied correctly in [21]. Here we extend this result for other type of center families.

4. Bifurcation diagrams for local cyclicity in families of cubic centers

In this section we use Theorem 3.1 to study the bifurcation diagram for some families of cubic centers, lying in components of the center variety of codimension 11, 10, and 9. The first, in Proposition 4.1, is the family labeled $CD_{31}^{(12)}$ that has generically cyclicity 11 and was studied previously by Christopher in [5], for only one parameter value $a = 2$ in (1), and by Yu and Tian in [21]. This proposition proves partially the main Theorem 1.1. The family labeled as $CR_{17}^{(12)}$ in [25], which depends also of a parameter a, is studied in Proposition 4.2. We have studied the local cyclicity for some values of this parameter a of the family $CR_{17}^{(12)}$ up to order 4 in the Lyapunov constants and we have found only 10 limit cycles. But using Theorem 3.1 we can get an extra limit cycle. Up to our knowledge this is the first time that the cyclicity of this family has been studied. The last cubic family has 2 free parameters, see Proposition 4.4, and we show that generically the origin has cyclicity at least 9 and that there are curves with cyclicity at least 10 and some special points with cyclicity at least 11. According Gasull, Garijo and Jarque, in [7], any holomorphic center is also a Darboux center. Liang and Torregrosa in [17] show that, for some values of the cubic family the cyclicity is as least 9. Here we explain that the cyclicity will increase depending on the specific center that we select. Up to our knowledge the studies of the bifurcation diagrams are new for these families. At the end of the section we show a center family with two parameters and not maximal codimension, labeled as $CD_{10}^{(11)}$ in [25], such that, perturbing with cubic polynomials, bifurcate also 12 limit cycles of small amplitude.

Proposition 4.1. *Consider system* (3) *with $n = 3$ and the unperturbed center*

$$\begin{cases} \dot{x} &= -10(256a^3xy + 384a^3y - 96a^2x^2 - 384a^2y^2 - 16a^2x - 600axy \\ &\quad -480ay + 225x^2 + 900y^2 - 225x)(32a^2x + 48a^2 - 75x + 150), \\ \dot{y} &= 16384a^5xy^2 + 24576a^5y^2 - 61440a^4y^3 + 16384a^5x + 56320a^4xy \\ &\quad -76800a^3xy^2 - 7680a^4y - 384000a^3y^2 + 288000a^2y^3 - 32000a^3x \\ &\quad -96000a^2xy + 90000axy^2 - 132000a^2y + 765000ay^2 - 337500y^3 \\ &\quad +168750ax - 84375xy - 337500y, \end{cases} \quad (7)$$

with a such that $32a^2 - 75 > 0$. Then, there exist only six parameter values a^* such that 12 limit cycles of small amplitude bifurcate from the origin. They are approximately $\pm 2.019925086, \pm 7.444369217,$ and ± 15.62631048.

We have computed the linear part of the first Lyapunov constants for some values of a, different from the a^* stated in the above proposition, obtaining always maximal rank 11. Then, using Theorem 2.2, we can obtain, after perturbation, only 11 limit cycles of small amplitude. We think that this situation will be generic.

Proof of Theorem 4.1. The system corresponding to the rational first integral (1) has a center at the point $(x, y) = (6(8a^2 + 25)/(32a^2 - 75), 70a/(32a^2 - 75))$. Then, translating it to the origin we get system (7).

Let us consider (3) with $b_{30} = 0, b_{12} = 0$, and $b_{03} = 0$. After computing the first 12 Lyapunov constants up to order 1, we have that, generically for every a, $L_1^{(1)}, \ldots, L_{10}^{(1)}$ are linearly independent with respect to the parameters

$$a_{02}, a_{03}, a_{11}, a_{12}, a_{20}, a_{21}, a_{30}, b_{02}, b_{11}, b_{20}.$$

Then, we can write, after a linear change of parameters, $L_k = u_k + O_2(u)$, for $k = 1, \ldots, 10$, where $u_{11} = b_{21}$ and $O_2(u)$ denotes the monomials in u of degree higher than 2 with coefficients rational functions in the parameter a. Moreover, we have that L_j write as (4) with

$$L_{11}^{(1)} = \sum_{l=1}^{10} g_{10,l}(a)u_l + g(a)f_0(a)u_{11}, \quad L_{12}^{(1)} = \sum_{l=1}^{10} g_{11,l}(a)u_l + g(a)f_1(a)u_{11},$$

where f_0 and f_1 are polynomials of degree 26 and 39 in a^2, respectively, g is a rational function without common factors with f_0 nor f_1. Additionally, the numerator and denominator of g are polynomials of degrees 69 and 90 in a^2 and $g_{10,l}$ and $g_{11,l}$ are also rational functions. All the involved polynomials are polynomials with rational coefficients.

The proof follows applying Theorem 3.1. To do that, we need to check that f_0 has real simple zeros and that the resultant of f_0 and f_1 with respect to a is a non zero rational number. So, there should be at least a special value $a = a^*$ such that $f_0(a^*) = 0, f_0'(a^*) \neq 0$ and $f_1(a^*) \neq 0$. Finally, it can be checked that there are only six possible values for a^*. The numerical approximation values for a^* are the ones given in the statement.   □

In the proof of the existence of the extra limit cycle done in [21] the computations of $L_k^{(1)}$ are the same that we obtain. As we have understood, the mistake is that his proof is not based directly in a result like Theorem 3.1 which we have perfectly identified the perturbation parameters and we have restricted the perturbation in order to apply it. Their proof is based in the fact that $L_{11}^{(1)}$ vanishes and $L_{12}^{(1)}$ not. This is not enough because the terms of order 2 of L_{11} can appear and the weak-focus order does not increase. In fact, if we only consider $f_0(a^*) = 0$ then $L_{11}^{(2)} = u_{11}^2 g_1(a)/g_2(a)$, with g_1 and g_2 polynomials of degree 66 and 103 in a^2. Moreover, in [21] the control of the number of relevant parameters as we have commented in Remark 3.2 is not clear.

Proposition 4.2. *Let $a \notin \{0, -1/6\}$ a real parameter. Consider the system*

$$\begin{cases} \dot{x} &= (x - ay + a + 2)(2\eta - 3\eta_y + 3x^2 + 6x + 6) - 3\eta\eta_y \\ &\quad -9x^2\eta_y + 9(2ax^2 + (2a - 1)x + 2a), \\ \dot{y} &= 3(y(x - ay + a + 2)(-3x + y + 2) + 3(x^2 + x - 2)), \end{cases} \tag{8}$$

with $\eta = xy - ay^2 + 2x + 2(1 + a)y + 1 - a$, $-1/6 < a < 0$, $1/3 < a < 1$, or $1 < a$. Then, it has a center at

$$(x^*, y^*) = \left(\frac{3(a - 1)}{6a + 1}, -\frac{3a^2 - 4a + 1}{a(6a + 1)} \right),$$

and the next properties hold.

  (i) *If $g(a) \neq 0$ and $f_0(a) \neq 0$ the local cyclicity, perturbing with polynomials of degree 3, is at least 10.*
 (ii) *If $g(a) \neq 0$, $f_0(a) = 0$ the local cyclicity, perturbing with polynomials of degree 3, is at least 11. Moreover, f_0 has only 4 simple roots in the considered intervals. The numerical approximation are $\{-0.12245, 0.39672, 0.61983, 2.70517\}$.*

The expressions of polynomials f_0 and g are

$$\begin{aligned}
f_0(a) = &\ 11556711608903120520a^{26} - 82791934329314091672a^{25} \\
&+ 228195405046186847010a^{24} + 9049153312278017424a^{23} \\
&- 157081144205847843464a^{22} + 335918075048147398203 9a^{21} \\
&- 315147810716332642769 4a^{20} - 325955324399233829796a^{19} \\
&+ 142113712204693890075 06a^{18} - 38670367283669710621611a^{17} \\
&+ 56868934982665036265406a^{16} - 54377179326178644006963a^{15} \\
&+ 30803908784073506907336a^{14} - 90192770456966323834 77a^{13} \\
&- 664922996737568168778a^{12} + 2963892390472140000813a^{11} \\
&- 1762296309778946693076a^{10} + 408343189249696331943a^{9} \\
&- 53423768941943519592a^{8} + 36887231065315303647a^{7} \\
&- 13263836783633911152a^{6} + 1484165815203151098a^{5} \\
&+ 85191877643707008a^{4} - 114163404746428485a^{3} \\
&+ 1130289090405930a^{2} + 1973552231555520a + 103574370739840,
\end{aligned}$$

$$\begin{aligned}
g(a) = &\ 44130128757997201642800a^{31} - 252501315621254559684000a^{30} \\
&+ 567997250848916245020180a^{29} - 813793828511873349837180a^{28} \\
&+ 239927936294998889113869 0a^{27} - 277720336430898312874527 0a^{26} \\
&- 1117982977709921462960878 5a^{25} + 51100343128278769201023051a^{24} \\
&- 96722734568856169055589531a^{23} + 101072414237147073155782098a^{22} \\
&- 81911167892441981812923273a^{21} + 91543737997225903881665763a^{20} \\
&- 123464208935758068586525599a^{19} + 135385335579943472406867144a^{18} \\
&- 10747031666134250947603527 0a^{17} + 5932298567720321123817612 6a^{16} \\
&- 22468443503910229293603606a^{15} + 6323085724047239916867708a^{14} \\
&- 16560396455903787612385 26a^{13} + 35134627516718478043473 0a^{12} \\
&+ 124075546922063688717 24a^{11} - 2921779219862791558927 8a^{10}
\end{aligned}$$

$$+ 3200041670276393240067a^9 + 933095466480821343399a^8$$
$$- 81964651107172872879a^7 - 23554321764806596878a^6$$
$$- 526449753238950189a^5 + 210455326225541295a^4 + 20323154636412705a^3$$
$$+ 375301845557100a^2 - 28137453964620a - 1083684121520.$$

Different phase portraits of system (8) are given in Figure 1.

Remark 4.3. *We notice that, although the family (8) is considered of codimension 12 by Zoladek in [25], we have not found more than 10 limit cycles of small amplitude as it is stated in the above result for $a = 2$ and computing up to order 10. We think that the same will happen for other values of a except the ones in Proposition 4.2 such that vanishes f_0.*

Proof of Proposition 4.2. Doing a translation in order that the center (x^*, y^*) of system (8) moves to the origin, we get

$$
\begin{cases}
\dot{x} = & -(6a+1)(648a^7y^3 - 1944a^7y^2 - 2430a^6x^2y - 1944a^6xy^2 + 216a^6y^3 \\
& +1458a^7y + 729a^6x^2 - 2916a^6xy - 3564a^6y^2 + 972a^5x^3 - 1701a^5x^2y \\
& -1296a^5xy^2 + 18a^5y^3 - 972a^6y + 2187a^5x^2 + 486a^5xy - 1404a^5y^2 \\
& +1134a^4x^3 - 216a^4x^2y - 270a^4xy^2 - 486a^5y + 1053a^4x^2 + 1782a^4xy \\
& -144a^4y^2 + 486a^3x^3 + 72a^3x^2y - 18a^3xy^2 + 54a^4y - 162a^3x^2 \\
& +594a^3xy + 90a^2x^3 + 18a^2x^2y - 36a^3y - 189a^2x^2 + 54a^2xy \\
& +6x^3a + ax^2y - 18a^2y - 33ax^2 - x^2), \\
\dot{y} = & 3(3a+1)^4(6a^2y - 3a^2 + ay + 4a - 1)(6a^2y - 9a^2 + ay + 3a - 1)x.
\end{cases}
$$

Then we can consider equation (3). The proof that this family has a center follows from a rational symmetry and it can be found in [22, 25].

Next step is the computation of $L_k^{(1)}$, for $k = 1, \ldots, 9$ and we consider them as linear functions depending on $a_{02}, a_{03}, a_{11}, a_{12}, a_{20}, a_{21}, b_{02}, b_{03}, b_{20}$. Hence, we write, after a linear change of coordinates adding $b_{21} = u_{10}$,

$$L_j = u_j + O_2(u), \text{ for } j = 1, \ldots, 9.$$

The other parameter values in (3) have been taken as zero. In $O_2(u)$ appear some denominators in a which are non zero under the hypotheses of the statement. In particular the condition $g(a) \neq 0$ appears solving the above linear change. It can be seen also in the following expressions of the next two Lyapunov constants. After
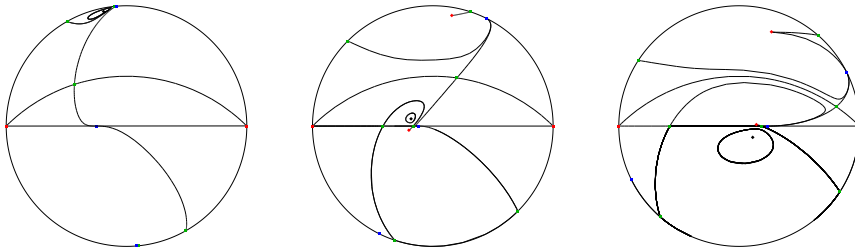

FIGURE 1. Phaseportraits in the Poincaré disk of the center (8) for
$a = -1/12$, $a = 1/2$, and $a = 2$

using the Implicit Function Theorem to vanish the first nine Lyapunov constants, the simplified expressions of the next two, are, except non zero mult'iplicative constants,

$$L_{10} = \frac{(3a+1)^{13}(6a+1)^{18}}{a^3(9a^2-3a+1)^{10}(a-1)^9}\frac{f_0(a)}{g(a)}u_{10} + O_2(u_{10}),$$

$$L_{11} = \frac{(3a+1)^{13}(6a+1)^{19}}{a^4(3a-1)(9a^2-3a+1)^{12}(a-1)^{11}}\frac{f_1(a)}{g(a)}u_{10} + O_2(u_{10}),$$

where f_0 and g are defined in the statement and f_1 is

$$\begin{aligned}
f_1(a) = {} & 724536477608572237880880a^{32} + 64058932577894477741378280a^{31}\\
& - 6101444818597575862234015556a^{30} + 236097300897845421009384137 4a^{29}\\
& - 31060724819721052795609422 06a^{28} - 78475483467839244558712159 44a^{27}\\
& + 37350465281198340430573666575a^{26} - 659129499127957031533491415 83a^{25}\\
& + 55213834912911379234932558885a^{24} + 656248908141307740026500310 70a^{23}\\
& - 38609751041628138548356885717 5a^{22} + 8522590404897635458648691244 60a^{21}\\
& - 11935083324604459005626435840 16a^{20} + 11399642857067111355287114550 09a^{19}\\
& - 73072623362574084436187732226 6a^{18} + 2808175102250410893158987037 66a^{17}\\
& - 214872020845367124995261195 40a^{16} - 5430421915006086060825210811 2a^{15}\\
& + 42894589880370044683717289676a^{14} - 16098081186021186459359445174a^{13}\\
& + 284185709232916197633344204 4a^{12} - 401707003814285433422087250a^{11}\\
& + 27851952088407689270492120 1a^{10} - 89922626165488742408968047a^{9}\\
& - 21943437319421124107681 7a^{8} + 52400450768774913989591 22a^{7}\\
& - 11410625546053058922089 85a^{6} - 15124036595328170215596a^{5}\\
& + 42928143800073303753090a^{4} - 98239754146992695055a^{3}\\
& - 576005750186099035950a^{2} - 28747061161858522560a + 21647043484626560.
\end{aligned}$$

Clearly $9a^2 - 3a + 1$ is non vanishing and, with the restriction on a given in the statement, all the rational functions are well defined.

Statement (i) follows from Theorem 2.2. Statement (ii) follows as the proof of Proposition 4.1. That is, computing the resultant of f_0 and f_1 and resultant of f_0 and f_0' with respect to a, and checking that f_0 has real zeros, which will be simple, such that f_1 does not vanish at them. From Theorem 3.1 we know that for the values of a such that f_0 vanishes we have 11 limit cycles of small amplitued bifurcating from the origin. □

The next result provides a complete bifurcation diagram for all holomorphic cubic centers having the coefficient of z^2 non vanishing. In this case it is not restrictive, rescaling if necessary, to assume that it is 1. In complex coordinates they write as

$$\dot{z} = iz + z^2 + (a+ib)z^3. \tag{9}$$

Proposition 4.4. *Consider system* (3) *with* $n = 3$ *and the unperturbed center*

$$\begin{cases} \dot{x} = ax^3 - 3axy^2 - 3bx^2y + by^3 + x^2 - y^2 - y, \\ \dot{y} = 3ax^2y - ay^3 + bx^3 - 3bxy^2 + 2xy + x, \end{cases}$$

for every value of the parameters $(a, b) \in \mathbb{R}^2 \setminus \{(0,0)\}$ and the polynomials

$$
\begin{aligned}
f_0(a, b) = {} & 8a^6 + 24a^4b^2 + 24a^2b^4 + 8b^6 + 282a^4b + 564a^2b^3 + 282b^5 - 37569a^4 \\
& - 45954a^2b^2 - 8385b^4 - 91924a^2b - 162484b^3 - 646020a^2 - 37860b^2, \\
f_1(a, b) = {} & 2448a^6b + 7344a^4b^3 + 7344a^2b^5 + 2448b^7 + 3208a^6 + 95916a^4b^2 \\
& + 182208a^2b^4 + 89500b^6 - 12055032a^4b - 15179760a^2b^3 - 3124728b^5 \\
& - 19489169a^4 - 64437898a^2b^2 - 66540089b^4 - 285166044a^2b \\
& - 92688444b^3 - 310735620a^2 - 18210660b^2, \\
f_2(a, b) = {} & 145864a^8 - 3776a^6b^2 - 886512a^4b^4 - 1178240a^2b^6 - 441368b^8 \\
& + 3892522a^6b - 9022362a^4b^3 - 29722290a^2b^5 - 16807406b^7 \\
& - 708522105a^6 + 1379959497a^4b^2 + 2743262973a^2b^4 + 654781371b^6 \\
& + 8068743920a^4b + 18906063664a^2b^3 + 16016705984b^5 \\
& - 5202830396a^4 + 86382442952a^2b^2 + 48606733828b^4 \\
& + 185131413648a^2b + 33791194128b^3 + 93466173600a^2 + 5477584800b^2, \\
g(a, b) = {} & 27936a^6 + 83808a^4b^2 + 83808a^2b^4 + 27936b^6 - 162180a^4b \\
& - 324360a^2b^3 - 162180b^5 - 199825a^4 - 227714a^2b^2 - 27889b^4 \\
& - 347172a^2b - 23172b^3 + 30636a^2 - 5364b^2.
\end{aligned}
$$

Then,

(i) *if $f_0(a, b)g(a, b) \neq 0$ there are 9 limit cycles of small amplitude bifurcating from the origin;*

(ii) *if $f_0(a, b) = 0$ and $f_1(a, b)g(a, b) \neq 0$ there are 10 limit cycles of small amplitude bifurcating from the origin;*

(iii) *if $f_0(a, b) = f_1(a, b) = 0$ and $f_2(a, b)g(a, b) \neq 0$ there are 11 limit cycles of small amplitude bifurcating from the origin.*

Moreover, there exist only two transversal intersection points of the curves $f_0(a, b) = 0$ and $f_1(a, b) = 0$ which are $(\pm a^*, b^*) \approx (\pm 69.66852455, -6.617950485)$.

The above result provides the bifurcation diagram for the local cyclicity of the 2-parameter holomorphic family (9). The curves f_0, f_1 and f_2 are drawn in Figure 2 in red, green and blue, respectively. Generically, the local cyclicity is 9. On the red curve, generically, the cyclicity is 10 and in the intersection point of the curves red and green the cyclicity is 11.

Proof of Proposition 4.4. After a change of sign if necessary we can restrict our analysis to $a > 0$. For every (a, b) different from $(0, 0)$ and taking zero the parameters $b_{20}, b_{11}, b_{30}, b_{12}$, we compute, with the parallelized algorithm described in [14], the linear terms of the first 11 Lyapunov constants, with respect to the relevant parameters $a_{20}, a_{11}, a_{02}, b_{02}, a_{30}, a_{21}, a_{12}, a_{03}, b_{21}$. If $g(a, b) \neq 0$ then, up to a linear change
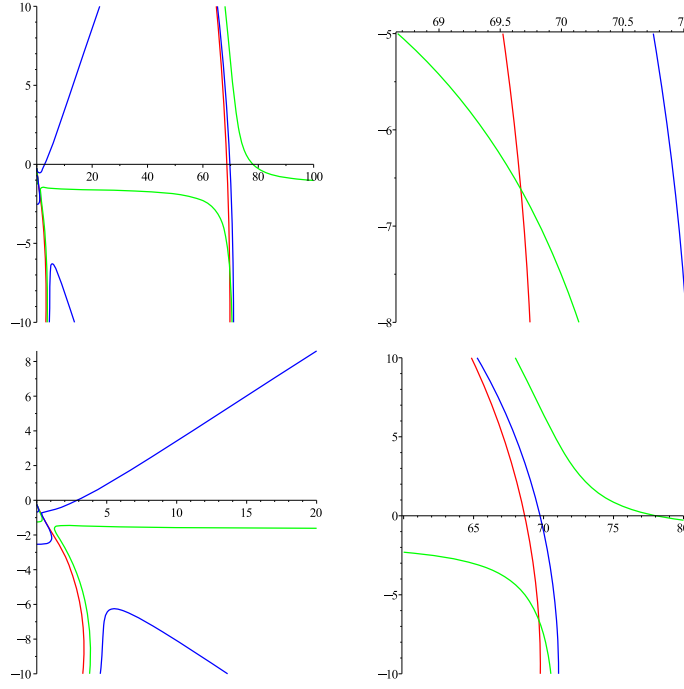
FIGURE 2. The curves, with some zooms, $f_0(a,b) = 0$, $f_1(a,b) = 0$, and $f_2(a,b) = 0$ given in Proposition 4.4 in red, green, blue, respectively

of parameters, we can write $L_j^{(1)} = u_j$, for $j = 1, \ldots, 8$, and

$$L_9^{(1)} = \frac{(81a^2 + (9b+2)^2)(a^2+b^2)^3}{g(a,b)} f_0(a,b) u_9,$$

$$L_{10}^{(1)} = \frac{(81a^2 + (9b+2)^2)(a^2+b^2)^3}{g(a,b)} f_1(a,b) u_9,$$

$$L_{11}^{(1)} = \frac{(81a^2 + (9b+2)^2)(a^2+b^2)^3}{g(a,b)} f_2(a,b) u_9.$$

To simplify we have divided, if necessary, by non zero multiplicative constants.

Computing the resultants of the pairs (f_0, f_1) and (f_0, f_2) with respect to a, we get

$b^6(4b-9)^2(9b-59)^2(512192700b^4 + 13330993797b^3 + 61034982291b^2$

$\quad - 33028358509b - 10270019239)^2,$

$b^6(4b-9)^2(9b-59)^2(5706989125856705507000b^7 + 22990976281237387495014b^6$

$\quad + 3688157828483981431 7085b^5 - 4478880915283836703764940b^4$

$\quad - 9505227203153802766492979b^3 + 3847660913988093703065912b^2$

$\quad + 13351954188119085151405788b + 2696188868201530577480960)^2.$

Removing the common factors, the above two polynomials in b of degrees 4 and 7 have no common roots, because it resultant, with respect to b is non vanishing. Then, any intersection point of the curves $f_0 = 0$ and $f_1 = 0$ is not in the curve $f_2 = 0$. Then straightforward computations shows that the curves $\{f_0 = 0, f_1 = 0\}$ have

only one real intersection point $(a^*, b^*) \approx (69.66852455, -6.617950485)$. Moreover, it is a transversal intersection and $f_2(a^*, b^*) \neq 0$.

The proof follows using Theorem 3.1 in each item in the statement. $\qquad\square$

Finally, we remark that the existence of centers with local cyclicity bigger than or equal to 12 is not restricted to families of codimension 12, some of them given in [25]. With the technique developed in this work we can approach the lower bounds for cyclicity problem in families with more than one parameter. But the systems of polynomial equations that appear have a very high degree and the difficulties in solving them are beyond the reach of our computers. As an example, we provide a new family of cubic centers that we have proved, only numerically, this lower bound. This family is inspired from the one labelled by $CD_{10}^{(11)}$ in [25] with codimension 11. In particular, it is defined by the rational first integral

$$H = \frac{(9x^3 + 9x^2 + 9bx + 9ay + 6a^2 + 6b^2 - b)^7}{(h_0(x) + h_1(x)y + h_2(x)y^2)^3}$$

where

$$
\begin{aligned}
h_0(x) =\ & 2187bx^7 + 5103bx^6 + 1701b(3b+2)x^5 + 189b(18a^2 + 18b^2 + 33b + 2)x^4 \\
& + 63b(72a^2 + 126b^2 + 6b - 1)x^3 + 21b(216a^2b + 216b^3 + 36a^2 + 54b^2 \\
& - 18b + 1)x^2 + 21b(18a^2 + 18b^2 + 6b - 1)(6a^2 + 6b^2 - b)x \\
& + (18a^2 + 18b^2 + 6b - 1)(6a^2 + 6b^2 - b)^2, \\
h_1(x) =\ & 5103abx^4 + 6804abx^3 + 1134ab(6b+1)x^2 + 252ab(18a^2 + 18b^2 + 6b - 1)x \\
& + 21a(18a^2 + 18b^2 + 6b - 1)(6a^2 + 6b^2 - b), \\
h_2(x) =\ & 1134a^2b(3x + 1).
\end{aligned}
$$

Using similar arguments as in the previous proofs, up to a linear change of parameters, we can write the linear developments of the Lyapunov constants as $L_j^{(1)} = u_j$, for $j = 1, \ldots, 9$, and $L_{j+10}^{(1)} = f_j(a, b)/g(a, b)\, u_{10}$, for $j = 0, 1, 2$. Where the f_0, f_1, f_2, and g are polynomials of degrees 254, 266, 278, and 288, respectively. The resultants of the noncommon factors of f_0, f_1 and f_0, f_2 with respect to a factorize in some polynomials. The noncommon factors of both are polynomials in a^2 of degrees 506 and 771, respectively, that we denote them by $f_{01}(a)$ and $f_{02}(a)$. Similarly for the resultants with respect to b having also degrees 506 and 771, and we denote them by $\bar{f}_{01}(b)$ and $\bar{f}_{02}(b)$. We find numerically the real zeros of $f_{01}(a)$ and $\bar{f}_{01}(b)$ and check which pairs (a, b) provide the transversal intersection points (a^*, b^*) of the curves $f_0(a, b) = f_1(a, b) = 0$ such that $f_2(a^*, b^*) \neq 0$. We have found 13 pairs with $a^* > 0$. One of them is $(a^*, b^*) \approx (0.393618957, 0.738793590)$. The transversality condition is guaranteed checking that $\mathrm{Jac}(f_0, f_1)$ is non vanishing at (a^*, b^*). The numerical computations have been made with more than 1000 digits, seeing the stabilization of the nonzero values when we increase the precision.

5. BIFURCATION DIAGRAMS FOR LOCAL CYCLICITY IN FAMILIES OF QUARTIC CENTERS

This section is devoted to prove the second part of the statement of our main result, Theorem 1.1. It follows from the next proposition. We provide the bifurcation diagram of local cyclicity of the cubic center given by Bondar and Sadovski in [4]

adding a straight line of equilibria. This problem can be studied to get 19 limit cycles. Here show a curious fact, the cyclicity depends on the selected straight line. We work with two parameters (a, b), showing the existence of a curve with cyclicity at least 20 and a point with at least 21. Our servers need around one day to get the expressions of all necessary Lyapunov constants. Moreover the size of each text file containing them has size higher than 170 MB. We use a Computer Assisted Proof using the Poincaré–Miranda Theorem (Theorem 2.5), the Gershgorin Theorem (Theorem 2.6) and technical Lemmas 7.1 and 7.2.

Finally, we do a partial study of the bifurcation diagram of the local cyclicity of the holomorphic center of degree $n = 4$, depending on 4 parameters. We prove the existence of a holomorphic center with 20 limit cycles of small amplitude bifurcating from the origin. We have strong numerical evidences that there are values of the parameters such that 21 limit cycles bifurcate from the origin, but the calculus are hard and an analytical proof has been impossible to be obtained. For the moment and for the perturbation of this family, we only present the analytical proof for the bifurcation of 20 limit cycles.

Proposition 5.1. *Consider equation* (3) *for $n = 4$ with the unperturbed system*

$$\begin{cases} \dot{x} = -y(1183x^2 - 68x + 1)(1 - ax - by), \\ \dot{y} = (672x^3 + 1484x^2y - 945xy^2 - 84y^3 - 58x^2 - 44xy + 30y^2 + x) \times \\ \quad (1 - ax - by). \end{cases} \quad (10)$$

Then, there exists a pair $(a^*, b^*) \approx (-0.8159251773700849, 0.55062996428210239)$ such that, for parameters α, λ small enough in (3), *at least* 21 *limit cycles of small amplitude bifurcate from origin.*

Proof. System (10), without the straight line of equilibria, has a center at the origin because it has a rational first integral, see [4]. We restrict our study to $b_{11} = b_{21} = b_{30} = b_{31} = b_{40} = 0$ in (3). After a linear change of coordinates we move from $a_{02}, a_{03}, a_{04}, a_{11}, a_{12}, a_{13}, a_{20}, a_{21}, a_{22}, a_{30}, a_{31}, a_{40}, b_{02}, b_{03}, b_{04}, b_{11}, b_{12}, b_{13}$ to $u_1, \ldots, u_{18}$ and write $L_k^{(1)} = u_k$ for $k = 1, \ldots, 18$. As we have done in the previous proofs, writing $b_{20} = u_{19}$, and removing the common factors, which are rational functions in (a, b), in the linear development of the next Lyapunov constants we can write

$$L_{19}^{(1)} = f_0(a, b)u_{19}, \quad L_{20}^{(1)} = f_1(a, b)u_{19}, \quad L_{21}^{(1)} = f_2(a, b)u_{19}. \quad (11)$$

The numerators of f_0, f_1, f_2 are polynomials with rational coefficient of degrees 180, 182, and 184, respectively. The total number of monomials are, respectively, 16329, 16694, and 17063. We have not added here the expressions because of their size.

Numerically we can find the solution (a^*, b^*) in the statement of the algebraic system $\{f_0 = 0, f_1 = 0\}$. Moreover, the intersection is transversal because the determinant of the Jacobian matrix at the intersection point is $-8.7569521108153076 \cdot 10^{570}$. At this point we have $f_2(a^*, b^*) = -1.7191356490086216 \cdot 10^{290}$.

To get an analytic proof we will use a Computer Assisted Proof with the help of Lemmas 7.1 and 7.2. We use Theorem 2.5 for the existence of the intersection point of f_0 and f_1 and Theorem 2.6 to prove the transversality. The technical lemmas also are used to check that at point f_2 is non vanishing. We fix a square $\mathcal{Q} = [-h, h]^2$ with $h = 10^{-12}$ and we do a rational affine change of coordinates such that a good rational approximation of (a^*, b^*) be inside $\mathcal{Q}$. This affine change of variables is

chosen such that the Taylor series of degree 1 of f_0 and f_1 at (a^*, b^*) will be the new coordinates. Then

$$\tilde{f}_0(S_0^-) \subset [-1.31146 \times 10^{-12}, -8.44847 \times 10^{-13}],$$
$$\tilde{f}_0(S_0^+) \subset [1.15471 \times 10^{-12}, 6.89142 \times 10^{-13}],$$
$$\tilde{f}_1(S_0^-) \subset [-1.15545 \times 10^{-12}, -6.90604 \times 10^{-13}],$$
$$\tilde{f}_1(S_0^+) \subset [1.30878 \times 10^{-12}, 8.44982 \times 10^{-13}],$$
$$\tilde{f}_2(\mathcal{Q}) \subset [0.9035737600, 1.096426240],$$

and we have proved the existence of (a^*, b^*) such that f_2 is non vanishing. In the computations we have worked with rational numbers with numerators and denominators of around 15000 digits. To simplify the computations we have worked with the functions $\tilde{f}_j(a, b) = f_j(a, b)/f_j(0, 0)$.

The last part is to check the transversality. Instead of compute the determinant of the Jacobian matrix of (f_0, f_1) with respect to (a, b), we use the technical lemmas to get that the elements in the Jacobian matrix for the transformed variables are, varying in $\mathcal{Q}$, $A_{11}, A_{22} \subset (0.84568065, 1.15431935)$, $A_{12} \subset (-0.15535611, 0.15535611)$ and $A_{21} \subset (-0.15498852, 0.15498852)$. Then with Theorem 2.6, both eigenvalues are positive and belong to the interval $(0.74, 1.25)$. Therefore, the determinant is different from zero. $\square$

Remark 5.2. *We remark the computational difficulties of the numeric in the above result. We should work with very high precision. In fact, working with 1000 digits the evaluations of f_0 and f_1 at (a^*, b^*) are $-2.19920305995245 \cdot 10^{-397}$ and $3.595005930091451 \cdot 10^{-390}$, respectively. Moreover, the necessary affine change of variables has need more than one computation day. Finally, the curves in Figure 3 has been drawn computing the points one by one working with very high precision and then using polynomial interpolation. In fact, the first time that we got (a^*, b^*) was from the intersection of this polynomial interpolation curves.*
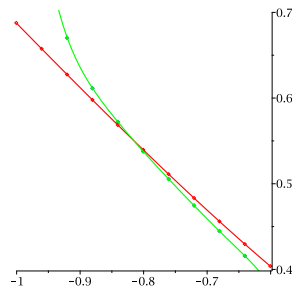

FIGURE 3. Drawing the zero level sets of f_0 and f_1 in (11) in red and green, respectively

Proposition 5.3. *Consider equation (3) for $n = 4$ with the unperturbed system written in complex coordinates, $z = x + iy$, as*

$$\dot{z} = iz + z^2 + (a_1 + ia_2)z^3 + (a_3 + ia_4)z^4. \tag{12}$$

If $a_1 = 1$ and $a_3 = 3$, there exist two algebraic curves $f_0(a_2, a_4)$ and $f_1(a_2, a_4)$ such that, generically on $f_0(a_2, a_4)$, there are small parameters λ for which (3) has at

least 19 *limit cycles of small amplitude bifurcating from the origin. Moreover, there are at least three transversal intersection points,*

$$(a_2^*, a_4^*) \in \{(-6.788836, 2.856062), (-4.387174, 4.549274), (-4.619905, -4.565876)\},$$

of f_0 and f_1 for which there are perturbations of degree four of (3) *such that at least* 20 *limit cycles of small amplitude bifurcate from the origin.*

Proof. In cartesian coordinates, taking $a_1 = 1$ and $a_3 = 3$, system (12) writes as

$$\begin{cases} \dot{x} &= 3x^4 - 4a_4x^3y - 18x^2y^2 + 4a_4xy^3 + 3y^4 + x^3 - 3a_2x^2y - 3xy^2 \\ & \quad + a_2y^3 + x^2 - y^2 - y, \\ \dot{y} &= a_4x^4 + 12x^3y - 6a_4x^2y^2 - 12xy^3 + a_4y^4 + a_2x^3 + 3x^2y - 3a_2xy^2 \\ & \quad - y^3 + 2xy + x. \end{cases}$$

We will restrict our analysis to $b_{11} = b_{20} = b_{21} = b_{30} = b_{31} = b_{40} = 0$. The Lyapunov constants up to order 1, with the algorithm explained in [14] and similarly as the proof of Proposition 5.1, can be computed and written as $L_k^{(1)} = u_k$, for $k = 1, \ldots, 17$. Here we have done a linear change of coordinates in the parameter space changing the linear independent parameters

$$a_{12}, a_{02}, a_{03}, a_{04}, a_{11}, a_{13}, a_{20}, a_{21}, a_{22}, a_{30}, a_{31}, a_{40}, b_{02}, b_{03}, b_{04}, b_{12}, b_{13},$$

by $u_1, \ldots, u_{17}$. Changing the last one b_{22} to u_{18} we have, as in the previous proofs and except a multiplicative rational function in a_2, a_4 as a common factor,

$$L_{18}^{(1)} = f_0(a_2, a_4)u_{18}, \quad L_{19}^{(1)} = f_1(a_2, a_4)u_{18}, \quad L_{20}^{(1)} = f_2(a_2, a_4)u_{18}. \tag{13}$$

The proof follows similarly as the proof of Theorem 4.4 to get the transversal intersection points in the statement. Computing the necessary resultants with respect to a_2 and a_4 to apply Theorem 3.1.

In Figure 4, we have drawn the algebraic curves $f_k(a_2, a_4) = 0$ for $k = 0, 1, 2$ in red, blue, and green, respectively. Notice, that in the pictures it is clear the existence of transversal intersections of $f_0 = 0$ and $f_1 = 0$ where f_2 is non vanishing.
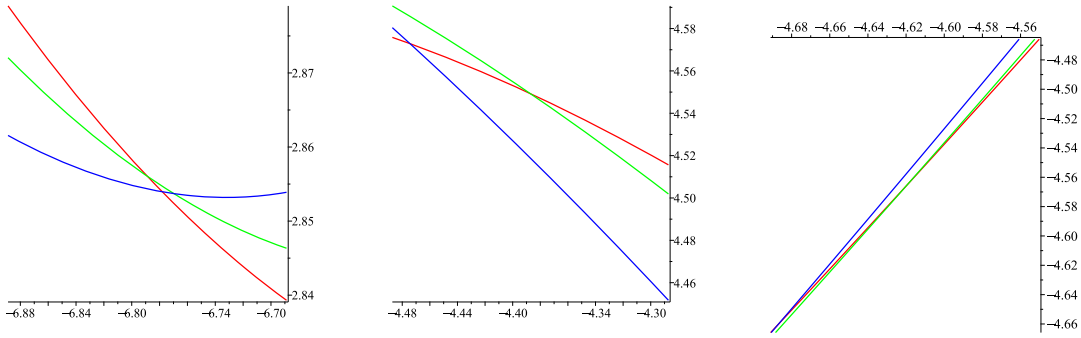

FIGURE 4. Drawing the zero level sets of f_0, f_1, and f_2 in (13) in red, green, and blue, respectively

$\square$

Remark 5.4. *Following the same procedure as for the Lyapunov constants given in* (13) *we compute also the next Lyapunov constant that can be written as*

$$L_{21}^{(1)} = f_3(a_2, a_4)u_{18}.$$

Now taking $a_1 = 1$ in (12) we can compute the corresponding algebraic functions f_0, f_1, f_2, f_3 depending only on (a_2, a_3, a_4). They have around 10^5 monomials and degrees $100, 101, 102, 103$, respectively. Then, we can solve numerically with high precision the first three obtaining

$$a^* = (a_2^*, a_3^*, a_4^*) \approx (0.26423354653702, 2.06583351382191, 2.26983478766641).$$

The evaluation at this point gets

$$f_0(a^*) \approx 4.35 \cdot 10^{-281}, \quad f_1(a^*) \approx 3.2 \cdot 10^{-275},$$
$$f_2(a^*) \approx 3.67 \cdot 10^{-272}, \quad f_3(a^*) \approx 1.091295989718 \cdot 10^{126},$$

and the determinant of the Jacobian matrix of (f_0, f_1, f_2) with respect to a at the intersection point a^* is $-3.82703230760 \cdot 10^{363}$. This gives a numerical evidence that the holomorphic family of degree 4 exhibits also 21 limit cycles of small amplitude bifurcating from the origin. The estabilization of the digits of the non vanishing values of f_3 and the determinant has been obtained working with enough precision.

6. Final comments

The computations in this work are quite high although basically we have worked only with developments of order 1 in the Lyapunov constants but center depending on parameters. This is because the existence of parameters in the unperturbed centers makes the things more complicated. Before the simplifications, the polynomials appearing as coefficients of the perturbation parameters are of very high degree and with rational coefficients with high number of digits. In fact this is why we have only considered vector fields of degrees $n = 3$ and $n = 4$. The important gaps found in the proof of the main theorem of [21] and corrected here are also made in the later work [20]. Using the same techniques developed in the present work the gaps can also be corrected.

It is possible to find high order focus especially for systems of degree even. The high order of these focus does not implies that automatically we can find more limit cycles. What really happens is that the unfolding of the possible limit cycles is not guaranteed and it is usually not possible. For instance, from the work [20], it can be seen that there are homogeneous perturbations of degree four of a linear type center exhibiting a weak foci curve of order 24. However the complete unfolding of these weak foci in the general class of degree four vector fields, that has only 24 parameters, is not guaranteed. In fact, for degree six class, there exist weak foci curves of order larger than the number of parameters of the perturbation.

In [17] the holomorphic centers are considered and it is proved that for low degree $4 \leq n \leq 13$ the cyclicity of each center is at least $n^2 + n - 2$ and for $n = 3$ it is at least 9. The results of this chapter provides higher values of the cyclicity but only for $n = 3$ and $n = 4$. Obtaining as new lower bounds 11 and 21, respectively, even though this last value has a proof which is not analytic. We have also worked with other holomorphic centers, $n = 5, 6, 7$ but only with one parameter. In all cases we have found at least one extra limit cycle than the ones obtained in [17]. But as the obtained lower bounds for $M(n)$ are worse than other obtained in [14] we have not added here.

In all the proofs it is very important to restrict our studies to exactly the number of parameters k and ℓ in Theorem 3.1. Then we will have always only lower bound

for the cyclicity in the considered families. This restriction ensures that the higher order terms do no affect in the expressions of the first order developments.

7. APPENDIX

Next two technical results will help us to find upper and lower bounds for a polynomial of n variables in a n-dimensional cube. The proofs of them can be found in [6].

Lemma 7.1 ([6]). *Consider $h > 0$, $p > 0$, q real numbers such that $p \in [\underline{p}, \overline{p}]$, with $\underline{p}\,\overline{p} > 0$, and $q \in [\underline{q}, \overline{q}]$, with $\underline{q}\,\overline{q} > 0$.*

  (i) *Then, $\sigma^\ell(q, p) \le qp \le \sigma^r(q, p)$, where*

$$\sigma^\ell(q, p) = \begin{cases} q\underline{p}, & \text{if } q > 0, \\ q\overline{p}, & \text{if } q < 0, \end{cases} \qquad \sigma^r(q, p) = \begin{cases} q\overline{p}, & \text{if } q > 0, \\ q\underline{p}, & \text{if } q < 0. \end{cases}$$

  (ii) *If $u_j \in [-h, h]$, for $j = 1, \ldots, n$ and denoting $u^i = u_1^{i_1} \ldots u_n^{i_n}$, for the multiindex $i = (i_1, \ldots, i_n) \ne 0$, we have $\mathcal{X}^\ell(q, u^i) \le qu^i \le \mathcal{X}^r(q, u^i)$, where*

$$\mathcal{X}^\ell(q, u^i) = \begin{cases} 0, & \text{if } q > 0 \text{ and } i_k \text{ even for all } k = 1, \ldots, n, \\ -\overline{q}h^{i_1 + \cdots + i_n}, & \text{if } q > 0 \text{ and } i_k \text{ odd for some } k = 1, \ldots, n, \\ \underline{q}h^{i_1 + \cdots + i_n}, & \text{if } q < 0, \end{cases}$$

and

$$\mathcal{X}^r(q, u^i) = \begin{cases} -\overline{q}h^{i_1 + \cdots + i_n}, & \text{if } q > 0 \text{ and } i_k \text{ even for all } k = 1, \ldots, n, \\ 0, & \text{if } q < 0 \text{ and } i_k \text{ odd for some } k = 1, \ldots, n, \\ \underline{q}h^{i_1 + \cdots + i_n}, & \text{if } q < 0. \end{cases}$$

Furthermore, $\mathcal{X}^\ell(q, 1) = \underline{q}$ and $\mathcal{X}^r(q, 1) = \overline{q}$.

Lemma 7.2 ([6]). *Let $h > 0$ and p_j be a positive non rational numbers such that $p_j \in [\underline{p}_j, \overline{p}_j]$ with $\underline{p}_j, \overline{p}_j$ rational numbers satisfying $\underline{p}_j, \overline{p}_j > 0$, for $j = 1, \ldots, m$. Consider the polynomial*

$$\mathcal{U}(u_1, \ldots, u_n) = \sum_{i_1 + \cdots + i_n = 0}^{M} \left(\sum_{j=1}^{m} U_{j,i} p_j \right) u^i$$

with $u^i = u_1^{i_i} \ldots u_n^{i_n}$, $i = (i_i, \ldots, i_n)$ and $U_{j,i}$ rational numbers. Then

$$U_i^\ell \le \sum_{j=1}^{M} U_{j,i} p_j \le U_i^r,$$

with $U_i^\ell = \sum_{j=1}^{m} U_{j,i} \sigma^\ell(U_{j,i}, p_j)$ and $U_i^r = \sum_{j=1}^{m} U_{j,i} \sigma^r(U_{j,i}, p_j)$. Moreover, if $u_j \in [-h, h]$, for $j = 1, \ldots, n$ and $U_i^\ell > U_i^r$ then

$$\underline{\mathcal{U}} = \sum_{i_1 + \cdots + i_n = 0}^{M} \mathcal{X}^\ell(U_i^\ell, u^i) \le \mathcal{U}(u_1, \ldots, u_n) \le \sum_{i_1 + \cdots + i_n = 0}^{M} \mathcal{X}^r(U_i^r, u^i) = \overline{\mathcal{U}}.$$

Acknowledgments

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