Limit cycles of continuous piecewise differential systems formed by linear and quadratic isochronous centers I

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First we study the planar continuous piecewise differential systems separated by the straight line $x = 0$ formed by a linear isochronous center in $x > 0$ and an isochronous quadratic center in $x < 0$. We prove that these piecewise differential systems cannot have crossing periodic orbits, and consequently they do not have crossing limit cycles.
Second we study the crossing periodic orbits and limit cycles of the planar continuous piecewise differential systems separated by the straight line $x = 0$ having in $x > 0$ the general quadratic isochronous center $\dot{x} = -y + x^2 - y^2$, $\dot{y} = x(1 + 2y)$ after an affine transformation, and in $x < 0$ an arbitrary quadratic isochronous center. For these kind of continuous piecewise differential systems the maximum number of crossing limit cycles is one, and there are examples having one crossing limit cycles
In short for these families of continuous piecewise differential systems we have solved the extension of the 16th Hilbert problem to themselves.

Keywords: Limit cycles, isochronous quadratic centers, continuous piecewise linear differential systems, first integrals

1. Introduction

In the qualitative theory of planar differential systems a *limit cycle* is an isolated periodic solution in the set of all periodic solutions, which remained the most sought solutions when modeling physical systems in the plane. As far as we known the notion of limit cycle appeared in the year 1885 in the work of Poincaré [Poincaré, 1928].

Most of the early examples in the theory of limit cycles in planar differential systems were commonly related to practical problems with mechanical and electronic systems, but periodic behavior appears in all branches of the sciences. To determine the existence of non–existence of limit cycles is one of the more difficult objects in the qualitative theory of planar differential equations. A large amount of references deals with the subject of limit cycles, many of them motivated for the famous second part of the Hilbert's 16th problem, which ask for the maximum number of limit cycles that the planar polynomial differential systems of a given degree can exhibit, see for details [Hilbert, 1900; Ilyashenko, 2002; Li, 2003].

Since 1930's the study of the limit cycles also became important in the continuous and discontinuous piecewise differential systems separated by a straight line, due to their applications to mechanics, electrical circuits, ... see for instance the books [Andronov *et al.*, 1966; di Bernardo *et al.*, 2008; Simpson, 2010] and the references therein.

As usual a *center* p of a planar differential system is a singular point for which there is a neighborhood U such that $U \setminus \{p\}$ is filled with periodic orbits. When all the periodic orbits surrounding a center have the same period this center is called *isochronous*. The centers started to be studied by Poincaré [Poincaré, 1881] and Dulac [Dulac, 1908], but the notion of isochronocity goes back to Huygens [Huygens, 2002] in 1673.

In this paper we consider continuous piecewise differential systems separated by the straight line $x = 0$ having in $x \leq 0$ and in $x \geq 0$ linear or quadratic isochronous centers, and we want to study the non–existence, and the existence of crossing periodic orbits and of crossing limit cycles, and in this last case we want also to know the maximum number of crossing limit cycles for these systems.

Here a *crossing periodic orbit* or a *crossing limit cycle* is a periodic orbit or a limit cycle which intersects exactly in two points the discontinuity line $x = 0$.

The *continuity* of a piecewise differential system separated by the straight line $x = 0$ formed by two centers means that the vector fields defined by these two centers (linear or quadratic) coincide on the line of discontinuity $x = 0$. So a continuous piecewise differential system is a continuous differential system in $\mathbb{R}^2$ and is an analytic differential system in $\mathbb{R}^2 \setminus \{x = 0\}$.

1.1. *Linear centers*

It is well known that the linear differential centers are isochronous and that the general expression of such centers is as follows, see for a proof [Llibre & Teixeira, 2018].

Lemma 1. *A linear differential system having a center can be written in the form*

$$\dot{x} = -\beta x - \frac{4\beta^2 + \omega^2}{4\alpha} y + \delta, \qquad \dot{y} = \alpha x + \beta y + \gamma, \tag{1}$$

with $\alpha > 0$ *and* $\omega > 0$.

The linear differential system (1) has the first integral

$$H_L(x, y) = 4(\alpha x + \beta y)^2 + 8\alpha(\gamma x - \delta y) + \omega^2 y^2.$$

1.2. *Quadratic isochronous centers*

We consider the quadratic polynomial differential systems having an isochronous center. This kind of centers were classified by Loud in the paper [Loud, 1964]. Those systems after an affine change of coordinates become one of the following four systems:

$$\dot{x} = -y + x^2 - y^2, \qquad \dot{y} = x(1 + 2y), \tag{2}$$

$$\dot{x} = -y + x^2, \qquad \dot{y} = x(1 + y), \tag{3}$$

$$\dot{x} = -y - \frac{4x^2}{3}, \qquad \dot{y} = x\left(1 - \frac{16}{3}y\right), \tag{4}$$

$$\dot{x} = -y + \frac{16}{3}x^2 - \frac{4}{3}y^2, \qquad \dot{y} = x\left(1 + \frac{8}{3}y\right). \tag{5}$$

We are interested in the general expressions of the quadratic isochronous centers. So we transform their normal forms (2), (3), (4) and (5) through the following general affine change of variables

$$(x, y) \to (a_1 x + b_1 y + c_1, a_2 x + b_2 y + c_2), \tag{6}$$

with

$$a_1 b_2 - a_2 b_1 \neq 0. \tag{7}$$

Generalized isochronous system (2). Using the change of variables (6) the quadratic system (2) becomes

$$
\begin{aligned}
\dot{x} = (& - b_2c_1^2 + b_1c_1 + 2b_1c_2c_1 + b_2c_2^2 + b_2c_2 \\
& + (2a_2b_1c_1 + 2a_1b_1c_2 - 2a_1b_2c_1 + 2a_2b_2c_2 + a_1b_1 + a_2b_2)\, x \\
& + \left(2b_1^2c_2 + 2b_2^2c_2 + b_1^2 + b_2^2\right) y + \left(2a_2b_1^2 + 2a_2b_2^2\right) xy \\
& + \left(a_1^2(-b_2) + 2a_2a_1b_1 + a_2^2b_2\right) x^2 + \left(b_2^3 + b_1^2b_2\right) y^2) /(a_2b_1 - a_1b_2), \\
\dot{y} = (& \, a_2c_1^2 - a_1c_1 - 2a_1c_2c_1 - a_2c_2^2 - a_2c_2 + \left(-2a_1^2c_2 - 2a_2^2c_2 - a_1^2 - a_2^2\right) x \\
& + (2a_2b_1c_1 - 2a_1b_1c_2 - 2a_1b_2c_1 - 2a_2b_2c_2 - a_1b_1 - a_2b_2)\, y \\
& + \left(-2a_1^2b_2 - 2a_2^2b_2\right) xy + \left(-a_2^3 - a_1^2a_2\right) x^2 \\
& + \left(a_2b_1^2 - 2a_1b_2b_1 - a_2b_2^2\right) y^2) /(a_2b_1 - a_1b_2).
\end{aligned}
\tag{8}
$$

Since $(x^2+y^2)/(2y+1)$ is a first integral of system (2), doing to it the change of variables (6) we get the following first integral of the generalized isochronous quadratic system (8)

$$
H_2(x,y) = \frac{(a_1x + b_1y + c_1)^2 + (a_2x + b_2y + c_2)^2}{2\,(a_2x + b_2y + c_2) + 1}.
$$

Generalized isochronous system (3). System (3) is equivalent to the following generalized isochronous system after the linear change of variables (6)

$$
\begin{aligned}
\dot{x} = (& \, b_2c_1^2 - b_1c_1 - b_1c_2c_1 - b_2c_2 \\
& + (-a_2b_1c_1 - a_1b_1c_2 + 2a_1b_2c_1 - a_1b_1 - a_2b_2)\, x \\
& + \left(b_1^2(-c_2) + b_2b_1c_1 - b_1^2 - b_2^2\right) y + \left(a_1b_1b_2 - a_2b_1^2\right) xy \\
& + \left(a_1^2b_2 - a_1a_2b_1\right) x^2) /(a_1b_2 - a_2b_1), \\
\dot{y} = (& \, a_2c_1^2 - a_1c_1 - a_1c_2c_1 - a_2c_2 + \left(a_1^2(-c_2) + a_2a_1c_1 - a_1^2 - a_2^2\right) x \\
& + (2a_2b_1c_1 - a_1b_1c_2 - a_1b_2c_1 - a_1b_1 - a_2b_2)\, y \\
& + \left(a_1a_2b_1 - a_1^2b_2\right) xy + \left(a_2b_1^2 - a_1b_1b_2\right) y^2) /(a_1b_2 - a_2b_1).
\end{aligned}
\tag{9}
$$

The quadratic system (3) has the first integral $(x^2+y^2)/(y+1)^2$. Therefore a first integral of system (9) is

$$
H_3(x,y) = \frac{(a_1x + b_1y + c_1)^2 + (a_2x + b_2y + c_2)^2}{(a_2x + b_2y + c_2 + 1)^2}.
$$

Generalized isochronous system (4). The quadratic system (4) is equivalent to the following generalized quadratic system after the linear change of variables (6)

$$
\begin{aligned}
\dot{x} = (& \, 4b_2c_1^2 + 3b_1c_1 - 16b_1c_2c_1 + 3b_2c_2 \\
& + (-16a_2b_1c_1 - 16a_1b_1c_2 + 8a_1b_2c_1 + 3a_1b_1 + 3a_2b_2)\, x \\
& + \left(-16b_1^2c_2 - 8b_2b_1c_1 + 3b_1^2 + 3b_2^2\right) y + \left(-16a_2b_1^2 - 8a_1b_2b_1\right) xy \\
& + \left(4a_1^2b_2 - 16a_1a_2b_1\right) x^2 - 12b_1^2b_2y^2) /3\,(a_2b_1 - a_1b_2)\,, \\
\dot{y} = (& - 4a_2c_1^2 - 3a_1c_1 + 16a_1c_2c_1 - 3a_2c_2 \\
& + \left(16a_1^2c_2 + 8a_2a_1c_1 - 3a_1^2 - 3a_2^2\right) x \\
& + (-8a_2b_1c_1 + 16a_1b_1c_2 + 16a_1b_2c_1 - 3a_1b_1 - 3a_2b_2)\, y \\
& + \left(16a_1^2b_2 + 8a_2a_1b_1\right) xy + 12a_1^2a_2x^2 \\
& + \left(16a_1b_1b_2 - 4a_2b_1^2\right) y^2) /3\,(a_2b_1 - a_1b_2)\,.
\end{aligned}
\tag{10}
$$

Since $(32x^2 - 24y + 9)^2/(3 - 16y)$ is a first integral of system (10), then a first integral of system (10) is

$$H_4(x,y) = \frac{\left(32\,(a_1x + b_1y + c_1)^2 - 24\,(a_2x + b_2y + c_2) + 9\right)^2}{3 - 16\,(a_2x + b_2y + c_2)}.$$

Generalized isochronous system (5). Doing the affine change of variables (6) the generalized isochronous system for the quadratic system (5) is

$$\begin{aligned}
\dot{x} = \big(& -16b_2c_1^2 + 3b_1c_1 + 8b_1c_2c_1 + 4b_2c_2^2 + 3b_2c_2 \\
&+ (8a_2b_1c_1 + 8a_1b_1c_2 - 32a_1b_2c_1 + 8a_2b_2c_2 + 3a_1b_1 + 3a_2b_2)\,x \\
&+ \left(8b_1^2c_2 - 24b_2b_1c_1 + 8b_2^2c_2 + 3b_1^2 + 3b_2^2\right) y \\
&+ \left(8a_2b_1^2 - 24a_1b_2b_1 + 8a_2b_2^2\right) xy + \left(-16a_1^2b_2 + 8a_2a_1b_1 + 4a_2^2b_2\right) x^2 \\
&+ \left(4b_2^3 - 8b_1^2b_2\right) y^2\big)/3\,(a_2b_1 - a_1b_2), \\
\dot{y} = \big(& 16a_2c_1^2 - 3a_1c_1 - 8a_1c_2c_1 - 4a_2c_2^2 - 3a_2c_2 \\
&+ \left(-8a_1^2c_2 + 24a_2a_1c_1 - 8a_2^2c_2 - 3a_1^2 - 3a_2^2\right) x \\
&+ (32a_2b_1c_1 - 8a_1b_1c_2 - 8a_1b_2c_1 - 8a_2b_2c_2 - 3a_1b_1 - 3a_2b_2)\,y \\
&+ \left(-8a_1^2b_2 + 24a_2a_1b_1 - 8a_2^2b_2\right) xy + \left(8a_1^2a_2 - 4a_2^3\right) x^2 \\
&+ \left(16a_2b_1^2 - 8a_1b_2b_1 - 4a_2b_2^2\right) y^2\big)/3\,(a_2b_1 - a_1b_2).
\end{aligned} \tag{11}$$

The quadratic system (5) has the first integral $(-256x^2 + 128y^2 + 96y + 9)/(8y + 3)^4$, which gives the following first integral for system (11)

$$H_5(x,y) = \frac{-256\,(a_1x + b_1y + c_1)^2 + 128\,(a_2x + b_2y + c_2)^2 + 96\,(a_2x + b_2y + c_2) + 9}{(8\,(a_2x + b_2y + c_2) + 3)^4}.$$

1.3. *Statement of the main results*

It has been proved in [Llibre & Teixeira, 2018] that continuous piecewise linear differential systems separated by one straight line formed by two linear centers have no crossing limit cycles. Now we shall see that this result extends to continuous piecewise linear differential systems separated by one straight line formed by one linear center and one quadratic isochronous center have no crossing limit cycles.

Theorem 1. *The continuous piecewise differential systems formed by a linear differential center (which is isochronous) and an isochronous quadratic center separated by the straight line $x = 0$ have no crossing periodic orbits, and consequently no crossing limit cycles.*

Theorem 1 is proved in section 2.

In what follows we characterize the existence and non-existence of crossing periodic orbits and crossing limit cycles for continuous piecewise linear differential systems separated by one straight line formed by two quadratic isochronous centers.

Theorem 2. *The following statements hold for the continuous piecewise differential systems formed by two generalized isochronous quadratic centers separated by the straight line $x = 0$.*

(a) If the generalized centers are (8) and (8), then the piecewise differential systems can have crossing periodic orbits but they cannot have crossing limit cycles.

(b) If the generalized centers are (8) and (9), then the piecewise differential systems can have crossing periodic orbits but they cannot have crossing limit cycles.

(c) If the generalized centers are (8) and (10), then the piecewise differential systems has at most one crossing limit cycle and there are systems realizing this limit cycle.

(d) If the generalized centers are (8) and (11), then the piecewise differential systems has at most one crossing limit cycle and there are systems realizing this limit cycle.

Theorem 2 is proved in section 3. So in particular Theorems 1 and 2 solve the 16th Hilbert problem extended to these piecewise differential systems.

This a relatively long paper for this reason we have left the study on the maximum number of crossing limit cycles of the continuous piecewise differential systems of the type (i)-(j) with (i) and (j) $\in \{(9),(10),(11)\}$ for a future new article.

2. Proof of Theorem 1

The first objective is to study periodic orbits and limit cycles of continuous piecewise differential systems formed by a linear center (1) and a generalized quadratic system (2), (3), (4) or (5). To find the crossing periodic orbits and the crossing limit cycles of such piecewise differential systems, we must solve the following algebraic system

$$H_k(0,y_1)-H_k(0,y_2)=0, \qquad H_L(0,y_1)-H_L(0,y_2)=0, \tag{12}$$

where $H_k(x,y)$ and $H_L(x,y)$ are the first integrals of the quadratic isochronous center and of the linear center respectively, and $(0,y_1)$ and $(0,y_2)$ with $y_1 \neq y_2$ are the two intersection points of the crossing periodic orbits with the straight line $x=0$.

In order that we have a continuous piecewise differential system (1)-(8), we impose that both systems coincide on $x=0$, and then both systems must verify the following algebraic system

$$\dot{x}_k-\dot{x}_L|_{x=0}=0, \qquad \dot{y}_k-\dot{y}_L|_{x=0}=0, \tag{13}$$

where $\dot{x}_L$, $\dot{y}_L$, $\dot{x}_k$ and $\dot{y}_k$ are the derivatives with respect to time t of x and y for linear system and quadratic system, respectively. Thus we get the following algebraic system

$$\begin{aligned}
&-a_2b_1\delta+a_1b_2\delta-b_2c_1^2+b_1c_1+2b_1c_2c_1+b_2c_2^2+b_2c_2=0,\\
&4a_2b_1\beta^2-4a_1b_2\beta^2+a_2b_1\omega^2-a_1b_2\omega^2+4\alpha b_1^2+4\alpha b_2^2+8\alpha b_1^2c_2+8\alpha b_2^2c_2=0,\\
&-b_2\left(b_1^2+b_2^2\right)=0,\\
&-a_2b_1\gamma+a_1b_2\gamma+a_2c_1^2-a_1c_1-2a_1c_2c_1-a_2c_2^2-a_2c_2=0,\\
&-a_2\beta b_1+a_1\beta b_2+2a_2b_1c_1-2a_1b_1c_2-2a_1b_2c_1-2a_2b_2c_2-a_1b_1-a_2b_2=0,\\
&a_2b_1^2-2a_1b_2b_1-a_2b_2^2=0.
\end{aligned}$$

Since when we evaluate $a_2b_1-a_1b_2$ in all real solutions (See supplementary section **5.1**) for this algebraic system we obtain zero, we get a contradiction with (7). Therefore there are no continuous piecewise differential systems (1)-(8).

In order that the piecewise differential systems (1)-(9) be continuous they must coincide on $x=0$, then the algebraic system (13) must be satisfied, which gives the following algebraic system

$$\begin{aligned}
&-a_2b_1\delta+a_1b_2\delta-b_2c_1^2+b_1c_1+b_1c_2c_1+b_2c_2=0,\\
&4a_2b_1\beta^2-4a_1b_2\beta^2+a_2b_1\omega^2-a_1b_2\omega^2+4\alpha b_1^2+4\alpha b_2^2-4\alpha b_1b_2c_1+4\alpha b_1^2c_2=0,\\
&-a_2b_1\gamma+a_1b_2\gamma+a_2c_1^2-a_1c_1-a_1c_2c_1-a_2c_2,=0,\\
&-a_2\beta b_1+a_1\beta b_2+2a_2b_1c_1-a_1b_1c_2-a_1b_2c_1-a_1b_1-a_2b_2=0,\\
&b_1=0.
\end{aligned}$$

All solutions (See supplementary section **5.2**) of this algebraic system give $a_2b_1-a_1b_2=0$, except the following solution $\{\alpha=\frac{4a_1^2c_1^2+8a_2a_1c_1+a_1^2\omega^2+4a_2^2}{4a_1b_2}, \beta=\frac{a_1c_1+a_2}{a_1}, \gamma=\frac{-a_2c_1^2+a_1c_1+a_1c_2c_1+a_2c_2}{a_1b_2}, \delta=\frac{c_1^2-c_2}{a_1}, b_1=0\}$ for which $a_2b_1-a_1b_2=-a_1b_2$.

Solving the algebraic system (12) for the values of the solution s_4 we get $y_1=y_2$. So the continuous piecewise differential system(1)-(9) with the values of s_4 has no crossing periodic orbits.

The piecewise differential systems (1)-(10) is continuous if and only if system (13) is verified, i.e.

$$
\begin{aligned}
&-3a_2b_1\delta+3a_1b_2\delta+4b_2c_1^2+3b_1c_1-16b_1c_2c_1+3b_2c_2=0,\\
&12a_2b_1\beta^2-12a_1b_2\beta^2+3a_2b_1\omega^2-3a_1b_2\omega^2+12\alpha b_1^2+12\alpha b_2^2-32\alpha b_1b_2c_1-64\alpha b_1^2c_22c_2=0,\\
&-3a_2\beta b_1+3a_1\beta b_2-8a_2b_1c_1+16a_1b_1c_2+16a_1b_2c_1-3a_1b_1-3a_2b_2=0,\\
&-3a_2b_1\gamma+3a_1b_2\gamma-4a_2c_1^2-3a_1c_1+16a_1c_2c_1-3a_2c_2=0,\\
&-4b_1\left(a_2b_1-4a_1b_2\right)=0,\\
&-4b_1^2b_2=0.
\end{aligned}
$$

But all real solutions (See supplementary section **5.3**) of this algebraic system give $a_2b_1-a_1b_2=0$, except the solution $\{\alpha=\frac{1024a_1^2c_1^2-384a_2a_1c_1+9a_1^2\omega^2+36a_2^2}{36a_1b_2}, \beta=\frac{3a_2-16a_1c_1}{3a_1}, \gamma=\frac{4a_2c_1^2+3a_1c_1-16a_1c_2c_1+3a_2c_2}{3a_1b_2}, \delta=\frac{-4c_1^2-3c_2}{3a_1}, b_1=0\}$ which gives $a_2b_1-a_1b_2=-a_1b_2$. By solving system (12) for the values of this solution we get $y_1=y_2$. Therefore the continuous piecewise differential systems (1)-(10) cannot have crossing periodic orbits.

To get a continuous piecewise differential system we impose that both systems (1)-(11) coincide on $x=0$ by using (13), and we obtain the following algebraic system

$$
\begin{aligned}
&-3a_2b_1\delta+3a_1b_2\delta-16b_2c_1^2+3b_1c_1+8b_1c_2c_1+4b_2c_2^2+3b_2c_2=0,\\
&12a_2b_1\beta^2-12a_1b_2\beta^2+3a_2b_1\omega^2-3a_1b_2\omega^2+12\alpha b_1^2+12\alpha b_2^2-96\alpha b_1b_2c_1+32\alpha b_1^2c_2+32\alpha b_2^2c_2=0,\\
&-3a_2b_1\gamma+3a_1b_2\gamma+16a_2c_1^2-3a_1c_1-8a_1c_2c_1-4a_2c_2^2-3a_2c_2=0,\\
&-3a_2\beta b_1+3a_1\beta b_2+32a_2b_1c_1-8a_1b_1c_2-8a_1b_2c_1-8a_2b_2c_2-3a_1b_1-3a_2b_2=0,\\
&4\left(4a_2b_1^2-2a_1b_2b_1-a_2b_2^2\right)=0,\\
&-4b_2\left(b_2^2-2b_1^2\right)=0.
\end{aligned}
$$

Since all solutions (See supplementary section **5.4**) of this algebraic system give $a_2b_1-a_1b_2=0$, then there are no continuous piecewise differential systems (1)-(11).

In summary Theorem 1 is proved.

3. Proof of Theorem 2

In what follows we consider in one side of the straight line $x=0$ the generalized isochronous systems of (2) and in the other side a generalized isochronous system (2), (3), (4) and (5). In the first generalized systems (8) we rename the parameters a_2, b_2 and c_2 by α_1, β_1 and γ_1, respectively; and in the second generalized systems (j) for (j=8, 9, 10 or 11) we rename the parameters a_1, b_1, c_1, a_2, b_2 and c_2 by a_2, b_2, c_2, α_2, β_2 and γ_2, respectively. Doing this condition (7) becomes

$$\alpha_1b_1-a_1\beta_1\neq 0 \quad \text{and} \quad \alpha_2b_2-a_2\beta_2\neq 0. \tag{14}$$

In order to study the crossing periodic orbits and limit cycles of a piecewise differential system (8)-(j) formed by two generalized isochronous systems with $(j)\in\{(8,9,10,11\}$, we must solve the following algebraic system

$$H_2(0,y_1)-H_2(0,y_2)=0, \qquad H_j(0,y_1)-H_j(0,y_2)=0, \tag{15}$$

where H_2 and H_j are the first integral of the generalized systems (8) and (j).

Proof. [Proof of statement (a) of Theorem 2] In order that the piecewise differential systems (8)-(8) be continuous, they must coincide on $x=0$, which means the following algebraic system must be satisfied.

$$
\begin{aligned}
&-2a_2b_1\beta_2\gamma_1c_1+2a_1b_2\beta_1c_2\gamma_2-a_2b_1\beta_2c_1+a_1b_2\beta_1c_2-a_2\beta_1\beta_2\gamma_1^2+a_1\beta_1\beta_2\gamma_2^2-a_2\beta_1\beta_2\gamma_1\\
&+a_1\beta_1\beta_2\gamma_2+a_2\beta_1\beta_2c_1^2-a_1\beta_1\beta_2c_2^2+\alpha_1b_2\beta_1\gamma_1^2-\alpha_1b_1\beta_2\gamma_2^2+\alpha_1b_2\beta_1\gamma_1-\alpha_1b_1\beta_2\gamma_2\\
&+\alpha_1\left(-b_2\right)\beta_1c_1^2+\alpha_1b_1\beta_2c_2^2+2\alpha_1b_1b_2\gamma_1c_1-2\alpha_1b_1b_2c_2\gamma_2+\alpha_1b_1b_2c_1-\alpha_1b_1b_2c_2=0,\\
&-2a_2\beta_2b_1^2\gamma_1+2a_1b_2^2\beta_1\gamma_2-a_2\beta_2b_1^2+a_1b_2^2\beta_1-2a_2\beta_1^2\beta_2\gamma_1+2a_1\beta_1\beta_2^2\gamma_2+a_1\beta_1\beta_2^2-a_2\beta_1^2\beta_2\\
&-2\alpha_1\beta_2^2b_1\gamma_2+2\alpha_1b_2\beta_1^2\gamma_1-\alpha_1\beta_2^2b_1+\alpha_1b_2\beta_1^2+2\alpha_1b_2b_1^2\gamma_1-2\alpha_1b_2^2b_1\gamma_2+\alpha_1b_2b_1^2\\
&-\alpha_1b_2^2b_1=0,\\
&-a_2b_1^2\beta_2\beta_1+a_1b_2^2\beta_2\beta_1-a_2\beta_2\beta_1^3+a_1\beta_2^3\beta_1+\alpha_1b_2\beta_1^3+\alpha_1b_1^2b_2\beta_1-\alpha_1b_1\beta_2^3-\alpha_1b_1b_2^2\beta_2=0,\\
&a_2\alpha_1\beta_2\gamma_1^2-a_1\alpha_1\beta_1\gamma_2^2+a_2\alpha_1\beta_2\gamma_1-a_1\alpha_1\beta_1\gamma_2-2a_1\alpha_1b_2\gamma_1c_1+2a_2\alpha_1b_1c_2\gamma_2-a_1\alpha_1b_2c_1\\
&+a_2\alpha_1b_1c_2-a_2\alpha_1\beta_2c_1^2+a_1\alpha_1\beta_1c_2^2+2a_1a_2\beta_2\gamma_1c_1-2a_1a_2\beta_1c_2\gamma_2+a_1a_2\beta_2c_1-a_1a_2\beta_1c_2\\
&-\alpha_1^2b_2\gamma_1^2+\alpha_1^2b_1\gamma_2^2-\alpha_1^2b_2\gamma_1+\alpha_1^2b_1\gamma_2+\alpha_1^2b_2c_1^2-\alpha_1^2b_1c_2^2=0,\\
&2a_2\alpha_1\beta_1\beta_2\gamma_1-2a_1\alpha_1\beta_1\beta_2\gamma_2-a_1\alpha_1\beta_1\beta_2+a_2\alpha_1\beta_1\beta_2-2a_1\alpha_1b_1b_2\gamma_1+2a_2\alpha_1b_1b_2\gamma_2\\
&-a_1\alpha_1b_1b_2+a_2\alpha_1b_1b_2+2a_1a_2b_1\beta_2\gamma_1-2a_1a_2b_2\beta_1\gamma_2-a_1a_2b_2\beta_1+a_1a_2b_1\beta_2\\
&-2a_1\alpha_1b_2\beta_1c_1+2a_1\alpha_1b_2\beta_1c_2-2a_2\alpha_1b_1\beta_2c_1+2a_2\alpha_1b_1\beta_2c_2+2a_1a_2\beta_1\beta_2c_1\\
&-2a_1a_2\beta_1\beta_2c_2-2\alpha_1^2b_2\beta_1\gamma_1+2\alpha_1^2b_1\beta_2\gamma_2-\alpha_1^2b_2\beta_1+\alpha_1^2b_1\beta_2+2\alpha_1^2b_1b_2c_1-2\alpha_1^2b_1b_2c_2=0,\\
&-a_1\alpha_1\beta_1\beta_2^2+a_2\alpha_1\beta_1^2\beta_2-a_2\alpha_1\beta_2b_1^2-2a_1\alpha_1b_2\beta_1b_1+2a_2\alpha_1b_2\beta_2b_1+a_1\alpha_1b_2^2\beta_1\\
&+2a_1a_2\beta_1\beta_2b_1-2a_1a_2b_2\beta_1\beta_2+\alpha_1^2\beta_2^2b_1-\alpha_1^2b_2\beta_1^2+\alpha_1^2b_2b_1^2-\alpha_1^2b_2^2b_1=0,
\end{aligned}
\tag{16}
$$

together with conditions (14).

From the sixth equation of (16) we get

$$
a_1=\frac{\alpha_1\left(a_2\beta_2\left(-b_1^2+2b_2b_1+\beta_1^2\right)+\alpha_2b_1\beta_2^2-\alpha_2b_2\left(\beta_1^2+b_1\left(b_2-b_1\right)\right)\right)}{\beta_1\left(2a_2\left(b_2-b_1\right)\beta_2+\alpha_2\beta_2^2+\alpha_2\left(2b_1-b_2\right)b_2\right)},
\tag{17}
$$

if the denominator of a_1 is non-zero. If it is zero then $\beta_1=0$ or $2a_2\left(b_2-b_1\right)\beta_2+\alpha_2\beta_2^2+\alpha_2\left(2b_1-b_2\right)b_2=0$. This last equation is equivalent to the following solution

$$
d_4=\{b_1=\frac{-2a_2\beta_2b_2-\alpha_2\beta_2^2+\alpha_2b_2^2}{2\left(\alpha_2b_2-a_2\beta_2\right)}\}.
$$

by neglecting all solutions (See supplementary section **5.5**) which do not satisfying conditions (14). So it remains to study this solution.

If the denominator of (17) is non-zero and $\alpha_1=0$, then from (17) we obtain that $a_1=0$, and consequently $\alpha_1b_1-a_1\beta_1=0$, which is a contradiction with first condition in (14).

In summary for solving system (16) we consider the following cases.

Case 1: Suppose that $b_1\neq(-2a_2\beta_2b_2-\alpha_2\beta_2^2+\alpha_2b_2^2)/2\left(\alpha_2b_2-a_2\beta_2\right)$, $\beta_1\neq0$ and $\alpha_1\neq0$. Since a_1 is defined in (17). From the fifth equation of (16), we get

$$
\begin{aligned}
c_1=&\,(b_1(2\gamma_1+1)(\alpha_2b_2-a_2\beta_2)+b_2(2\alpha_2\beta_1c_2-a_2(2\beta_1\gamma_2-4\beta_2\gamma_1+\beta_1-2\beta_2))+\beta_2(\alpha_2(-2\beta_1\gamma_2\\
&+2\beta_2\gamma_1-\beta_1+\beta_2)-2a_2\beta_1c_2)+b_2^2(-(2\alpha_2\gamma_1+\alpha_2)))/2\beta_1(\alpha_2b_2-a_2\beta_2).
\end{aligned}
$$

Now we solve simultaneously the second and the third equations of (16), and we obtain the only solution $\{\beta_1=\frac{\alpha_1\beta_2(b_2^2+\beta_2^2)}{2a_2(b_2-b_1)\beta_2+\alpha_2\beta_2^2+\alpha_2(2b_1-b_2)b_2},$
$\gamma_2=\frac{2b_2(2\gamma_1+1)(a_2\beta_2+\alpha_2b_1)+\beta_2(\beta_2(2\alpha_2\gamma_1-\alpha_1+\alpha_2)-2a_2(2b_1\gamma_1+b_1))+b_2^2(-(2\alpha_2\gamma_1+\alpha_1+\alpha_2))}{2\alpha_1(b_2^2+\beta_2^2)}\}$,
among all real solutions (See supplementary section **5.6**) for which $b_1\neq(-2a_2\beta_2b_2-\alpha_2\beta_2^2+\alpha_2b_2^2)/2\left(\alpha_2b_2-a_2\beta_2\right)$, $\beta_1\neq0$, $\alpha_1\neq0$ and conditions (14) are satisfied.

Solving now simultaneously the first and the fourth equations of (16). The only real solutions (See supplementary section **5.7**) for which $b_1\neq(-2a_2\beta_2b_2-\alpha_2\beta_2^2+\alpha_2b_2^2)/2(\alpha_2b_2-a_2\beta_2)$, $\beta_1\neq0$, $\alpha_1\neq0$ and conditions (14) are satisfied are solutions
$s_1=\{b_1=b_2,\ \alpha_1=-\alpha_2\}$ and $s_2=\{b_1=b_2,\ \alpha_1=\alpha_2\}$.
Then we must discuss these two solutions s_1 and s_2 in the following subcases to detect whether the continuous piecewise differential systems (8)-(8) have or do not have limit cycles.

Subcase 1.1: Consider s_1. Solving system (15) we get $y_1 = y_2$, or the following solution

$$y_1 = \frac{2b_1\left(2\gamma_1 c_2 + c_2\right) + b_1^2\left(2\gamma_1 y_2 + y_2\right) + \beta_2\left(2c_2^2 - 2\gamma_1\left(\gamma_1+1\right) + y_2\left(2\beta_2\gamma_1 + \beta_2\right)\right)}{\left(b_1^2+\beta_2^2\right)\left(-2\gamma_1 + 2\beta_2 y_2 - 1\right)}.$$

Since in order to have a crossing periodic orbit we must have $y_1 \neq y_2$. Therefore these continuous piecewise differential systems have a continuum of crossing periodic orbits and then no limit cycles.
Subcase 1.2: Now we consider s_2, so system (15) gives $y_1 = y_2$, or

$$y_1 = -\frac{2b_1\left(2\gamma_1 c_2 + c_2\right) + b_1^2\left(2\gamma_1 y_2 + y_2\right) + \beta_2\left(-2c_2^2 + 2\gamma_1\left(\gamma_1+1\right) + y_2\left(2\beta_2\gamma_1 + \beta_2\right)\right)}{\left(b_1^2+\beta_2^2\right)\left(2\gamma_1 + 2\beta_2 y_2 + 1\right)}.$$

As in the previous subcase no limit cycles.
Case 2: We suppose that $b_1 = (-2a_2\beta_2 b_2 - \alpha_2\beta_2^2 + \alpha_2 b_2^2)/2(\alpha_2 b_2 - a_2\beta_2)$ and $\beta_1 \neq 0$. With this value for b_1 the sixth equation of (16) gives the only following four real solutions $s_1 = \{\alpha_1 = 0\}$, $s_2 = \{a_2 = 0,\ \alpha_2 = 0\}$, $s_3 = \{b_2 = 0,\ \beta_2 = 0\}$ and $s_4 = \{\alpha_2 = 0,\ \beta_2 = 0\}$. The only solution that satisfying conditions (14) is s_1, which is equivalent to $\alpha_1 = 0$. Now from the third equation of (16), regarding all the real solutions (See supplementary section **5.8**), the only solution verifying (14) is
$a_1 = \frac{4a_2\alpha_2 b_2\beta_2^3 + 2\beta_2^2\left(2a_2^2\left(b_2^2+\beta_1^2\right) - \alpha_2^2 b_2^2\right) - 4a_2\alpha_2 b_2\beta_2\left(b_2^2+2\beta_1^2\right) + \alpha_2^2\beta_2^4 + \alpha_2^2 b_2^2\left(b_2^2+4\beta_1^2\right)}{4\beta_2\left(b_2^2+\beta_2^2\right)\left(a_2\beta_2 - \alpha_2 b_2\right)}$.

By solving the fifth equation of (16) we obtain from all real solutions (See supplementary section **5.9**) the following solutions s_1, s_2 and s_8
$s_1 = \{\gamma_2 = \frac{2a_2b_2(2\beta_2\gamma_1 - \beta_1 + \beta_2) + \alpha_2\beta_2(2\beta_2\gamma_1 - 2\beta_1 + \beta_2) + 4a_2\beta_1\beta_2(c_1 - c_2) + b_2^2(-(2\alpha_2\gamma_1 + \alpha_2)) + 4\alpha_2\beta_1 b_2(c_2 - c_1)}{4\beta_1(a_2b_2 + \alpha_2\beta_2)}\}$,
$s_2 = \{a_2 = -\alpha_2\beta_2/b_2,\ c_2 = \frac{2b_2\gamma_1 + b_2 + 4\beta_1 c_1}{4\beta_1}\}$, and
$s_3 = \{b_2 = 0,\ c_2 = c_1,\ \alpha_2 = 0\}$,
which satisfying (14). Then we divide case 2 into the following three subcases.
Subcase 2.1: We consider the solution s_1. Solving the second equation of (16) we get the following real solutions
$u_1 = \{\gamma_1 = -\frac{2\beta_1(c_1-c_2)(\alpha_2 b_2 - a_2\beta_2)}{\alpha_2\left(b_2^2+\beta_2^2\right)} - \frac{1}{2}\}$, $u_2 = \{c_2 = c_1,\ \alpha_2 = 0\}$ and $u_3 = \{\alpha_2 = 0,\ \beta_1 = 0\}$.
But u_1 and u_2 are the only solutions satisfying (14) which must be studied separately into two subcases.
Subcase 2.1.1: Consider u_1. Solving simultaneously the first and the fourth equations of (16), we get the only real solution $\beta_1 = 0$, in contradiction with the hypothesis of Case 2.
Subcase 2.1.2: Consider u_2. Then all equations of (16) are verified except the first one, which becomes

$$a_2\beta_2\left(\beta_2^2 - \beta_1^2\right)\left(-4b_2\beta_1 c_1\left(2\gamma_1+1\right) + \left(2b_2\gamma_1 + b_2\right)^2 + \beta_1^2\left(4c_1^2+1\right)\right) = 0.$$

Solving this equation we obtain the following set of real solutions $v_1 = \{a_2 = 0\}$, $v_2 = \{\beta_2 = 0\}$, $v_3 = \{\beta_2 = -\beta_1\}$ and $v_4 = \{\beta_2 = \beta_1\}$.
The solutions for which conditions (14) are verified are v_3 and v_4. Now we must discuss these two subcases as follows.
Subcase 2.1.2.1: Consider v_3. Then we have a continuous piecewise differential system. We prove that there are a continuum of crossing periodic orbits and no limit cycles, because solving the algebraic system (15) we get $y_1 = y_2$, or the solution

$$y_1 = -\frac{2b_2\left(2\gamma_1 c_1 + c_1\right) + b_2^2\left(2\gamma_1 y_2 + y_2\right) + \beta_1\left(-2c_1^2 + 2\gamma_1\left(\gamma_1+1\right) + y_2\left(2\beta_1\gamma_1 + \beta_1\right)\right)}{\left(b_2^2+\beta_1^2\right)\left(2\gamma_1 + 2\beta_1 y_2 + 1\right)}.$$

Subcase 2.1.2.2: Consider v_4. Then we get a continuous piecewise differential system. Solving the algebraic system (15) we obtain $y_1 = y_2$, or

$$y_1 = -\frac{2b_2\left(2\gamma_1 c_1 + c_1\right) + b_2^2\left(2\gamma_1 y_2 + y_2\right) + \beta_1\left(-2c_1^2 + 2\gamma_1\left(\gamma_1+1\right) + y_2\left(2\beta_1\gamma_1 + \beta_1\right)\right)}{\left(b_2^2+\beta_1^2\right)\left(2\gamma_1 + 2\beta_1 y_2 + 1\right)}.$$

This gives a continuum of crossing periodic orbits.
Subcase 2.2: Assume that solution s_2 holds. Solving the second equation of (16) we get the following sets of solutions $\{\alpha_2 = 0\}$, $\{\gamma_2 = (2\beta_2\gamma_1 - \beta_1 + \beta_2)/(2\beta_1)\}$ and $\{\beta_1 = 0, \gamma_1 = -1/2\}$. The only solution

for which conditions (14) are satisfied is $\gamma_2 = (2\beta_2\gamma_1 - \beta_1 + \beta_2)/(2\beta_1)$. The remaining unsolved equations of (16) are the first and the fourth equations, we solve them simultaneously and we get only one real solution $\alpha_2 = 0$, but this solution gives a contradiction with (14). Then there are no continuous piecewise differential systems (8)-(8) in this case.

Subcase 2.3: Consider the solution s_3. Then the remaining unsolved equations of system (16) are the first, the second and the fourth equations. Solving these three equations simultaneously we get one of the following sets of real solutions $t_1 = \{a_2 = 0\}$, $t_2 = \{\beta_1 = 0\}$, $t_3 = \{\beta_2 = -\beta_1,\ \gamma_2 = -\gamma_1 - 1\}$ and $t_4 = \{\beta_2 = \beta_1,\ \gamma_2 = \gamma_1\}$.

The sets of solutions t_3 and t_4 are the only for which conditions (14) are verified. We now have a continuous piecewise differential system (8)-(8). Consider these two cases t_3 and t_4 separately.

Subcase 2.3.1: Consider solution t_3. Then for showing whether the piecewise differential system (8)-(8) has a limit cycle or not, we solve the algebraic system (15). Here we get $y_1 = y_2$, or

$$y_1 = \frac{2c_1^2 - 2\gamma_1(\gamma_1+1) - y_2(2\beta_1\gamma_1 + \beta_1)}{\beta_1(2\gamma_1 + 2\beta_1 y_2 + 1)}.$$

We have then a continuum of crossing periodic orbits.

Subcase 2.3.2: Consider the solution t_4. I In a similar way solving the algebraic system (15). We get $y_1 = y_2$, or

$$y_1 = \frac{2c_1^2 - 2\gamma_1(\gamma_1+1) - y_2(2\beta_1\gamma_1 + \beta_1)}{\beta_1(2\gamma_1 + 2\beta_1 y_2 + 1)}.$$

Then we get a continuum of crossing periodic orbits.

Case 3: Assume that $b_1 \neq (-2a_2\beta_2 b_2 - \alpha_2\beta_2^2 + \alpha_2 b_2^2)/2(\alpha_2 b_2 - a_2\beta_2)$, $\beta_1 = 0$. The third equation of (16) becomes $-\alpha_1 b_1 \beta_2 (b_2^2 + \beta_2^2) = 0$, and since $\beta_1 = 0$ we must have $\beta_2 = 0$, because $b_1 = 0$ or $\alpha_1 = 0$ gives $b_1\alpha_1 - a_1\beta_1 = 0$. Solving the five remaining equations of (16), the only solutions from all real solutions (See supplementary section **5.10**) for which (14) and $b_1 \neq (-2a_2\beta_2 b_2 - \alpha_2\beta_2^2 + \alpha_2 b_2^2)/(2(\alpha_2 b_2 - a_2\beta_2))$ are verified are s_1, s_2, s_3, s_4 and s_5 given as follows

$s_1 = \{b_1 = b_2,\ c_1 = c_2,\ \gamma_1 = -1/2,\ \gamma_2 = -1/2\}$,
$s_2 = \{a_2 = a_1,\ b_1 = b_2,\ c_1 = c_2,\ \alpha_1 = -\alpha_2,\ \gamma_1 = -\gamma_2 - 1\}$,
$s_3 = \{b_1 = b_2,\ c_1 = c_2,\ \alpha_1 = -\alpha_2,\ \gamma_1 = -1/2,\ \gamma_2 = -1/2\}$,
$s_4 = \{a_2 = a_1,\ b_1 = b_2,\ c_1 = c_2,\ \alpha_1 = \alpha_2,\ \gamma_1 = \gamma_2\}$ and
$s_5 = \{b_1 = b_2,\ c_1 = c_2,\ \alpha_1 = \alpha_2,\ \gamma_1 = -1/2,\ \gamma_2 = -1/2\}$.

For these five cases we have a continuous piecewise differential systems (8)-(8). Now we discuss these five subcases.

Case 3.1: Consider the solution s_1. Then the first integrals of systems (8)-(8) are

$$H_1(x,y) = [8a_1b_1xy + 8a_1c_1x + 4a_1^2x^2 + 8b_1c_1y + 4b_1^2y^2 + 4c_1^2 + 4\alpha_1^2x^2 - 4\alpha_1x + 1]/[8\alpha_1x],$$

and

$$H_1(x,y) = [8a_2b_1xy + 8a_2c_1x + 4a_2^2x^2 + 8b_1c_1y + 4b_1^2y^2 + 4c_1^2 + 4\alpha_2^2x^2 - 4\alpha_2x + 1]/[8\alpha_2x],$$

respectively. So they are not defined on the y-axis, and consequently we cannot have continuous piecewise differential systems.

Case 3.2: Consider the solution s_2. Then solving the algebraic system (14) we obtain $y_1 = y_2$, or $y_1 = -y_2 - 2c_1/b_1$. We conclude that there is a continuum of crossing periodic orbits.

Case 3.3: Consider the solution s_3. Then the first integrals of systems (8) are

$$H_1(x,y) = -[8a_1b_2xy + 8a_1c_2x + 4a_1^2x^2 + 8b_2c_2y + 4b_2^2y^2 + 4c_2^2 + 4\alpha_2^2x^2 + 4\alpha_2x + 1]/[8\alpha_2x],$$

and

$$H_1(x,y) = [8a_2b_2xy + 8a_2c_2x + 4a_2^2x^2 + 8b_2c_2y + 4b_2^2y^2 + 4c_2^2 + 4\alpha_2^2x^2 - 4\alpha_2x + 1]/[8\alpha_2x].$$

The same conclusion than in case 3.1.

Case 3.4: Consider the solution s_4. Now solving the algebraic system (14) we obtain $y_1 = y_2$, or $y_1 = -y_2 - 2c_2/b_2$. There are a continuum of crossing periodic orbits.

Case 3.5: Consider the solution s_5. Then the first integrals of systems (8) are

$$H_1(x,y) = [8a_1b_2xy + 8a_1c_2x + 4a_1^2x^2 + 8b_2c_2y + 4b_2^2y^2 + 4c_2^2 + 4\alpha_2^2x^2 - 4\alpha_2x + 1]/[8\alpha_2x],$$

and

$$H_1(x,y) = [8a_2b_2xy + 8a_2c_2x + 4a_2^2x^2 + 8b_2c_2y + 4b_2^2y^2 + 4c_2^2 + 4\alpha_2^2x^2 - 4\alpha_2x + 1]/[8\alpha_2x].$$

Again as case 3.1.
Case 4: Assume that $b_1 = (-2a_2\beta_2b_2 - \alpha_2\beta_2^2 + \alpha_2b_2^2)/(2(\alpha_2b_2 - a_2\beta_2))$ and $\beta_1 = 0$. Then the sixth equation of (16) becomes

$$\frac{\alpha_1\left(\alpha_2{\beta_2}^2 + 2a_2\beta_2b_2 - \alpha_2{b_2}^2\right)^2}{4(a_2\beta_2 - \alpha_2b_2)} = 0.$$

Since we have $\beta_1 = 0$ we cannot take $\alpha_1 = 0$ otherwise we get a contradiction with the first condition in (14). So we must take $\alpha_2{\beta_2}^2 + 2a_2\beta_2b_2 - \alpha_2{b_2}^2 = 0$. By solving this equation, we obtain $a_2 = ({b_2}^2\alpha_2 - \alpha_2{\beta_2}^2)/2b_2\beta_2$, and by replacing a_2 in $b_1 = (-2a_2\beta_2b_2 - \alpha_2\beta_2^2 + \alpha_2b_2^2)/(2(\alpha_2b_2 - a_2\beta_2))$ we get $b_1 = 0$, which is a contradiction with the first condition in (14). Then we cannot have continuous piecewise differential system int this case. ■

Proof. [Proof of statement (*b*) of Theorem 2] In order that the piecewise differential system (8)-(9) be continuous they must coincide on $x = 0$, so the following algebraic system must be satisfied

$$\begin{aligned}
&-2a_2\beta_2b_1c_1\gamma_1 + a_1\beta_1b_1c_2\gamma_2 + a_1\beta_1b_1c_2 - a_2\beta_2b_1c_1 - a_2\beta_1\beta_2\gamma_1^2 - a_2\beta_1\beta_2\gamma_1\\
&+a_1\beta_1\beta_2\gamma_2 + a_2\beta_1\beta_2c_1^2 - a_1\beta_1\beta_2c_2^2 + \alpha_2\beta_1b_1\gamma_1^2 + \alpha_2\beta_1b_1\gamma_1 - \alpha_1\beta_2b_1\gamma_2\\
&-\alpha_2\beta_1b_1c_1^2 + \alpha_1\beta_2b_1c_2^2 + 2\alpha_2b_1^2c_1\gamma_1 - \alpha_1b_1^2c_2\gamma_2 + \alpha_2b_1^2c_1 - \alpha_1b_1^2c_2 = 0,\\
&-2a_2\beta_2b_1^2\gamma_1 + a_1\beta_1b_1^2\gamma_2 + a_1\beta_1b_1^2 - a_2\beta_2b_1^2 - a_1\beta_1\beta_2b_1c_2 - 2a_2\beta_1^2\beta_2\gamma_1 + a_1\beta_1\beta_2^2\\
&-a_2\beta_1^2\beta_2 + 2\alpha_2\beta_1^2b_1\gamma_1 + \alpha_2\beta_1^2b_1 - \alpha_1\beta_2^2b_1 + 2\alpha_2b_1^3\gamma_1 - \alpha_1b_1^3\gamma_2 + \alpha_2b_1^3\\
&-\alpha_1b_1^3 + \alpha_1\beta_2b_1^2c_2 = 0,\\
&-\beta_1\left(b_1^2 + \beta_1^2\right) = 0,\\
&2a_2\alpha_1\beta_1\beta_2\gamma_1 + a_2\alpha_1\beta_1\beta_2 - a_1\alpha_2\beta_1\beta_2 + a_2\alpha_1b_1b_2\gamma_2 - 2a_1\alpha_2b_1b_2\gamma_1 + a_2\alpha_1b_1b_2\\
&-a_1\alpha_2b_1b_2 + 2a_1a_2b_1\beta_2\gamma_1 - a_1a_2b_2\beta_1\gamma_2 - a_1a_2b_2\beta_1 + a_1a_2b_1\beta_2 - 2a_2\alpha_1b_1\beta_2c_1\\
&+a_2\alpha_1b_1\beta_2c_2 - 2a_1\alpha_2b_2\beta_1c_1 + 2a_1\alpha_2b_2\beta_1c_2 + 2a_1a_2\beta_1\beta_2c_1 - a_1a_2\beta_1\beta_2c_2\\
&-2\alpha_2\alpha_1b_2\beta_1\gamma_1 - \alpha_2\alpha_1b_2\beta_1 + \alpha_2\alpha_1b_1\beta_2 + 2\alpha_2\alpha_1b_1b_2c_1 - 2\alpha_2\alpha_1b_1b_2c_2 = 0,\\
&a_2\alpha_1\beta2\gamma_1^2 + a_2\alpha_1\beta_2\gamma_1 - a_1\alpha_2\beta_1\gamma_2 - 2a_1\alpha_2b_2\gamma_1c_1 + a_2\alpha_1b_1c_2\gamma_2 - a_1\alpha_2b_2c_1\\
&+a_2\alpha_1b_1c_2 - a_2\alpha_1\beta_2c_1^2 + a_1\alpha_2\beta_1c_2^2 + 2a_1a_2\beta_2\gamma_1c_1 - a_1a_2\beta_1c_2\gamma_2 + a_1a_2\beta_2c_1\\
&-a_1a_2\beta_1c_2 - \alpha_1\alpha_2b_2\gamma_1^2 - \alpha_1\alpha_2b_2\gamma_1 + \alpha_1\alpha_2b_1\gamma_2 + \alpha_1\alpha_2b_2c_1^2 - \alpha_1\alpha_2b_1c_2^2 = 0,\\
&-2a_1\beta_1b_1 + a_1b_2\beta_1 - \alpha_1\beta_1^2 + \alpha_1b_1^2 - \alpha_1b_2b_1 = 0,
\end{aligned} \tag{18}$$

with

$$\alpha_1b_1 - a_1\beta_1 \neq 0 \quad \text{and} \quad \alpha_2b_2 - a_2\beta_2 \neq 0.$$

We see easily that a necessary condition for a continuous piecewise system is $\beta_1 = 0$ and this implies that $\alpha_1 \neq 0$ and $b_1 \neq 0$, otherwise we have a contradiction with the condition $\alpha_1b_1 - a_1\beta_1 \neq 0$. Substituting $\beta_1 = 0$ into the last equation of (18) we obtain $b_1(b_1 - b_2)\alpha_1 = 0$, and consequently $b_2 = b_1$, because b_1 and α_1 cannot be zero. Now in all cases of this proof we take $\beta_1 = 0$, $b_2 = b_1$, $\alpha_1 \neq 0$ and $b_1 \neq 0$.
Case 1: $b_1c_2 + \beta_2 \neq 0$, $\beta_2 \neq 0$ and $\gamma_1(\gamma_1 + 1) \neq 0$. Then from the first equation of (18) we get

$$\gamma_2 = \frac{-2a_2\beta_2c_1\gamma_1 - a_2\beta_2c_1 + 2\alpha_2b_1c_1\gamma_1 - \alpha_1b_1c_2 + \alpha_2b_1c_1 + \alpha_1\beta_2c_2^2}{\alpha_1\left(b_1c_2 + \beta_2\right)}.$$

From the second equation of (18) we obtain

$$\alpha_1 = \frac{b_1\left(2\gamma_1 + 1\right)\left(b_1\left(c_2 - c_1\right) + \beta_2\right)\left(\alpha_2b_1 - a_2\beta_2\right)}{\beta_2\left(b_1^2 + \beta_2^2\right)}.$$

Using the fifth equation of (18) we have

$$a_1 = -b_1(-a_2b_1\beta_2 - 2a_2b_1\beta_2c_2^2 + 4a_2b_1\beta_2c_1c_2 - 2a_2b_1\beta_2c1^2 - \alpha_2\beta_2^2 + a_2\beta_2^2c_1 - a_2\beta_2^2c_2$$
$$-\alpha_2b_1\beta_2c_1 + \alpha_2b_1\beta_2c_2 + 2\alpha_2b_1^2c_1^2 + 2\alpha_2b_1^2c_2^2 - 4\alpha_2b_1^2c_1c_2)/(\beta_2(b_1^2 + \beta_2^2)).$$

Finally from the fourth equation of (18) we get

$$\beta_2 = \frac{b_1(c_1 - c_2)(c_1^2 - 2c_2c_1 + c_2^2 + \gamma_1^2 + \gamma_1 + 1)}{\gamma_1(\gamma_1 + 1)}.$$

With these values of the parameters the piecewise differential systems (8)-(9) are continuous.

Now we must solve the system

$$H_2(0, y_1) - H_2(0, y_2) = 0, \qquad H_3(0, y_1) - H_3(0, y_2) = 0, \tag{19}$$

for studying the crossing periodic orbits and the crossing limit cycles of these continuous piecewise differential systems.

The first equation of (19) in this case is

$$\frac{b_1(y_1 - y_2)(2c_1 + b_1y_1 + b_1y_2)}{1 + 2\gamma_1} = 0. \tag{20}$$

Since in order to have a crossing periodic orbit we must have $y_1 \neq y_2$, we obtain that $y_2 = -(2c_1 + b_1y_1)/b_1$. Substituting y_2 into the second equation of (19) we get that $y_1 = y_2 = -c_1/b_1$. Therefore this piecewise differential systems has no crossing periodic orbits.

Case 2: Assume that $b_1c_2 + \beta_2 \neq 0$, $\beta_2 \neq 0$ and $\gamma_1(1 + \gamma_1) = 0$. Then from the case 1 we have the same values for γ_2, α_1 and a_1 substituting in them $\gamma_1 = 0$ or $\gamma_1 = -1$. Then from the fourth equation of (18) we should have $c_2 = c_1$ otherwise we have non–continuous piecewise differential system. With these values the piecewise differential systems (8)-(9) are continuous. Again from (20) we obtain $y_2 = -(2c_1 + b_1y_1)/b_1$. Substituting y_2 into the second equation of (19) we get that $y_1 = y_2 = -c_1/b_1$. So no crossing periodic orbits.

Case 3: $b_1c_2 + \beta_2 \neq 0$ and $\beta_2 = 0$. Then $c_2 \neq 0$. Now as in the case 1, from first equation of system (18) we obtain $\gamma_2 = (c_1\alpha_2(1 + 2\gamma_1) - c_2\alpha_1)/(c_2\alpha_1)$. Then the second equation of (18) becomes

$$-\frac{b_1^3(c_2 - c_1)\alpha_2(1 + 2\gamma_1)}{c_2} = 0. \tag{21}$$

Which α_2 and b_1 cannot be zero from the condition of (14). So in this case we have three subcases satisfy $(c_2 - c_1)(1 + 2\gamma_1) = 0$.

Subcase 3.1: $1 + 2\gamma_1 \neq 0$. Then $c_2 = c_1$ to verify equation (21). The fifth equation of (18) becomes $-(a_1 - a_2)b_1^2\alpha 2(1 + 2\gamma_1) = 0$, so we must take $a_2 = a_1$, otherwise the piecewise differential system cannot be continuous. Finally from the fourth equation of (18) we get $\alpha_2 = (\alpha_1(1 + \gamma_1 + \gamma_1^2))/(1 + 2\gamma_1)$. With these values of the parameters the piecewise differential systems (8)-(9) are continuous. Now the first and second equation of (19) become

$$\frac{b_1(y_1 - y_2)(2c_1 + b_1y_1 + b_1y_2)}{1 + 2\gamma_1} = 0, \qquad \frac{b_1(y_1 - y_2)(2c_1 + b_1y_1 + b_1y_2)}{(1 + \gamma_1 + \gamma_1^2)^2} = 0,$$

respectively. Since the numerator of both equations coincide we have a continuum of crossing periodic orbits in this subcase.

Subcase 3.2: $c_2 \neq c_1$. Then we must take $1 + 2\gamma_1 = 0$ to satisfy the equation (21). Using the fifth equation of (18) we obtain $2b_1^2(c_1 - c_2)\alpha_1\alpha_2 = 0$, we see that cannot verify this equation under the conditions of this subcase. Then we cannot have continuous piecewise differential system in subcase.

Subcase 3.3: $1 + 2\gamma_1 = c_2 - c_1 = 0$. Then the fifth equation of (18) is satisfied, and the fourth equation of (18) becomes $-3b_1\alpha_1\alpha_2/4 = 0$, which cannot be satisfied otherwise we have a contradiction with (14). So in this subcase the piecewise differential system cannot be continuous.

Case 4: $b_1c_2+\beta_2 = 0$ and $c_2 \neq 0$. The first equation of (18) gives $\alpha_1 = (a_2c_1c_2+c_1\alpha_2)(1+2\gamma_1)/(c_2(1+c_2^2))$. The second equation of (18) becomes

$$\frac{b_1^3(a_2c_2+\alpha_2)(1+2\gamma_1)(c_2+c_2^3-c_1(1+2c_2^2+\gamma_2))}{c_2(1+c_2^2)} = 0.$$

From the condition $b_1c_2+\beta_2 = 0$ of this case we get $c_2 = -\beta_2/b_1$ and from the starting of the proof of this statement we have $b_1 = b_2$, by substituting in $a_2c_2+\alpha_2 = 0$ gives $a_2\beta_2 - b_2\alpha_2 = 0$; which is a contradiction with second condition in (14). If $1+2\gamma_1 = 0$, from the above formula of α_1 we get $\alpha_1 = 0$ and since $\beta_1 = 0$ we obtain $a_1\beta_1 - b_1\alpha_1 = 0$; which is a contradiction with first condition in (14). So $a_2c_2+\alpha_2$ and $1+2\gamma_1$ cannot be zero, we must take

$$c_2 + c_2^3 - c_1(1+2c_2^2+\gamma_2) = 0. \tag{22}$$

Subcase 4.1: $c_1 = 0$. Then from the second equation of (18) we have

$$b_1^3(a_2c_2+\alpha_2)(1+2\gamma_1) = 0,$$

Since $a_2c_2+\alpha_2$ and $1+2\gamma_1$ cannot be zero as we have seen above this last equation cannot be satisfy, and consequently the piecewise differential system cannot be continuous.
Subcase 4.2: $c_1 \neq 0$. Then from (22) we get $\gamma_2 = (-c_1+c_2-2c_1c_2^2+c_2^3)/c_1$. The fifth equation of (18) is

$$\frac{b_1^2(a_2c_2+\alpha_2)(a_2c_2(1+2c_1^2-3c_1c_2+c_2^2)-a_1(c_2+c_2^3)+c_1(2c_1-3c_2)\alpha_2)(1+2\gamma_1)}{c_2(1+c_2^2)} = 0.$$

As we have seen in the previous case $a_2c_2+\alpha_2$ and $1+2\gamma_1$ cannot be zero. So we should have

$$a_2c_2(1+2c_1^2-3c_1c_2+c_2^2)-a_1(c_2+c_2^3)+c_1(2c_1-3c_2)\alpha_2 = 0.$$

Solving this equation we get

$$a_1 = \frac{a_2c_2+2a_2c_1^2c_2-3a_2c_1c_2^2+a_2c_2^3+2c_1^2\alpha_2-3c_1c_2\alpha_2}{c_2(1+c_2^2)}.$$

And solving the fourth equation of (18) we obtain

$$\gamma_1 = \frac{-c_1 \pm \sqrt{-c_1(3c_1+4c_1^3-4c_2-12c_1^2c_2+12c_1c_2^2-4c_2^3)}}{2c_1}.$$

We assume that the squareroot which appears in the expression of γ_1 is real, otherwise the piecewise differential system is not continuous.

With these values of the parameters we have satisfied all equations of system (18). Then the piecewise differential systems (8)-(9) are continuous. Now the first equation of (19) becomes

$$-\frac{b_1c_1(y_1-y_2)(2c_1+b_1y_1+b_1y_2)}{\sqrt{-c_1(3c_1+4c_1^3-4c_2-12c_1^2c_2+12c_1c_2^2-4c_2^3)}} = 0.$$

In order to get crossing periodic orbits we need that $2c_1+b_1y_1+b_1y_2 = 0$, or equivalently $y_2 = (-2c_1 - b_1y_1)/b_1$. Substituting y_2 into the second equation of (19) we get

$$\frac{4c_1^3(1+c_2^2)(c_1+b_1y_1)^3}{c_2^2(-1+2c_1c_2-c_2^2+b_1c_1y_1)^2(1+2c_1^2-2c_1c_2+c_2^2+b_1c_1y_1)^2} = 0.$$

By solving this equation we get $y_1 = -c_1/b_1$ and yield to $y_2 = -c_1/b_1$. Then in this case the piecewise differential system cannot have crossing periodic orbits.
Case 5: $b_1c_2+\beta_2 = 0$ and $c_2 = 0$. Then $\beta_2 = 0$, and the first equation of system (18) becomes $b_1^2c_1\alpha_2(1+2\gamma_1) = 0$. Since b_1 and α_2 cannot be zero, so we have the cases verify $c_1(1+2\gamma_1) = 0$, otherwise we have non-continuous piecewise differential system piecewise. Then we have the two following subcases.
Subcase 5.1: $c_1 = 0$ and $1+2\gamma_1 \neq 0$. Then the second equation of system (18) becomes $b_1^3(\alpha_2+2\alpha_2\gamma_1 - \alpha_1(1+\gamma_2)) = 0$. Since b_1 cannot be zero we have that $\alpha_2+2\alpha_2\gamma_1-\alpha_1(1+\gamma_2) = 0$. By solving this equation we

get $\gamma_2 = (-\alpha_1 + \alpha_2 + 2\alpha_2\gamma_1)/\alpha_1$. Thus system (18) is satisfied in this subcase and the piecewise differential system is continuous.

Now the first and second equation of (19) become

$$\frac{b_1^2(y_1 - y_2)(y_1 + y_2)}{1 + 2\gamma_1} = 0 \quad \text{and} \quad \frac{b_1^2(y_1 - y_2)(y_1 + y_2)^2}{1 + \gamma_1 + \gamma_1^2} = 0$$

respectively. This system has the two solutions $y_1 = y_2$ and $y_1 = -y_2$. The last solution means that the piecewise differential system has a continuum of periodic orbits. Then no crossing limit cycles.
Subcase 5.2: $c_1 \neq 0$ and $1 + 2\gamma_1 = 0$. The second equation of system (18) becomes $-b_1^3\alpha_1(1 + \gamma_2) = 0$. Since b_1 and α_1 cannot be zero, we have that $\gamma_2 = -1$. The fifth equation becomes $\sqrt{3}b_1^2\alpha_1\alpha_2 = 0$, but this is a contradiction because all the parameters which appear in it are nonzero. Therefore the piecewise differential system cannot be continuous.
Subcase 5.3: $c_1 = 1 + 2\gamma_1 = 0$. The same proof and result as in subcase 5.2. ■

Proof. [Proof of statement (*c*) of Theorem 2] For studying the maximum number of limit cycles of the piecewise differential systems (8)-(10), we must solve system (15). Solving the first equation of (15) with respect to y_1 we obtain $y_1 = y_2$ and

$$y_1 = -\frac{2b_1(2\gamma_1 c_1 + c_1) + b_1^2(2\gamma_1 y_2 + y_2) + \beta_1\left(-2c_1^2 + 2\gamma_1(\gamma_1 + 1) + y_2(2\beta_1\gamma_1 + \beta_1)\right)}{\left(b_1^2 + \beta_1^2\right)(2\gamma_1 + 2\beta_1 y_2 + 1)}. \tag{23}$$

In order that the piecewise differential systems (8)-(10) be continuous they must coincide on $x = 0$, which means the following algebraic system must be satisfied.

$$\begin{aligned}
&-6a_2b_1\beta_2\gamma_1c_1 - 16a_1b_2\beta_1c_2\gamma_2 - 3a_2b_1\beta_2c_1 + 3a_1b_2\beta_1c_2 - 3a_2\beta_1\beta_2\gamma_1^2 - 3a_2\beta_1\beta_2\gamma_1\\
&+3a_1\beta_1\beta_2\gamma_2 + 3a_2\beta_1\beta_2c_1^2 + 4a_1\beta_1\beta_2c_2^2 + 3\alpha_2b_2\beta_1\gamma_1^2 + 3\alpha_2b_2\beta_1\gamma_1 - 3\alpha_1b_1\beta_2\gamma_2\\
&-3\alpha_2b_2\beta_1c_1^2 - 4\alpha_1b_1\beta_2c_2^2 + 6\alpha_2b_1b_2\gamma_1c_1 + 16\alpha_1b_1b_2c_2\gamma_2 + 3\alpha_2b_1b_2c_1 - 3\alpha_1b_1b_2c_2 = 0,\\[1ex]
&-6a_2\beta_2b_1^2\gamma_1 - 16a_1b_2^2\beta_1\gamma_2 - 3a_2\beta_2b_1^2 + 3a_1b_2^2\beta_1 - 8a_1b_2\beta_1\beta_2c_2 - 6a_2\beta_1^2\beta_2\gamma_1 + 3a_1\beta_1\beta_2^2\\
&-3a_2\beta_1^2\beta_2 + 6\alpha_2b_2\beta_1^2\gamma_1 - 3\alpha_1\beta_2^2b_1 + 3\alpha_2b_2\beta_1^2 + 6\alpha_2b_2b_1^2\gamma_1 + 16\alpha_1b_2^2b_1\gamma_2 + 3\alpha_2b_2b_1^2\\
&-3\alpha_1b_2^2b_1 + 8\alpha_1b_2\beta_2b_1c_2 = 0,\\[1ex]
&a_2b_1^2\beta_2\beta_1 + 4a_1b_2^2\beta_2\beta_1 + a_2\beta_2\beta_1^3 - \alpha_2b_2\beta_1^3 - \alpha_2b_1^2b_2\beta_1 - 4\alpha_1b_1b_2^2\beta_2 = 0,\\[1ex]
&3a_2\alpha_1\beta_2\gamma_1^2 + 3a_2\alpha_1\beta_2\gamma_1 - 3a_1\alpha_2\beta_1\gamma_2 - 6a_1\alpha_2b_2\gamma_1c_1 - 16a_2\alpha_1b_1c_2\gamma_2 - 3a_1\alpha_2b_2c_1\\
&+3a_2\alpha_1b_1c_2 - 3a_2\alpha_1\beta_2c_1^2 - 4a_1\alpha_2\beta_1c_2^2 + 6a_1a_2\beta_2\gamma_1c_1 + 16a_1a_2\beta_1c_2\gamma_2 + 3a_1a_2\beta_2c_1\\
&-3a_1a_2\beta_1c_2 - 3\alpha_1\alpha_2b_2\gamma_1^2 - 3\alpha_1\alpha_2b_2\gamma_1 + 3\alpha_1\alpha_2b_1\gamma_2 + 3\alpha_1\alpha_2b_2c_1^2 + 4\alpha_1\alpha_2b_1c_2^2 = 0,\\[1ex]
&6a_2\alpha_1\beta_1\beta_2\gamma_1 + 3a_2\alpha_1\beta_1\beta_2 - 3a_1\alpha_2\beta_1\beta_2 - 16a_2\alpha_1b_1b_2\gamma_2 - 6a_1\alpha_2b_1b_2\gamma_1 + 3a_2\alpha_1b_1b_2\\
&-3a_1\alpha_2b_1b_2 + 6a_1a_2b_1\beta_2\gamma_1 + 16a_1a_2b_2\beta_1\gamma_2 - 3a_1a_2b_2\beta_1 + 3a_1a_2b_1\beta_2 - 6a_2\alpha_1b_1\beta_2c_1\\
&-16a_2\alpha_1b_1\beta_2c_2 - 6a_1\alpha_2b_2\beta_1c_1 - 8a_1\alpha_2b_2\beta_1c_2 + 6a_1a_2\beta_1\beta_2c_1 + 16a_1a_2\beta_1\beta_2c_2\\
&-6\alpha_2\alpha_1b_2\beta_1\gamma_1 - 3\alpha_2\alpha_1b_2\beta_1 + 3\alpha_2\alpha_1b_1\beta_2 + 6\alpha_2\alpha_1b_1b_2c_1 + 8\alpha_2\alpha_1b_1b_2c_2 = 0,\\[1ex]
&3a_2\alpha_1\beta_1^2\beta_2 - 3a_2\alpha_1\beta_2b_1^2 - 6a_1\alpha_2b_2\beta_1b_1 - 16a_2\alpha_1b_2\beta_2b_1 - 4a_1\alpha_2b_2^2\beta_1 + 6a_1a_2\beta_1\beta_2b_1\\
&+16a_1a_2b_2\beta_1\beta_2 - 3\alpha_1\alpha_2b_2\beta_1^2 + 3\alpha_1\alpha_2b_2b_1^2 + 4\alpha_1\alpha_2b_2^2b_1 = 0,
\end{aligned} \tag{24}$$

together with conditions (14).

Solving the sixth equation of (24) we get

$$a_1 = \frac{a_2\alpha_1\beta_2\left(-3b_1^2 - 16b_2b_1 + 3\beta_1^2\right) + \alpha_1\alpha_2b_2\left(3b_1^2 + 4b_2b_1 - 3\beta_1^2\right)}{2\alpha_2b_2(3b_1 + 2b_2)\beta_1 - 2a_2(3b_1 + 8b_2)\beta_1\beta_2}, \tag{25}$$

except when denominator $2\alpha_2b_2(3b_1 + 2b_2)\beta_1 - 2a_2(3b_1 + 8b_2)\beta_1\beta_2 = 0$. This denominator vanishes if and only if one of the following solutions holds:

$$\begin{array}{l} d_1 = \{\beta_1 = 0\}, \\ d_2 = \{\beta_2 = \frac{\alpha_2 b_2(3b_1+2b_2)}{a_2(3b_1+8b_2)}\}, \\ d_3 = \{a_2 = 0,\ b_1 = -2b_2/3\}, \\ d_4 = \{a_2 = 0,\ \alpha_2 = 0\}, \\ d_5 = \{b_1 = -8b_2/3,\ \alpha_2 = 0\}, \\ d_6 = \{a_2 = 0,\ b_2 = 0\}, \\ d_7 = \{b_1 = 0,\ b_2 = 0\}. \end{array} \tag{26}$$

To continue solving the algebraic system (24) we must discuss all cases when $2\alpha_2 b_2 (3b_1 + 2b_2) \beta_1 - 2a_2 (3b_1 + 8b_2) \beta_1\beta_2 = 0$ and $2\alpha_2 b_2 (3b_1 + 2b_2) \beta_1 - 2a_2 (3b_1 + 8b_2) \beta_1\beta_2 \neq 0$.

Case 1: Assume that

$$2\alpha_2 b_2 (3b_1 + 2b_2) \beta_1 - 2a_2 (3b_1 + 8b_2) \beta_1\beta_2 \neq 0. \tag{27}$$

Using a_1 given by (25) we solve the third equation of the algebraic system (24) and we obtain the following solution
$\alpha_1 = \frac{a_2(3b_1+8b_2)\beta_1\beta_2-\alpha_2 b_2(3b_1+2b_2)\beta_1}{6b_2^2\beta_2}$,
the only allowed solution (See supplementary section **5.11**) for which conditions (27) and (14) are verified, then we take this value of α_1 and solve the second equation of (24) and we get the solution (See supplementary section **5.12**)
$\gamma_2 = \frac{3b_2^2(8\beta_2\gamma_1+\beta_1+4\beta_2)-8\beta_1\beta_2 b_2 c_2+3\beta_1\beta_2^2}{16b_2^2\beta_1}$,
which does not contradict conditions (27) or (14), so for this value of γ_2 solving the fifth equation of system (24) we obtain
$\beta_2 = \frac{b_2(8b_2\gamma_1+b_1(6\gamma_1+3)+4b_2-6\beta_1 c_1-8\beta_1 c_2)}{3\beta_1}$,
All real solutions (See supplementary section **5.13**) are in contradiction with (27) or (14) except the solution v_1. Replacing β_2 in the remaining equations of the algebraic system (24) and by solving the first equation of this system we obtain the following allowed solutions (See supplementary section **5.14**) w_1 and w_2
$w_1 = \{c_2 = \frac{(15b_1+56b_2)\beta_1(2\gamma_1+1)-\frac{\sqrt{3}\mathcal{S}}{b_1^2+\beta_1^2}-30\beta_1^2 c_1}{112\beta_1^2}\}$ and
$w_2 = \{c_2 = \frac{\sqrt{3}\mathcal{S}+\beta_1(b_1^2+\beta_1^2)(15b_1(2\gamma_1+1)+56b_2(2\gamma_1+1)-30\beta_1 c_1)}{112\beta_1^2(b_1^2+\beta_1^2)}\}$,
whith
$\mathcal{S} = \beta_1^2(b_1^2+\beta_1^2)(36\beta_1 b_1^3 c_1(2\gamma_1+1) + b_1^2(448(2b_2\gamma_1+b_2)^2 - 3\beta_1^2(12c_1^2+12\gamma_1(\gamma_1+1)+31)) + 4\beta_1 b_1 c_1(2\gamma_1+1)(9\beta_1^2 - 448b_2^2) + 448b_2^2\beta_1^2(4c_1^2+1) - 9b_1^4(2\gamma_1+1)^2 - 12\beta_1^4(3c_1^2+7))$.
Without solving the fourth equation of the algebraic system (24) to get a piecewise continuous differential systems (8)-(10). We solve the algebraic system (15). In these two cases replacing the value of y_1 given by (23) into the second equation of (15) its left hand side is equivalent to a long polynomial of second degree with respect to y_2.

Subcase 1.1: If we consider w_1. Then the polynomial of second degree with respect to y_2, after equating it to zero (See supplementary section **5.15**), we compute the two roots of this polynomial we get two different values y_{21} and y_{22}. By replacing y_{21} we get y_{11}, and replacing y_{22} we get y_{12}. Since we obtain that $y_{11} = y_{22}$ and $y_{12} = y_{21}$, it follows that in this subcase the system has at most one limit cycle.

Subcase 1.2: If we consider w_2. Then the polynomial of second degree with respect to y_2, after equating it to zero (See supplementary section **5.16**), we compute the two roots of this polynomial we get two different values y_{21} and y_{22}. By replacing y_{21} we get y_{11}, and replacing y_{22} we get y_{12}. Due to the facts that $y_{11} = y_{22}$ and $y_{12} = y_{21}$, the system in this subcase has at most one limit cycle.

Case 2: Assume that $2\alpha_2 b_2 (3b_1 + 2b_2) \beta_1 - 2a_2 (3b_1 + 8b_2) \beta_1\beta_2 = 0$. Here we consider the subcases d_1, d_2, d_3, d_5 and d_7 defined by (26) except d_4 and d_6 because are in contradiction with (14).

Subcase 2.1: Consider the set d_1 of (26). In this case we have $\beta_1 = 0$. The third equation (24) becomes $4b_2^2\beta_2 = 0$, this gives $b_2 = 0$ or $\beta_2 = 0$. b_2 and β_2 cannot vanish at the same time otherwise we get a

contradiction with (14). If $b_2 = 0$ and $\beta_2 \neq 0$, then the sixth equation of (24) becomes $-3a_2b_1\beta_2 = 0$ which gives $a_2 = 0$ or $b_1 = 0$, since $\beta_1 = 0$ and $b_2 = 0$ we get a contradiction with (14). Then we have no continuous piecewise differential systems (8)-(10) in this case. Finally if we have $\beta_2 = 0$ and $b_2 \neq 0$, then the sixth equation of (24) becomes $b_2(3b_1 + 4b_2)\alpha_2 = 0$, which gives $b_2 = 0$, $\alpha_2 = 0$ or $3b_1 + 4b_2 = 0$, and since $\beta_2 = 0$, all these parameters give a contradiction with (14) except $b_2 = -3b_1/4$. Solving the four remaining equation of (24) we get the only solutions (See supplementary section **5.17**) s_1 and s_2 that verify (14) as follows
$s_1 = \{c_1 = -4c_2/3,\ \gamma_1 = -1/2,\ \gamma_2 = 3/16\}$ and
$s_2 = \{a_2 = -3a_1/4,\ c_1 = -4c_2/3,\ \gamma_1 = -\frac{2\alpha_2}{3\alpha_1} - 1/2,\ \gamma_2 = 3/16 - \frac{\alpha_2^2}{3\alpha_1^2}\}$.

Subcase 2.1.1: Consider the set s_1. In this case we have $c_1 = -4c_2/3$, $\gamma_1 = -1/2$ and $\gamma_2 = 3/16$. Now the algebraic system (24) is verified. But the first integrals of systems (8) and (10) are

$$H_1(x,y) = [72a_1b_1xy - 96a_1c_2x + 36a_1^2x^2 - 96b_1c_2y + 36b_1^2y^2 + 64c_2^2 + 36\alpha_1^2x^2 - 36\alpha_1x + 9]/[72\alpha_1x],$$

and

$$H_3(x,y) = -[\left(-96a_2b_1xy + 128a_2c_2x + 64a_2^2x^2 - 96b_1c_2y + 36b_1^2y^2 + 64c_2^2 - 48\alpha_2x + 9\right)^2]/[64\alpha_2x],$$

respectively, note that they are not defined on the y-axis, which means that there are no continuous piecewise differential systems formed by (8) and (10).

Subcase 2.1.2: Consider the set s_2. Which gives the parameters $a_2 = -3a_1/4$, $c_1 = -4c_2/3$, $\gamma_1 = -2\alpha_2/(3\alpha_1) - 1/2$ and $\gamma_2 = 3/16 - \alpha_2^2/3\alpha_1^2$. With these parameters we get continuous piecewise differential systems (8)-(10). Solving the algebraic system (15) we obtain $y_1 = y_2$ or $y_1 = 8c_2/(3b_1) - y_2$, this provided a continuum of periodic orbits.

Subcase 2.2: Consider the set d_2 of (26). In this case we have $\beta_2 = \alpha_2b_2(3b_1 + 2b_2)/(a_2(3b_1 + 8b_2))$. Since $b_2 = 0$ or $\alpha_2 = 0$ is in contradiction with $a_2\beta_2 - b_2\alpha_2 \neq 0$, then we have either

$$\beta_2 = 0 \qquad \text{and} \qquad b_2 = -3b_1/2, \tag{28}$$

or

$$\alpha_2 = \frac{a_2\,(3b_1 + 8b_2)\,\beta_2}{b_2\,(3b_1 + 2b_2)}. \tag{29}$$

Using (28) the third and the sixth equations of (24) become $-\beta_1\left(b_1^2 + \beta_1^2\right) = 0$ and $-\alpha_1\left(b_1^2 + \beta_1^2\right) = 0$, respectively. Therefore either $\beta_1 = \alpha_1 = 0$ or $b_1 = \beta_1 = 0$ and this contradicts (14). Then in this case we cannot have a continuous piecewise differential system.

Now using the value of α_2 from (29). The sixth equation of (24) becomes $-\alpha_1\left(b_1^2 + \beta_1^2\right) = 0$. Since $b_1 = \beta_1 = 0$ contradicts (14) we must take $\alpha_1 = 0$. Solving now the fifth equation of (24), all solutions (See supplementary section **5.18**) give a contradiction with (14) except s_1 and s_2 given as follows

$$\begin{aligned} s_1 &= \{\gamma_1 = \frac{2b_2\beta_1\left(-2\beta_2b_2(9c_1+8c_2)+b_2^2(16\gamma_2-3)-12\beta_2^2\right)+3b_1\left(b_2^2(\beta_1(16\gamma_2-3)-6\beta_2)+8\beta_1\beta_2b_2c_2-3\beta_1\beta_2^2\right)}{36b_1b_2^2\beta_2}\} \text{ and} \\ s_2 &= \{b_1 = 0, c_1 = \frac{1}{18}(\frac{b_2(16\gamma_2-3)}{\beta_2} - \frac{12\beta_2}{b_2} - 16c_2)\}. \end{aligned} \tag{30}$$

Then we must study these two cases separately.

Subcase 2.2.1: Consider the set s_1 of (30). By solving the third equation of (24), all solutions (See supplementary section **5.19**) give a contradiction with (14) except u_1 and u_2

$$\begin{aligned} u_1 &= \{\beta_1 = -\frac{\sqrt{2a_1b_2(3b_1+2b_2)-3a_2b_1^2}}{\sqrt{3}\sqrt{a_2}}\} \text{ and} \\ u_2 &= \{\beta_1 = \frac{\sqrt{2a_1b_2(3b_1+2b_2)-3a_2b_1^2}}{\sqrt{3}\sqrt{a_2}}\}. \end{aligned} \tag{31}$$

Then we must study these two cases separately.

Subcase 2.2.1.1: Consider the set u_1 of (31). Solving the second equation of (24) we get from the sets of all real solutions (See supplementary section **5.20**) the only allowed solution is

$$c_1 = \frac{3b_1(8\beta_2b_2c_2 + b_2^2(16\gamma_2 - 3) - 3\beta_2^2) + 4b_2(-16\beta_2b_2c_2 + b_2^2(16\gamma_2 - 3) - 12\beta_2^2)}{72b_2^2\beta_2}.$$

The first equation of (24) gives the only allowed solution among the sets of real solutions (See supplementary section **5.21**) which is given by

$$b_1 = \frac{4b_2a_2(32\beta_2 b_2^3 c_2(3-16\gamma_2) + \beta_2^2 b_2^2(256c_2^2 - 384\gamma_2 + 153) + 384\beta_2^3 b_2 c_2 + b_2^4(3-16\gamma_2)^2 + 144\beta_2^4)}{9a_1(32\beta_2 b_2^3 c_2(3-16\gamma_2) + 2\beta_2^2 b_2^2(128c_2^2 + 24\gamma_2 + 9) - 48\beta_2^3 b_2 c_2 + b_2^4(3-16\gamma_2)^2 + 9\beta_2^4)} - \frac{2}{3}b_2.$$

Solving the fourth equation of (24) gives from all real solutions (See supplementary section **5.22**), the only allowed solution is

$$a_2 = -\frac{3a_1\left(32\beta_2 b_2^3 c_2(3-16\gamma_2) + 8\beta_2^2 b_2^2\left(32c_2^2 + 6\gamma_2 + 9\right) - 48\beta_2^3 b_2 c_2 + b_2^4(3-16\gamma_2)^2 + 63\beta_2^4\right)}{4\left(32\beta_2 b_2^3 c_2(3-16\gamma_2) + \beta_2^2 b_2^2\left(256c_2^2 - 384\gamma_2 + 153\right) + 384\beta_2^3 b_2 c_2 + b_2^4(3-16\gamma_2)^2 + 144\beta_2^4\right)}.$$

Now we have a continuous piecewise differential systems (8)-(10). We solve the algebraic system (15) with respect to y_1 and y_2, we get one of the following sets of solutions

$s_0 = \{y_1 = -\frac{\sqrt{\mathcal{R}}+8\beta_2 b_2 c_2 + b_2^2(16\gamma_2-3)-3\beta_2^2}{24b_2^2\beta_2},\ y_2 = -\frac{\sqrt{\mathcal{R}}+8\beta_2 b_2 c_2 + b_2^2(16\gamma_2-3)-3\beta_2^2}{24b_2^2\beta_2}\}$,

$s_1 = \{y_1 = \frac{\sqrt{\mathcal{R}}-8\beta_2 b_2 c_2 - 16b_2^2\gamma_2 + 3b_2^2 + 3\beta_2^2}{24b_2^2\beta_2},\ y_2 = \frac{\sqrt{\mathcal{R}}-8\beta_2 b_2 c_2 - 16b_2^2\gamma_2 + 3b_2^2 + 3\beta_2^2}{24b_2^2\beta_2}\}$,

$s_2 = \{y_1 = \frac{27\beta_2^3 b_2^2 + \beta_2 b_2^4(128c_2^2+96\gamma_2-27) - \sqrt{3}\sqrt{\mathcal{T}} - 144\beta_2^2 b_2^3 c_2 + 8b_2^5 c_2(3-16\gamma_2)}{8b_2^4(-16\beta_2 b_2 c_2 + b_2^2(16\gamma_2-3)+6\beta_2^2)}$,
$y_2 = \frac{27\beta_2^3 b_2^2 + \beta_2 b_2^4(128c_2^2+96\gamma_2-27) + \sqrt{3}\sqrt{\mathcal{T}} - 144\beta_2^2 b_2^3 c_2 + 8b_2^5 c_2(3-16\gamma_2)}{8b_2^4(-16\beta_2 b_2 c_2 + b_2^2(16\gamma_2-3)+6\beta_2^2)}\}$,

$s_3 = \{y_1 = \frac{27\beta_2^3 b_2^2 + \beta_2 b_2^4(128c_2^2+96\gamma_2-27) + \sqrt{3}\sqrt{\mathcal{T}} - 144\beta_2^2 b_2^3 c_2 + 8b_2^5 c_2(3-16\gamma_2)}{8b_2^4(-16\beta_2 b_2 c_2 + b_2^2(16\gamma_2-3)+6\beta_2^2)}$,
$y_2 = \frac{27\beta_2^3 b_2^2 + \beta_2 b_2^4(128c_2^2+96\gamma_2-27) - \sqrt{3}\sqrt{\mathcal{T}} - 144\beta_2^2 b_2^3 c_2 + 8b_2^5 c_2(3-16\gamma_2)}{8b_2^4(-16\beta_2 b_2 c_2 + b_2^2(16\gamma_2-3)+6\beta_2^2)}\}$.

Where
$\mathcal{R} = (8\beta_2 b_2 c_2 + b_2^2(16\gamma_2 - 3) - 3\beta_2^2)^2 - 48b_2^2\beta_2(b_2c_2(16\gamma_2-3) - \beta_2(4c_2^2 + 3\gamma_2))$ and
$\mathcal{T} = b_2^4(-16\beta_2 b_2 c_2 + 2b_2^2(8\gamma_2 - 3) + 3\beta_2^2)(32\beta_2 b_2^3 c_2(3-16\gamma_2) + 2\beta_2^2 b_2^2(128c_2^2 + 120\gamma_2 - 27) - 240\beta_2^3 b_2 c_2 + b_2^4(3-16\gamma_2)^2 + 45\beta_2^4)$.
Here s_0 and s_1 give $y_1 = y_2$; y_1 of s_2 is equal to y_2 of s_3, and y_2 of s_2 is equal to y_1 of s_3. Then the continuous piecewise differential systems (8)-(10) can have at most one limit cycle.

Numerical example.
$a_1 = -1$, $a_2 = \frac{4395}{19252}$, $b_1 = \frac{370}{231}$, $b_2 = -\frac{1}{2}$, $c_1 = \frac{4022}{231}$, $c_2 = -\frac{11}{2}$, $\alpha_1 = 0$, $\alpha_2 = -\frac{465}{4813}$, $\beta_1 = -\frac{1}{77}\left(4\sqrt{\frac{5541}{5}}\right)$, $\beta_2 = 1$, $\gamma_1 = -\frac{1}{154}233\sqrt{\frac{1847}{15}} - \frac{1}{2}$ and $\gamma_2 = 10$.
Figure 1 shows the crossing limit cycle in this case. Here the points of intersection with the $y-$axis are $y_1 = -9.83073..$ and $y_2 = -15.1693...$

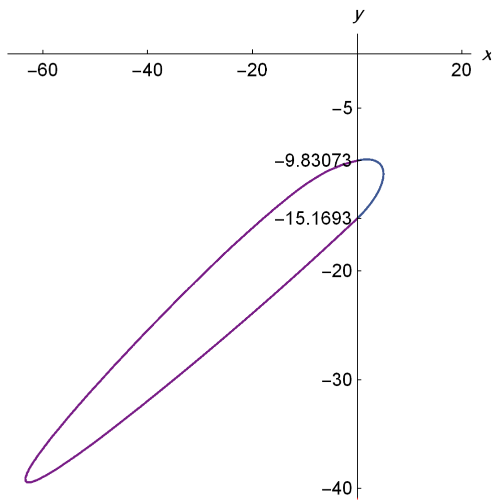

Fig. 1. Crossing limit cycle formed by systems (8) and (10).

Subcase 2.2.1.2: Consider the set u_2 of (31). We follow the same steps as in **Subcase 2.2.1.1** (See supplementary section **5.23**). We get a continuous piecewise differential systems (8)-(10). We solve the algebraic system (15) with respect to y_1 and y_2, and we get one of the same sets of solutions $s_0,\ s_1,\ s_2$ and s_3 as in **Subcase 2.2.1.1**. Then the continuous piecewise differential systems (8)-(10) can have at most one limit cycle.
Subcase 2.2.2: Consider the set s_2 of (30). Solving the second equation of (24) we get $a_2 = 4a_1b_2^2/(3\beta_1^2)$, then we solve the third equation of (24) and we get, among all sets of solutions (See supplementary section **5.24**), the only allowed solution

$$\gamma_2 = \frac{3b_2^2(8\beta_2\gamma_1 + \beta_1 + 4\beta_2) - 8\beta_1\beta_2b_2c_2 + 3\beta_1\beta_2^2}{16b_2^2\beta_1}.$$

Then for this value of γ_2 solving simultaneously the first and the fourth equation of (24) we get the only one real solution $\beta_2 = 0$, which gives a contradiction with (14). Then there are no continuous piecewise differential systems (8)-(10) in this case.
Subcase 2.3: Consider the set d_3 of (26). In this case we have $a_2 = 0$ and $b_1 = -2b_2/3$. With these parameters the sixth equation of (24) becomes $\alpha_1\left(4b_2^2 + 9\beta_1^2\right) = 0$. Solving now the third and the fourth equations of (24) we conclude that all sets of real solutions (See supplementary section **5.25**) give a contradiction with (14) except the following one
$\{\beta_2 = \frac{\alpha_2\left(4b_2^2+9\beta_1^2\right)}{36a_1b_2},\ \gamma_2 = -\frac{6b_2c_1\gamma_1+3b_2c_1+4\beta_1c_2^2}{3\beta_1}\}$.
Then for these values of β_2 and γ_2 solving the first, the second, and the fifth equations of (24) all obtained solutions are complex. Then we have no continuous piecewise differential systems (8)-(10) in this case.
Subcase 2.4: Consider the set d_5 of (26). Here we have $\alpha_2 = 0$ and $b_1 = -8b_2/3$. The sixth equation of (24) becomes $\alpha_1\left(64b_2^2 + 9\beta_1^2\right) = 0$. Since $\alpha_2 = 0$ we must have $b_2 \neq 0$, otherwise we get a contradiction with (14), which yields to $\alpha_1 = 0$. Solving now the third and the fourth equations of (24) simultaneously, all real solutions (See supplementary section **5.26**) give a contradiction with (14) except the following one

$$\begin{aligned} s_1 &= \{a_1 = \tfrac{1}{36}a_2(-\tfrac{9\beta_1^2}{b_2^2} - 64),\ \gamma_2 = \tfrac{3}{16}(1 - \tfrac{c_1(2\beta_2\gamma_1+\beta_2)}{\beta_1c_2})\},\\ s_2 &= \{a_1 = \tfrac{1}{36}a_2(-\tfrac{9\beta_1^2}{b_2^2} - 64),\ c_2 = 0,\ \gamma_1 = -1/2\} \quad \text{and}\\ s_3 &= \{a_1 = \tfrac{1}{36}a_2(-\tfrac{9\beta_1^2}{b_2^2} - 64),\ c_1 = 0,\ c_2 = 0\}, \end{aligned} \tag{32}$$

which must be discussed separately in the following subcases.
Subcase 2.4.1: Consider the set s_1 of (32). We solve now the remaining equations, the first, the second and the fifth equations of (24). All real solutions (See supplementary section **5.27**) give a contradiction with (14) except the following two solutions

$$\begin{aligned} u_1 &= \{c_1 = -8c_2/3,\ \tfrac{\sqrt{b_2^2\beta_1^2\left(128b_2^2-9\beta_1^2\right)\left(16b_2^2+9\beta_1^2\right)}}{128b_2^2\beta_1-9\beta_1^3},\ \gamma_1 = -\tfrac{3\beta_1^2\left(16b_2^2+9\beta_1^2\right)}{8\sqrt{b_2^2\beta_1^2\left(128b_2^2-9\beta_1^2\right)\left(16b_2^2+9\beta_1^2\right)}} + \tfrac{\beta_1c_2}{b_2} - \tfrac{1}{2}\} \text{ and}\\ u_2 &= \{c_1 = -8c_2/3,\ \beta_2 = \tfrac{\sqrt{b_2^2\beta_1^2\left(128b_2^2-9\beta_1^2\right)\left(16b_2^2+9\beta_1^2\right)}}{9\beta_1^3-128b_2^2\beta_1},\ \gamma_1 = \tfrac{3\beta_1^2\left(16b_2^2+9\beta_1^2\right)}{8\sqrt{b_2^2\beta_1^2\left(128b_2^2-9\beta_1^2\right)\left(16b_2^2+9\beta_1^2\right)}} + \tfrac{\beta_1c_2}{b_2} - \tfrac{1}{2}\}. \end{aligned} \tag{33}$$

Then this case is divided into two other subcases.
Subcase 2.4.1.1: Consider the set u_1 of (33). We have now a continuous piecewise differential systems (8)-(10). Solving the first equation of (15) with respect to y_1 and y_2 we get $y_2 = y_1$ which does not give limit cycles for the piecewise differential systems (8)-(11) in this subcase (See supplementary section **5.28**).
Subcase 2.4.1.2: Consider the set u_2 of (33). We have now a continuous piecewise differential systems (8)-(10). As in **Subcase 2.4.1.2** we get $y_2 = y_1$ which does not give limit cycles for the piecewise differential systems (8)-(11) in this subcase (See supplementary section **5.29**).
Subcase 2.4.2: Consider the set s_2 of (32). Solving now the remaining equations, the first, the second and the fifth equations of (24) simultaneously. All solutions (See supplementary section **5.30**) give a contradiction with (14). Then we have no continuous piecewise differential systems in this case.
Subcase 2.4.3: Consider the set s_3 of (32). We solve the remaining equations, the first, the second and the fifth equations of (24). All sets of solutions (See supplementary section **5.31**) give a contradiction with

(14) except the sets u_1 and u_2.

$$
\begin{aligned}
u_1 = \{\beta_2 &= \frac{16\beta_1 b_2^4+9\beta_1^3 b_2^2}{\sqrt{b_2^2\beta_1^2\left(128b_2^2-9\beta_1^2\right)\left(16b_2^2+9\beta_1^2\right)}},\ \gamma_1 = -\frac{-36\beta_1^2 b_2^2+3\sqrt{b_2^2\beta_1^2\left(1008\beta_1^2 b_2^2+2048b_2^4-81\beta_1^4\right)}+512b_2^4}{1024b_2^4-72b_2^2\beta_1^2},\\
\gamma_2 &= \frac{54b_2^2}{9\beta_1^2-128b_2^2}+\frac{9}{16}\},\\
u_2 = \{\beta_2 &= \frac{\sqrt{b_2^2\beta_1^2\left(128b_2^2-9\beta_1^2\right)\left(16b_2^2+9\beta_1^2\right)}}{9\beta_1^3-128b_2^2\beta_1},\ \gamma_1 = \frac{36\beta_1^2 b_2^2+3\sqrt{b_2^2\beta_1^2\left(128b_2^2-9\beta_1^2\right)\left(16b_2^2+9\beta_1^2\right)}-512b_2^4}{8b_2^2\left(128b_2^2-9\beta_1^2\right)},\\
\gamma_2 &= \frac{54b_2^2}{9\beta_1^2-128b_2^2}+\frac{9}{16}\}.
\end{aligned}
\tag{34}
$$

Then two other subcases bifurcate from this case.
Subcase 2.4.3.1: Consider the set u_1 of (34). Now we have a continuous piecewise differential systems (8)-(10). Solving the algebraic system (15) we get $y_1 = y_2$, then there are no periodic orbits, and consequently no limit cycles.
Subcase 2.4.3.2: Consider the set u_2 of (34). By solving the algebraic system (15) we get $y_1 = y_2$, then there are no periodic orbits in this case.
Subcase 2.5: Consider the set d_7 of (26), which means $b_1 = b_2 = 0$. By solving the algebraic system (24). All sets of real solutions (See supplementary section **5.32**) give no continuous piecewise differential systems can be found, because all solutions give a contradiction with (14). ■

Proof. [Proof of statement (d) of Theorem 2] In order that the piecewise differential systems (8)-(11) be continuous they must coincide on $x = 0$, so these systems must satisfy the following algebraic system

$$
\begin{aligned}
&-6a_2b_1\beta_2\gamma_1c_1+8a_1b_2\beta_1c_2\gamma_2-3a_2b_1\beta_2c_1+3a_1b_2\beta_1c_2-3a_2\beta_1\beta_2\gamma_1^2+4a_1\beta_1\beta_2\gamma_2^2\\
&-3a_2\beta_1\beta_2\gamma_1+3a_1\beta_1\beta_2\gamma_2+3a_2\beta_1\beta_2c_1^2-16a_1\beta_1\beta_2c_2^2+3\alpha_2b_2\beta_1\gamma_1^2-4\alpha_1b_1\beta_2\gamma_2^2\\
&+3\alpha_2b_2\beta_1\gamma_1-3\alpha_1b_1\beta_2\gamma_2-3\alpha_2b_2\beta_1c_1^2+16\alpha_1b_1\beta_2c_2^2+6\alpha_2b_1b_2\gamma_1c_1-8\alpha_1b_1b_2c_2\gamma_2\\
&+3\alpha_2b_1b_2c_1-3\alpha_1b_1b_2c_2=0,\\
&-6a_2\beta_2b_1^2\gamma_1+8a_1b_2^2\beta_1\gamma_2-3a_2\beta_2b_1^2+3a_1b_2^2\beta_1-24a_1b_2\beta_1\beta_2c_2-6a_2\beta_1^2\beta_2\gamma_1+8a_1\beta_1\beta_2^2\gamma_2\\
&+3a_1\beta_1\beta_2^2-3a_2\beta_1^2\beta_2-8\alpha_1\beta_2^2b_1\gamma_2+6\alpha_2b_2\beta_1^2\gamma_1-3\alpha_1\beta_2^2b_1+3\alpha_2b_2\beta_1^2+6\alpha_2b_2b_1^2\gamma_1\\
&-8\alpha_1b_2^2b_1\gamma_2+3\alpha_2b_2b_1^2-3\alpha_1b_2^2b_1+24\alpha_1b_2\beta_2b_1c_2=0,\\
&-3a_2b_1^2\beta_2\beta_1-8a_1b_2^2\beta_2\beta_1-3a_2\beta_2\beta_1^3+4a_1\beta_2^3\beta_1+3\alpha_2b_2\beta_1^3+3\alpha_2b_1^2b_2\beta_1-4\alpha_1b_1\beta_2^3\\
&+8\alpha_1b_1b_2^2\beta_2=0,\\
&3a_2\alpha_1\beta_2\gamma_1^2-4a_1\alpha_2\beta_1\gamma_2^2+3a_2\alpha_1\beta_2\gamma_1-3a_1\alpha_2\beta_1\gamma_2-6a_1\alpha_2b_2\gamma_1c_1+8a_2\alpha_1b_1c_2\gamma_2\\
&-3a_1\alpha_2b_2c_1+3a_2\alpha_1b_1c_2-3a_2\alpha_1\beta_2c_1^2+16a_1\alpha_2\beta_1c_2^2+6a_1a_2\beta_2\gamma_1c_1-8a_1a_2\beta_1c_2\gamma_2\\
&+3a_1a_2\beta_2c_1-3a_1a_2\beta_1c_2-3\alpha_1\alpha_2b_2\gamma_1^2+4\alpha_1\alpha_2b_1\gamma_2^2-3\alpha_1\alpha_2b_2\gamma_1+3\alpha_1\alpha_2b_1\gamma_2\\
&+3\alpha_1\alpha_2b_2c_1^2-16\alpha_1\alpha_2b_1c_2^2=0,\\
&6a_2\alpha_1\beta_1\beta_2\gamma_1-8a_1\alpha_2\beta_1\beta_2\gamma_2+3a_2\alpha_1\beta_1\beta_2-3a_1\alpha_2\beta_1\beta_2+8a_2\alpha_1b_1b_2\gamma_2-6a_1\alpha_2b_1b_2\gamma_1\\
&+3a_2\alpha_1b_1b_2-3a_1\alpha_2b_1b_2+6a_1a_2b_1\beta_2\gamma_1-8a_1a_2b_2\beta_1\gamma_2-3a_1a_2b_2\beta_1+3a_1a_2b_1\beta_2\\
&-6a_2\alpha_1b_1\beta_2c_1+8a_2\alpha_1b_1\beta_2c_2-6a_1\alpha_2b_2\beta_1c_1+32a_1\alpha_2b_2\beta_1c_2+6a_1a_2\beta_1\beta_2c_1-8a_1a_2\beta_1\beta_2c_2\\
&-6\alpha_2\alpha_1b_2\beta_1\gamma_1+8\alpha_2\alpha_1b_1\beta_2\gamma_2-3\alpha_2\alpha_1b_2\beta_1+3\alpha_2\alpha_1b_1\beta_2+6\alpha_2\alpha_1b_1b_2c_1-32\alpha_2\alpha_1b_1b_2c_2=0,\\
&-4a_1\alpha_2\beta_1\beta_2^2+3a_2\alpha_1\beta_1^2\beta_2-3a_2\alpha_1\beta_2b_1^2-6a_1\alpha_2b_2\beta_1b_1+8a_2\alpha_1b_2\beta_2b_1+16a_1\alpha_2b_2^2\beta_1\\
&+6a_1a_2\beta_1\beta_2b_1-8a_1a_2b_2\beta_1\beta_2+4\alpha_1\alpha_2\beta_2^2b_1-3\alpha_1\alpha_2b_2\beta_1^2+3\alpha_1\alpha_2b_2b_1^2-16\alpha_1\alpha_2b_2^2b_1=0,
\end{aligned}
\tag{35}
$$

together with conditions (14).

Solving the second and the fifth equation of (35). Regarding the sets of all real solutions (See supplementary section **5.33**), the only allowed solutions which do not contradict (14) are the following

$$
\begin{aligned}
s_1 = \{ & c_1 = [-3b_1^2(2\gamma_1+1)(4\alpha_2 b_2 - a_2\beta_2) + b_1(9a_1\beta_2 b_2(2\gamma_1+1) + \alpha_1\beta_2^2(8\gamma_2+3) \\
& +4\alpha_1 b_2^2(8\gamma_2+3)) - \beta_1(3(2\gamma_1+1)(-a_2\beta_2\beta_1 + 4\alpha_2 b_2\beta_1 - 3\alpha_1 b_2\beta_2) \\
& +a_1(8\gamma_2+3)(4b_2^2+\beta_2^2))]/[18b_2\beta_2(\alpha_1 b_1 - a_1\beta_1)] \text{ if } b_2\beta_2 \neq 0, \\
& c_2 = \tfrac{(-3b_1^2(2\gamma_1+1)(\alpha_2 b_2 - a_2\beta_2) - \beta_1(3\beta_1(2\gamma_1+1)(\alpha_2 b_2 - a_2\beta_2) + a_1(8\gamma_2+3)(b_2^2+\beta_2^2)) + \alpha_1 b_1(8\gamma_2+3)(b_2^2+\beta_2^2)}{(24b_2\beta_2(\alpha_1 b_1 - a_1\beta_1)}\}, \\
s_2 = \{ & b_2 = 0,\ c_1 = \tfrac{a_1\beta_2 b_1(6\gamma_1+3) - 3\alpha_2\beta_1(2\beta_1\gamma_1+\beta_1) + \beta_1\beta_2(-8a_1c_2+6\alpha_1\gamma_1+3\alpha_1) - 3b_1^2(2\alpha_2\gamma_1+\alpha_2) + 8\alpha_1\beta_2 b_1 c_2}{6\beta_2(\alpha_1 b_1 - a_1\beta_1)}, \\
& \gamma_2 = \tfrac{-3a_2(2\gamma_1+1)(b_1^2+\beta_1^2)}{8\beta_2(\alpha_1 b_1 - a_1\beta_1)} - 3/8\} \quad \text{and} \\
s_3 = \{ & c_1 = \tfrac{-3a_2(2\gamma_1+1)(b_1^2+\beta_1^2) + a_1 b_2(b_1(6\gamma_1+3) - 32\beta_1 c_2) + \alpha_1 b_2(32b_1 c_2 + 6\beta_1\gamma_1 + 3\beta_1)}{6b_2(\alpha_1 b_1 - a_1\beta_1)},\ \beta_2 = 0, \\
& \gamma_2 = \tfrac{3(\beta_1(\alpha_2\beta_1(2\gamma_1+1) + a_1 b_2) + b_1^2(2\alpha_2\gamma_1+\alpha_2) - \alpha_1 b_1 b_2)}{8b_2(\alpha_1 b_1 - a_1\beta_1)}\},
\end{aligned}
\tag{36}
$$

which are studied in the following three cases.

Case 1: Consider the set s_1 of (36). Solving the third equation of (35) we get one of the following sets of allowed real solutions (See supplementary section **5.34**)

$$
\begin{aligned}
u_1 &= \{\alpha_2 = \frac{\beta_2\left(4\beta_2^2(\alpha_1 b_1 - a_1\beta_1) + 3a_2 b_1^2\beta_1 + 8a_1 b_2^2\beta_1 + 3a_2\beta_1^3 - 8\alpha_1 b_1 b_2^2\right)}{3b_2\beta_1\left(b_1^2+\beta_1^2\right)}\} \text{ if } b_2\beta_1 \neq 0, \\
u_2 &= \{b_2 = -\frac{\beta_2}{\sqrt{2}},\ \beta_1 = 0\} \quad \text{and} \\
u_3 &= \{b_2 = \frac{\beta_2}{\sqrt{2}},\ \beta_1 = 0\},
\end{aligned}
\tag{37}
$$

which do not contradict (14) and $b_2\beta_2 \neq 0$.

Subcase 1.1: Consider the set u_1 of (37). We solve the sixth equation of (35) to get one of the following allowed real solution (See supplementary section **5.35**)

$$
a_2 = \frac{4\beta_2^2(a_1\beta_1 - \alpha_1 b_1) + b_2\left(6a_1\beta_1 b_1 - 16a_1 b_2\beta_1 + 3\alpha_1\beta_1^2 - 3\alpha_1 b_1^2 + 16\alpha_1 b_2 b_1\right)}{3\beta_1\left(b_1^2+\beta_1^2\right)}\},
\tag{38}
$$

which does not contradicts (14), $b_2\beta_2 \neq 0$ and $b_2\beta_1 \neq 0$. We solve the first equation of (35) to get one of the following real solutions (See supplementary section **5.36**)

$$
\begin{aligned}
z_1 = \{ & \gamma_1 = \tfrac{(\beta_1(8\gamma_2+3) - 4\beta_2)}{8\beta_2} \\
& - \tfrac{9(-b_2^2\beta_1^2\beta_2^4(9b_1^2+32b_2^2+9\beta_1^2-16\beta_2^2)(\beta_2^2-2b_2^2)(4\beta_2^2(9b_1^2-32b_2^2+9\beta_1^2)+b_2^2(9b_1^2-256b_2^2+9\beta_1^2)-16\beta_2^4))^{1/2}}{8\beta_2^2(\beta_2^2-2b_2^2)(-4\beta_2^2(9b_1^2-32b_2^2+9\beta_1^2)+b_2^2(-9b_1^2+256b_2^2-9\beta_1^2)+16\beta_2^4)}\}, \\
z_2 = \{ & \gamma_1 = \tfrac{(\beta_1(8\gamma_2+3) - 4\beta_2)}{8\beta_2} \\
& + \tfrac{9(-b_2^2\beta_1^2\beta_2^4(9b_1^2+32b_2^2+9\beta_1^2-16\beta_2^2)(\beta_2^2-2b_2^2)(4\beta_2^2(9b_1^2-32b_2^2+9\beta_1^2)+b_2^2(9b_1^2-256b_2^2+9\beta_1^2)-16\beta_2^4))^{1/2}}{8\beta_2^2(\beta_2^2-2b_2^2)(-4\beta_2^2(9b_1^2-32b_2^2+9\beta_1^2)+b_2^2(-9b_1^2+256b_2^2-9\beta_1^2)+16\beta_2^4)}\}, \\
z_3 = \{ & b_1 = -\tfrac{1}{3}\sqrt{\tfrac{32\beta_2^2}{3} - 9\beta_1^2},\ b_2 = -\tfrac{\beta_2}{\sqrt{6}}\}, \\
z_4 = \{ & b_1 = \tfrac{1}{3}\sqrt{\tfrac{32\beta_2^2}{3} - 9\beta_1^2},\ b_2 = -\tfrac{\beta_2}{\sqrt{6}}\}, \\
z_5 = \{ & b_1 = -\tfrac{1}{3}\sqrt{\tfrac{32\beta_2^2}{3} - 9\beta_1^2},\ b_2 = \tfrac{\beta_2}{\sqrt{6}}\} \quad \text{and} \\
z_6 = \{ & b_1 = \tfrac{1}{3}\sqrt{\tfrac{32\beta_2^2}{3} - 9\beta_1^2},\ b_2 = \tfrac{\beta_2}{\sqrt{6}}\}.
\end{aligned}
\tag{39}
$$

The only allowed solutions which do not contradict (14), $b_2\beta_2 \neq 0$ and $b_2\beta_1 \neq 0$. Again these solutions gives six other different cases.

Subcase 1.1.1: Consider the set z_1 of (39). Now the remaining fourth equation of (35) becomes

$$
\beta_2\left(b_2^2+\beta_2^2\right)(\alpha_1 b_1 - a_1\beta_1)^2\left(12b_1\left(\beta_2^2 - 6b_2^2\right) + b_2\left(128b_2^2 + 9\beta_1^2 - 32\beta_2^2\right) + 9b_2 b_1^2\right) = 0,
$$

since $\left(b_2^2+\beta_2^2\right)(\alpha_1 b_1 - a_1\beta_1) \neq 0$, otherwise we have a contradiction with (14), and from $b_2\beta_2 \neq 0$, we have $\beta_2 \neq 0$. This last equation is equivalent to the following one

$$
12b_1\left(\beta_2^2 - 6b_2^2\right) + b_2\left(128b_2^2 + 9\beta_1^2 - 32\beta_2^2\right) + 9b_2 b_1^2 = 0.
$$

From $b_2\beta_2 \neq 0$ we have $b_2 \neq 0$, then we can get the solution for this last equation by solving it with respect to b_1, and hence we get the solutions for the fourth equation of (35). In fact we have the two solutions

$$b_1 = \frac{\pm\sqrt{-\left(9\beta_1^2 + 16\beta_2^2\right) b_2^2 + 16b_2^4 + 4\beta_2^4} + 12b_2^2 - 2\beta_2^2}{3b_2}.$$

With every one of these two values for b_1 we have a continuous piecewise differential systems (8)-(11). Moreover, using all the parameters which provide the continuity of the system except the final value b_1, we solve the algebraic system (15), we get only one pair (y_1, y_2) excluding $y_1 = y_2$ and permutation between y_1 and y_2. Then the continuous piecewise differential systems (8)-(11) can have at most one limit cycle.

Numerical example.

$a_1 = 1$, $a_2 = \frac{173549\sqrt{13039}+147712067}{260215428}$, $b_1 = \frac{1}{450}\left(2308 - \sqrt{13039}\right)$, $b_2 = 3/2$,
$c_1 = -\frac{19(\sqrt{13039}-2308)}{5040} - \frac{1}{8}\sqrt{\frac{6995470812\sqrt{13039}+10285238052791}{76047982270}}$, $c_2 = \frac{3}{112}\left(95 - 14\sqrt{\frac{254(5052\sqrt{13039}+1718561)}{299401505}}\right)$,
$\alpha_1 = 1$, $\alpha_2 = \frac{7(34487\sqrt{13039}+7413985)}{477061618}$, $\beta_1 = 11/10$, $\beta_2 = 7/5$, $\gamma_1 = \frac{9}{112}\left(17 - 231\sqrt{\frac{10(5052\sqrt{13039}+1718561)}{7604798227}}\right)$ and $\gamma_2 = 2$.

The crossing limit cycle is shown in Figure 2.

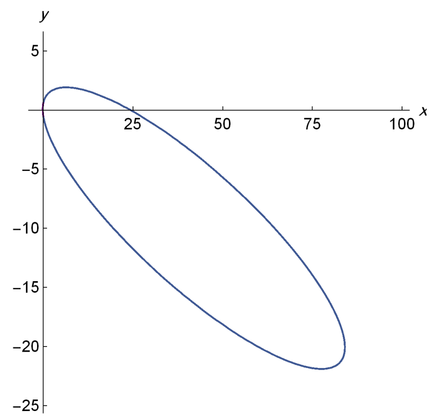

Fig. 2. Crossing limit cycle formed by systems (8)-(11).

Zooming in the graph near the points of intersection $y_1 = -0.493173$ and $y_2 = 0.58115$ with the $y-$axis, we get the graph as shown in Figure 3.

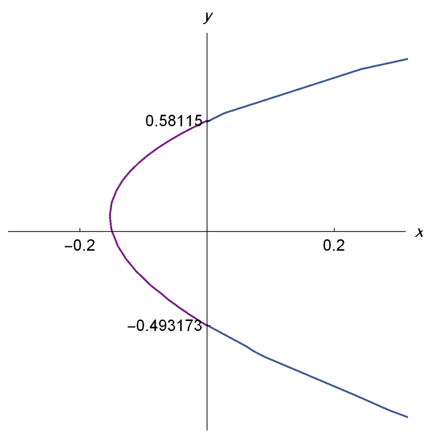

Fig. 3. Zooming in the graph of crossing limit cycle formed by systems (8)-(11).

Subcase 1.1.2: Consider the set z_2 of (39). The remaining fourth equation of (35) becomes

$$\beta_2 \left(b_2^2+\beta_2^2\right) \left(12b_1 \left(\beta_2^2-6b_2^2\right)+b_2 \left(128b_2^2+9\beta_1^2-32\beta_2^2\right)+9b_2b_1^2\right) \left(\alpha_1 b_1-a_1\beta_1\right)^2=0,$$

This last equation is equivalent to the following one

$$12b_1 \left(\beta_2^2-6b_2^2\right)+b_2 \left(128b_2^2+9\beta_1^2-32\beta_2^2\right)+9b_2b_1^2=0.$$

From $b_2\beta_2 \neq 0$ we have $b_2 \neq 0$, then we can get all solution for this last equation by solving it with respect to b_1. Hence we get the two solutions for the fourth equation of (35)

$$b_1=\frac{\pm\sqrt{-\left(9\beta_1^2+16\beta_2^2\right)b_2^2+16b_2^4+4\beta_2^4}+12b_2^2-2\beta_2^2}{3b_2}.$$

With every one of these two values for b_1 we have a continuous piecewise differential systems (8)-(11). Again using all parameters which provide the continuity except the final value b_1, we solve the algebraic system (15), and we get only one pair (y_1, y_2) excluding $y_1 = y_2$ and permutation between y_1 and y_2. Then the continuous piecewise differential systems (8)-(11) can have at most one limit cycle.
Subcase 1.1.3: Consider the set z_3 of (39). Here we do not need to solve the fourth equation of (35) to get a continuous piecewise differential systems (8)-(11), because solving the algebraic system (15) we get only one pair (y_1, y_2) except a permutation between y_1 and y_2 which are
$y_1 = [\beta_1^5(-(704\gamma_2(4\gamma_2+3)+639))(800\gamma_2(4\gamma_2+3)+693)+16\beta_2\beta_1^4(2\gamma_1+1)(8\gamma_2+3)(31264\gamma_2(4\gamma_2+3)+27549)-672\beta_1^3(3904\gamma_2(4\gamma_2+3)+2601)(2\beta_2\gamma_1+\beta_2)^2+374272\beta_1^2(8\gamma_2+3)(2\beta_2\gamma_1+\beta_2)^3-311296\beta_1(2\beta_2\gamma_1+\beta_2)^4+12\sqrt{6}(2\gamma_1+1)\sqrt{\mathcal{R}}]/[24\beta_1(32\beta_1\beta_2(\beta_1(8\gamma_2+3)-4(2\beta_2\gamma_1+\beta_2))(\beta_1^2(164\gamma_2(4\gamma_2+3)+153)-82\beta_2\beta_1(2\gamma_1+1)(8\gamma_2+3)+164(2\beta_2\gamma_1+\beta_2)^2)-\sqrt{6}\sqrt{\mathcal{R}})]$ and

$y_2 = [\sqrt{6}\sqrt{\mathcal{R}}+8\beta_1\beta_2(-4\beta_1^3(8\gamma_2+3)(164\gamma_2(4\gamma_2+3)+153)+9\beta_2\beta_1^2(2\gamma_1+1)(640\gamma_2(4\gamma_2+3)+387)-912\beta_1(8\gamma_2+3)(2\beta_2\gamma_1+\beta_2)^2+512(2\beta_2\gamma_1+\beta_2)^3)]/[48\beta_1^2\beta_2^2(\beta_1^2(704\gamma_2(4\gamma_2+3)+639)-352\beta_2\beta_1(2\gamma_1+1)(8\gamma_2+3)+704(2\beta_2\gamma_1+\beta_2)^2)]$,
where
$\mathcal{R} = -\beta_1^2\beta_2^2(\beta_1^2(224\gamma_2(4\gamma_2+3)+207)-112\beta_2\beta_1(2\gamma_1+1)(8\gamma_2+3)+224(2\beta_2\gamma_1+\beta_2)^2)(\beta_1^2(512\gamma_2(4\gamma_2+3)+531)-256\beta_2\beta_1(2\gamma_1+1)(8\gamma_2+3)+512(2\beta_2\gamma_1+\beta_2)^2)^2$.
And since from z_{10} of (39) we have $b_1 = -\sqrt{\mathcal{S}}/3$, with $\mathcal{S} = 32\beta_2^2/3 - 9\beta_1^2$. This provided that $\mathcal{S} \geq 0$ and $\mathcal{R} \geq 0$ which are verified if and only if $\beta_1 = 0$, this value of β_1 is not allowed because we have already from (38)

$$a_2=\frac{4\beta_2^2\left(a_1\beta_1-\alpha_1 b_1\right)+b_2\left(6a_1\beta_1 b_1-16a_1b_2\beta_1+3\alpha_1\beta_1^2-3\alpha_1 b_1^2+16\alpha_1 b_2 b_1\right)}{3\beta_1\left(b_1^2+\beta_1^2\right)}.$$

This proves that the possible continuous piecewise differential systems (8)-(11) have no limit cycles.
Subcase 1.1.4: Consider the set z_4 of (39). This case follows in a similar results to **Subcase 1.1.3**. Then we have no limit cycles for the possible continuous piecewise differential systems (8)-(11).
Subcase 1.1.5: Consider the set z_5 of (39). Again this case gives similar results to **Subcase 1.1.3**.
Subcase 1.1.6: Consider the set z_6 of (39). Also this case gives similar results to **Subcase 1.1.3**.
Subcase 1.2: Consider the set u_2 of (37). Before solving the remaining equations of (35) to get a continuous piecewise differential systems (8)-(11), we solve the algebraic system (15) by using only one parameters $\beta_1 = 0$ to get four pairs (y_1, y_2) as follows
$S_0 = \{y_1 = -\frac{\sqrt{\mathcal{R}_1}+16c_1}{16b_1},\ y_2 = \frac{\sqrt{\mathcal{R}_1}-16c_1}{16b_1}\}$,
$S_1 = \{y_1 = \frac{\sqrt{\mathcal{R}_1}-16c_1}{16b_1},\ y_2 = -\frac{\sqrt{\mathcal{R}_1}+16c_1}{16b_1}\}$,
$S_2 = \{y_1 = -\frac{\sqrt{2}\sqrt{\mathcal{R}_2}+16c_1}{16b_1},\ y_2 = \frac{\sqrt{2}\sqrt{\mathcal{R}_2}-16c_1}{16b_1}\}$,
$S_3 = \{y_1 = \frac{\sqrt{2}\sqrt{\mathcal{R}_2}-16c_1}{16b_1},\ y_2 = -\frac{\sqrt{2}\sqrt{\mathcal{R}_2}+16c_1}{16b_1}\}$,
where

$\mathcal{R}_1 = [2048c_1^3(\beta_2^2-2b_2^2)\beta_2^4+256b_1c_1^2(4(8\gamma_2+3)b_2^2+16c_2\beta_2b_2-3\beta_2^2(8\gamma_2+3))\beta_2^3+16b_1^2c_1(\beta_2(256c_2^2+96\gamma_2(4\gamma_2+3)+63)-128b_2c_2(8\gamma_2+3))\beta_2^3-2b_1^3(8\gamma_2+3)((256c_2^2+32\gamma_2(4\gamma_2+3)+27)\beta_2^2-96b_2c_2(8\gamma_2+3)\beta_2+$

$4b_2^2(8\gamma_2+3)^2)\beta_2-2b_1[\beta_2^2(b_1(8\gamma_2+3)-8c_1\beta_2)^2(512c_1^2(2(32\gamma_2(4\gamma_2+3)+9)b_2^2-64c_2\beta_2(8\gamma_2+3)b_2+(256c_2^2+9)\beta_2^2)\beta_2^4-128b_1c_1(8\gamma_2+3)(2(32\gamma_2(4\gamma_2+3)+9)b_2^2-64c_2\beta_2(8\gamma_2+3)b_2+(256c_2^2+9)\beta_2^2)\beta_2^3+b_1^2((256c_2^2+9)(256c_2^2+128\gamma_2(4\gamma_2+3)+81)\beta_2^4-128b_2c_2(8\gamma_2+3)(256c_2^2+64\gamma_2(4\gamma_2+3)+45)\beta_2^3+8b_2^2(8\gamma_2+3)^2(768c_2^2+(16\gamma_2+3)(16\gamma_2+9))\beta_2^2-512b_2^3c_2(8\gamma_2+3)^3\beta_2+16b_2^4(8\gamma_2+3)^4))]^{1/2}]/[\beta_2^3(8c_1\beta_2(\beta_2^2-2b_2^2)+b_1(4(8\gamma_2+3)b_2^2-16c_2\beta_2b_2-\beta_2^2(8\gamma_2+3)))]$ and

$\mathcal{R}_2=[1024c_1^3(\beta_2^2-2b_2^2)\beta_2^4+8b_1^2c_1(\beta_2(256c_2^2+96\gamma_2(4\gamma_2+3)+63)-128b_2c_2(8\gamma_2+3))\beta_2^3-b_1^3(8\gamma_2+3)((256c_2^2+32\gamma_2(4\gamma_2+3)+27)\beta_2^2-96b_2c_2(8\gamma_2+3)\beta_2+4b_2^2(8\gamma_2+3)^2)\beta_2+(128c_1^2(4(8\gamma_2+3)b_2^2+16c_2\beta_2b_2-3\beta_2^2(8\gamma_2+3))\beta_2^3+[\beta_2^2(b_1(8\gamma_2+3)-8c_1\beta_2)^2(512c_1^2(2(32\gamma_2(4\gamma_2+3)+9)b_2^2-64c_2\beta_2(8\gamma_2+3)b_2+(256c_2^2+9)\beta_2^2)\beta_2^4-128b_1c_1(8\gamma_2+3)(2(32\gamma_2(4\gamma_2+3)+9)b_2^2-64c_2\beta_2(8\gamma_2+3)b_2+(256c_2^2+9)\beta_2^2)\beta_2^3+b_1^2((256c_2^2+9)(256c_2^2+128\gamma_2(4\gamma_2+3)+81)\beta_2^4-128b_2c_2(8\gamma_2+3)(256c_2^2+64\gamma_2(4\gamma_2+3)+45)\beta_2^3+8b_2^2(8\gamma_2+3)^2(768c_2^2+(16\gamma_2+3)(16\gamma_2+9))\beta_2^2-512b_2^3c_2(8\gamma_2+3)^3\beta_2+16b_2^4(8\gamma_2+3)^4))]^{1/2})b_1]/[\beta_2^3(8c_1\beta_2(\beta_2^2-2b_2^2)+b_1(4(8\gamma_2+3)b_2^2-16c_2\beta_2b_2-\beta_2^2(8\gamma_2+3)))]$.

Since y_1 (y_2) of S_0 is equal y_2 (y_1) of S_1, and y_1 (y_2) of S_2 is equal to y_2 (y_1) of S_3, then we can choose two pairs S_0 and S_2. This proves that the possible continuous piecewise differential systems (8)-(11) can have at most two limit cycles in this case, but in what follows we will prove that they cannot have limit cycles because $\mathcal{R}_1<0$ and $\mathcal{R}_2<0$.

Return now to the continuity condition. Solving the sixth equation of (35) we get one allowed solution which does not contradict (14), then we have $\beta_2=-3b_1/(4\sqrt{2})$ (See supplementary section **5.37**).

Now the first equation of (35) becomes equivalent to

$$b_1^2(448a_2\alpha_2(2\gamma_1+1)^2+216a_1(2\gamma_1+1)^2(\sqrt{2}a_2+\alpha_2)-64\sqrt{2}(2a_2\gamma_1+a_2)^2+\sqrt{2}(256(2\alpha_2\gamma_1+\alpha_2)^2-81\alpha_1^2))=0,$$

and since $\beta_1=0$ we must have $b_1\neq 0$, otherwise they contradict (14). Solving this last equation with respect to α_1 we get two following values

$$\alpha_1=\pm\frac{2}{9}\sqrt{(2\gamma_1+1)^2\left(56\sqrt{2}\alpha_2a_2+27\sqrt{2}a_1\alpha_2-16a_2^2+54a_1a_2+64\alpha_2^2\right)}. \tag{40}$$

Without solving the fourth remaining equation of (35), we use all parameters we always obtain $\mathcal{R}_1<0$ and $\mathcal{R}_2<0$.

Subcase 1.3: Consider the set u_3 of (37). Working as in **Subcase 1.2** the piecewise differential system has no limit cycles in this subcase.

Case 2: Consider the set s_2 of (36). Solving the third and the sixth equation of (35) we get the only one solution $\{a_1=3a_2(b_1^2+\beta_1^2)/(4\beta_2^2)+\alpha_1b_1/\beta_1,\ \alpha_2=3a_2b_1/(2\beta_2)+\alpha_1\beta_2/\beta_1\}$ which verify the condition (14) (See supplementary section **5.38**).

Then solving the first and the fourth equation of (35) simultaneously, The solutions for which the condition (14) is verified are two (See supplementary section **5.39**)

$$\begin{array}{l} u_1=\{b_1=0,\ \beta_1=-4\beta_2\sqrt{64c_2^2+9}/(3\sqrt{256c_2^2+9})\} \quad \text{and} \\ u_2=\{b_1=0,\ \beta_1=4\beta_2\sqrt{64c_2^2+9}/(3\sqrt{256c_2^2+9})\}. \end{array} \tag{41}$$

For any of these solutions we have now a continuous piecewise differential systems (8)-(11).

Subcase 2.1: Consider the set u_1 of (41). Here we consider the first integrals for (8)-(11) which are given respectively as follows

$$\frac{(a_1x+b_1y+c_1)^2+(\gamma_1+\alpha_1x+\beta_1y)^2}{2(\gamma_1+\alpha_1x+\beta_1y)+1}=k_1, \tag{42}$$

and

$$\frac{-256(a_2x+b_2y+c_2)^2+128(\gamma_2+\alpha_2x+\beta_2y)^2+96(\gamma_2+\alpha_2x+\beta_2y)+9}{(8(\gamma_2+\alpha_2x+\beta_2y)+3)^4}=k_2, \tag{43}$$

where k_1 and k_2 are two arbitrary constants.

The points of intersections of orbits of system (8) with the $y-$axis is given by solving equation (42) with respect to y, then we get the set of solutions as follows

$$Y1=\left\{y=-\frac{3\beta_2\sqrt{64c_2^2+9}\sqrt{256c_2^2+9}(k_1-\gamma_1)+\sqrt{\beta_2^2\left(-64c_2^2-9\right)\left(256c_2^2+9\right)\left(16c_2^2-9k_1(k_1+1)\right)}}{4\beta_2^2\left(64c_2^2+9\right)},\right.$$

$$y=\frac{3\beta_2\sqrt{64c_2^2+9}\sqrt{256c_2^2+9}(\gamma_1-k_1)+\sqrt{\beta_2^2\left(-64c_2^2-9\right)\left(256c_2^2+9\right)\left(16c_2^2-9k_1(k_1+1)\right)}}{4\beta_2^2\left(64c_2^2+9\right)}\}\Bigg\}.$$

To obtain the points of intersections of orbits of system (9) with the $y-$axis, we solve equation (43) with respect to y, then we have the following solutions

$$Y2=\Bigg\{y=\frac{3\sqrt{256c_2^2+9}(2\gamma_1+1)}{8\beta_2\sqrt{64c_2^2+9}}-\frac{1}{8}\sqrt{\frac{1-\sqrt{-256c_2^2k_2-9k_2+1}}{\beta_2^2k_2}},$$

$$y=\frac{1}{8}\left(\frac{3\sqrt{256c_2^2+9}(2\gamma_1+1)}{\beta_2\sqrt{64c_2^2+9}}+\sqrt{\frac{1-\sqrt{-256c_2^2k_2-9k_2+1}}{\beta_2^2k_2}}\right),$$

$$y=\frac{3\sqrt{256c_2^2+9}(2\gamma_1+1)}{8\beta_2\sqrt{64c_2^2+9}}-\frac{1}{8}\sqrt{\frac{\sqrt{-256c_2^2k_2-9k_2+1}+1}{\beta_2^2k_2}},$$

$$y=\frac{1}{8}\left(\frac{3\sqrt{256c_2^2+9}(2\gamma_1+1)}{\beta_2\sqrt{64c_2^2+9}}+\sqrt{\frac{\sqrt{-256c_2^2k_2-9k_2+1}+1}{\beta_2^2k_2}}\right)\Bigg\}.$$

All possible cases for which the orbits of (8) coincide with the orbits of (11) on the $y-$ axis to forme a closed orbits are given by solving, with respect to k_1 and k_2, the algebraic system formed by connecting the two points of $Y1$ together with any possible two points of $Y2$. All such cases give the following different pairs (k_1,k_2) as follows

$$K1=\left\{k_1=-\tfrac{1}{2},\ k_2=-\frac{3}{256c_2^2+9}\right\},$$

$$K2=\left\{k_1=\tfrac{1}{6}(-\sqrt{64c_2^2+9}-3),\ k_2=\frac{1}{256c_2^2+9}\right\}\ \text{and}$$

$$K3=\left\{k_1=\tfrac{1}{6}(\sqrt{64c_2^2+9}-3),\ k_2=\frac{1}{256c_2^2+9}\right\}.$$

Using all these three pairs, $Y1$ becomes:

For $K1$**:** $Y1$ is given by

$$Y1=\Bigg\{y=-\frac{\sqrt{\beta_2^2\left(-\left(64c_2^2+9\right)^2\right)\left(256c_2^2+9\right)}-3\beta_2\sqrt{64c_2^2+9}\sqrt{256c_2^2+9}(2\gamma_1+1)}{8\beta_2^2\left(64c_2^2+9\right)},$$

$$y=\frac{3\beta_2\sqrt{64c_2^2+9}\sqrt{256c_2^2+9}(2\gamma_1+1)+\sqrt{\beta_2^2\left(-\left(64c_2^2+9\right)^2\right)\left(256c_2^2+9\right)}}{8\beta_2^2\left(64c_2^2+9\right)}\Bigg\},$$

which are complex.

For $K2$**:** $Y1$ is given by

$$Y1=\left\{y=\frac{\sqrt{256c_2^2+9}\left(\sqrt{64c_2^2+9}+6\gamma_1+3\right)}{8\beta_2\sqrt{64c_2^2+9}},y=\frac{\sqrt{256c_2^2+9}\left(\sqrt{64c_2^2+9}+6\gamma_1+3\right)}{8\beta_2\sqrt{64c_2^2+9}}\right\},$$

which gives only one element.

For $K3$**:** $Y1$ is given by

$$Y1=\left\{y=-\frac{\sqrt{256c_2^2+9}\left(\sqrt{64c_2^2+9}-6\gamma_1-3\right)}{8\beta_2\sqrt{64c_2^2+9}},y=-\frac{\sqrt{256c_2^2+9}\left(\sqrt{64c_2^2+9}-6\gamma_1-3\right)}{8\beta_2\sqrt{64c_2^2+9}}\right\},$$

which gives only one element.

This provided no limit cycle for the piecewise differential system (8)-(11) in this subcase.

Subcase 2.2: Consider the set u_2 of (41). Working as in **Subcase 2.1**. Again this provided no limit cycles.

Case 3: Consider the set s_3 of (36). Solving the third and the sixth equation of (35) we get the following set of real solutions $\{b_2=3b_1/16,\ \beta_1=0\}$ for which the condition (14) is verified (See supplementary section **5.40**).

Solving the first equation of (35) we obtain one of the following sets of real solutions (See supplementary section **5.41**)

$$\begin{array}{l} u_1=\{a_2=3a_1/16\}\quad\text{and}\\ u_2=\{\gamma_1=-1/2\},\end{array}\tag{44}$$

for which the condition (14) is verified. For any of these solutions we have now a continuous piecewise differential systems (8)-(11).

Subcase 3.1: Consider the set u_1 of (44). We shall solve the fourth equation of (35) which now is

$$\alpha_2 b_1 \left(256 \left(2\alpha_2\gamma_1 + \alpha_2\right)^2 - 9\alpha_1^2 \left(\gamma_1^2 + \gamma_1 + 1\right)\right) = 0.$$

Since $b_1 \neq 0$ and $\alpha_2 \neq 0$ (because $\beta_1 = \beta_2 = 0$), this equation has two solutions

$$\gamma_1^{\mp} = \frac{-9\alpha_1^2 + 1024\alpha_2^2 \mp 3\sqrt{3072\alpha_1^2\alpha_2^2 - 27\alpha_1^4}}{18\alpha_1^2 - 2048\alpha_2^2}.$$

Subcase 3.1.1: For γ_1^- we have a continuous piecewise differential systems (8)-(11). We solve the algebraic system (15). The first two equations become

$$\frac{(3\alpha_1 - 32\alpha_2)(3\alpha_1 + 32\alpha_2) b_1 (y_1 - y_2)(3b_1y_1 + 3b_1y_2 + 32c_2)}{9\sqrt{3}\sqrt{-\alpha_1^2 \left(9\alpha_1^2 - 1024\alpha_2^2\right)}} = 0,$$

$$\frac{(3\alpha_1 - 32\alpha_2)^2 (3\alpha_1 + 32\alpha_2)^2 b_1 (y_1 - y_2)(3b_1y_1 + 3b_1y_2 + 32c_2)}{15925248\alpha_2^4} = 0.$$

From these equations we get $y_1 = y_2$ or $y_1 = -\frac{32c_2}{3b_1} - y_2$. Then this provided a continuum of periodic orbits and then no limit cycles.
Subcase 3.1.2: For γ_1^+. We again have a continuous piecewise differential systems (8)-(11). We solve the algebraic system (15) its two first equations become

$$\frac{(3\alpha_1 - 32\alpha_2)(3\alpha_1 + 32\alpha_2) b_1 (y_1 - y_2)(3b_1y_1 + 3b_1y_2 + 32c_2)}{9\sqrt{3}\sqrt{-\alpha_1^2 \left(9\alpha_1^2 - 1024\alpha_2^2\right)}} = 0,$$

$$\frac{(3\alpha_1 - 32\alpha_2)^2 (3\alpha_1 + 32\alpha_2)^2 b_1 (y_1 - y_2)(3b_1y_1 + 3b_1y_2 + 32c_2)}{15925248\alpha_2^4} = 0.$$

From these equations we get $y_1 = y_2$ or $y_1 = -\frac{32c_2}{3b_1} - y_2$. Then this provided a continuum of periodic orbits and then no limit cycles.
Subcase 3.2: Consider the set u_2 of (41). Then the fourth equation of (35) becomes $-27\alpha_1\alpha_2 b_1 = 0$. This equation has $\alpha_1 = 0$, $\alpha_2 = 0$ or $b_1 = 0$ but all these solutions are not allowed because they contradict (14). Then in this subcase there are no continuous piecewise differential systems (8)-(11). ■

4. Conclusion

First we have studied the planar continuous piecewise differential systems formed by a quadratic isochronous center and an isochronous linear center separated by the straight line $x = 0$, and we have proved that they cannot neither crossing periodic orbits, nor limit cycles, see Theorem 1.

Second we study the crossing periodic orbits and limit cycles of the planar continuous piecewise differential systems separated by the straight line $x = 0$ having in $x > 0$ the quadratic isochronous center (8), and in $x < 0$ an arbitrary quadratic isochronous center, i.e. one of the systems (8), (9), (10) or (11). We remark that these last four quadratic polynomial differential systems are all the quadratic polynomial differential systems having an isochronous center. For these four families of continuous piecewise differential systems the maximum number of crossing limit cycles is one, and there are examples having one crossing limit cycles. See Theorem 2.

In short for the eight different families of planar continuous piecewise differential systems here studied we have proved the extension of the 16th Hilbert problem to them.

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5. SUPPLEMENTARY INFORMATION FOR REVIEW ONLY

5.1. *SECTION*

All real solutions
$s_1 = \{a_2 = 0, b_2 = 0, c_2 = -1/2\}$,
$s_2 = \{b_1 = 0, b_2 = 0, c_1 = \frac{-\sqrt{(2a_1c_2+a_1)^2+4a_2\left(a_2c_2^2+a_2c_2\right)}+2a_1c_2+a_1}{2a_2}\}$,
$s_3 = \{b_1 = 0, b_2 = 0, c_1 = \frac{\sqrt{(2a_1c_2+a_1)^2+4a_2\left(a_2c_2^2+a_2c_2\right)}+2a_1c_2+a_1}{2a_2}\}$,
$s_4 = \{a_1 = 0, a_2 = 0, b_1 = 0, b_2 = 0\}$,
$s_5 = \{a_2 = 0, b_1 = 0, b_2 = 0, c_1 = 0\}$,
$s_6 = \{a_2 = 0, b_1 = 0, b_2 = 0, c_2 = -1/2\}$ and
$s_7 = \{\alpha = 0, a_1 = 0, a_2 = 0, b_2 = 0, c_1 = 0\}$.

5.2. *SECTION*

The algebraic system has the following real solutions
$s_1 = \{b_1 = 0, b_2 = 0, c_2 = \frac{c_1(a_2c_1-a_1)}{a_1c_1+a_2}\}$,
$s_2 = \{a_1 = 0, a_2 = 0, b_1 = 0, b_2 = 0\}$,
$s_3 = \{a_2 = 0, b_1 = 0, b_2 = 0, c_1 = 0\}$,
$s_4 = \{\alpha = \frac{4a_1^2c_1^2+8a_2a_1c_1+a_1^2\omega^2+4a_2^2}{4a_1b_2}, \beta = \frac{a_1c_1+a_2}{a_1}, \gamma = \frac{-a_2c_1^2+a_1c_1+a_1c_2c_1+a_2c_2}{a_1b_2}, \delta = \frac{c_1^2-c_2}{a_1}, b_1 = 0\}$ and
$s_5 = \{\alpha = 0, a_1 = 0, a_2 = 0, b_1 = 0, c_2 = c_1^2\}$.

5.3. *SECTION*

The algebraic system has the following real solutions
$s_1 = \{b_1 = 0, b_2 = 0, c_2 = \frac{c_1(4a_2c_1+3a_1)}{16a_1c_1-3a_2}\}$,
$s_2 = \{a_2 = 0, b_2 = 0, c_2 = 3/16\}$,
$s_3 = \{a_1 = 0, a_2 = 0, b_1 = 0, b_2 = 0\}$,
$s_4 = \{a_2 = 0, b_1 = 0, b_2 = 0, c_1 = 0\}$,
$s_5 = \{a_2 = 0, b_2 = 0, c_1 = 0, c_2 = 3/16\}$,
$s_6 = \{\alpha = \frac{1024a_1^2c_1^2-384a_2a_1c_1+9a_1^2\omega^2+36a_2^2}{36a_1b_2}, \beta = \frac{3a_2-16a_1c_1}{3a_1}, \gamma = \frac{4a_2c_1^2+3a_1c_1-16a_1c_2c_1+3a_2c_2}{3a_1b_2}, \delta = \frac{-4c_1^2-3c_2}{3a_1}, b_1 = 0\}$,
$s_7 = \{\alpha = 0, a_1 = 0, a_2 = 0, b_1 = 0, c_2 = -4c_1^2/3\}$,
$s_8 = \{\alpha = 0, a_1 = 0, a_2 = 0, b_2 = 0, c_1 = 0\}$ and
$s_9 = \{a_1 = 0, a_2 = 0, b_2 = 0, c_1 = 0, c_2 = 3/16\}$.

5.4. *SECTION*

The algebraic system has the following real solutions
$s_1 = \{a_2 = 0, b_2 = 0, c_2 = -3/8\}$,
$s_2 = \{a_1 = 0, a_2 = 0, b_2 = 0, c_2 = -3/8\}$,
$s_3 = \{a_1 = 0, a_2 = 0, b_1 = 0, b_2 = 0\}$,
$s_4 = \{a_2 = 0, b_1 = 0, b_2 = 0, c_1 = 0\}$,
$s_5 = \{a_2 = 0, b_1 = 0, b_2 = 0, c_2 = -3/8\}$,
$s_6 = \{\alpha = 0, a_1 = 0, a_2 = 0, b_2 = 0, c_1 = 0\}$,
$s_7 = \{b_1 = 0, b_2 = 0, c_1 = \frac{-\sqrt{(8a_1c_2+3a_1)^2+64a_2\left(4a_2c_2^2+3a_2c_2\right)}+8a_1c_2+3a_1}{32a_2}\}$,
$s_8 = \{b_1 = 0, b_2 = 0, c_1 = \frac{\sqrt{(8a_1c_2+3a_1)^2+64a_2\left(4a_2c_2^2+3a_2c_2\right)}+8a_1c_2+3a_1}{32a_2}\}$,
$s_9 = \{\alpha = 0, a_1 = 0, a_2 = 0, b_1 = -b_2/\sqrt{2}, c_1 = (-\sqrt{288c_2^2 + 216c_2 + 9/2} - 4\sqrt{2}c_2 - 3/\sqrt{2})/32\}$,
$s_{10} = \{\alpha = 0, a_1 = 0, a_2 = 0, b_1 = -b_2/\sqrt{2}, c_1 = (\sqrt{288c_2^2 + 216c_2 + 9/2} - 4\sqrt{2}c_2 - 3/\sqrt{2})/32\}$,

$s_{11} = \{\alpha = 0, a_1 = 0, a_2 = 0, b_1 = b_2/\sqrt{2}, c_1 = (-\sqrt{288c_2^2 + 216c_2 + 9/2} + 4\sqrt{2}c_2 + 3/\sqrt{2})/32\}$ and
$s_{12} = \{\alpha = 0, a_1 = 0, a_2 = 0, b_1 = b_2/\sqrt{2}, c_1 = (\sqrt{288c_2^2 + 216c_2 + 9/2} + 4\sqrt{2}c_2 + 3/\sqrt{2})/32\}$.

5.5. *SECTION*

$d_1 = \{a_2 = 0, \alpha_2 = 0\}$,
$d_2 = \{b_2 = 0, \beta_2 = 0\}$,
$d_3 = \{\alpha_2 = 0, \beta_2 = 0\}$,
$d_4 = \{b_1 = \frac{-2a_2\beta_2 b_2 - \alpha_2\beta_2^2 + \alpha_2 b_2^2}{2(\alpha_2 b_2 - a_2\beta_2)}\}$.

5.6. *SECTION*

All real solutions
$r_1 = \{b_1 = \frac{2a_2\beta_2 b_2 + \alpha_2\beta_2^2 - \alpha_2 b_2^2}{2a_2\beta_2 - 2\alpha_2 b_2},\ \alpha_1 = 0\}$,
$r_2 = \{b_2 = 0,\ \beta_2 = 0\}$,
$r_3 = \{\alpha_1 = 0,\ \beta_1 = 0,\ \gamma_1 = -1/2\}$,
$r_4 = \{a_2 = 0,\ \alpha_1 = 0,\ \alpha_2 = 0\}$,
$r_5 = \{b_1 = b_2/2,\ \beta_2 = 0,\ \gamma_2 = -1/2\}$,
$r_6 = \{b_1 = b_2/2,\ \alpha_1 = 0,\ \beta_2 = 0\}$,
$r_7 = \{\alpha_1 = 0,\ \alpha_2 = 0,\ \beta_2 = 0\}$,
$r_9 = \{\beta_1 = 0,\ \beta_2 = 0,\ \gamma_2 = \frac{2\alpha_2 b_1(2\gamma_1+1) - b_2(2\alpha_2\gamma_1 + \alpha_1 + \alpha_2)}{2\alpha_1 b_2}\}$,
$r_{10} = \{\beta_1 = \frac{\alpha_1\beta_2(b_2^2+\beta_2^2)}{2a_2(b_2-b_1)\beta_2 + \alpha_2\beta_2^2 + \alpha_2(2b_1-b_2)b_2}$,
$\gamma_2 = \frac{2b_2(2\gamma_1+1)(a_2\beta_2+\alpha_2 b_1) + \beta_2(\beta_2(2\alpha_2\gamma_1 - \alpha_1 + \alpha_2) - 2a_2(2b_1\gamma_1 + b_1)) + b_2^2(-(2\alpha_2\gamma_1 + \alpha_1 + \alpha_2))}{2\alpha_1(b_2^2+\beta_2^2)}\}$.

5.7. *SECTION*

All real solutions
$s_1 = \{a_2 = \alpha_2 b_2/\beta_2,\ \alpha_1 = -\sqrt{\alpha_2^2(-b_1^2 + b_2^2 + \beta_2^2)}/\beta_2\}$,
$s_2 = \{a_2 = \alpha_2 b_2/\beta_2,\ \alpha_1 = \sqrt{\alpha_2^2(-b_1^2 + b_2^2 + \beta_2^2)}/\beta_2\}$,
$s_3 = \{a_2 = \frac{\alpha_2\beta_2^2 - \alpha_2 b_2^2 + 2\alpha_2 b_1 b_2}{2b_1\beta_2 - 2b_2\beta_2},\ \alpha_1 = 0\}$,
$s_4 = \{b_1 = b_2,\ \alpha_1 = -\alpha_2\}$,
$s_5 = \{b_1 = b_2,\ \alpha_1 = \alpha_2\}$,
$s_6 = \{b_2 = 0,\ \alpha_1 = 0\}$,
$s_7 = \{b_2 = 0,\ \beta_2 = 0\}$,
$s_8 = \{b_1 = b_2/2,\ \beta_2 = 0\}$,
$s_9 = \{b_1 = b_2,\ \beta_2 = 0\}$,
$s_{10} = \{b_2 = 0,\ \beta_2 = 0\}$,
$s_{11} = \{\alpha_2 = 0,\ \beta_2 = 0\}$,
$s_{12} = \{\beta_2 = 0,\ \gamma_1 = -1/2\}$,
$s_{13} = \{\alpha_1 = 0,\ \gamma_1 = -1/2\}$,
$s_{14} = \{a_2 = 0,\ b_2 = 0,\ \alpha_1 = -\sqrt{\alpha_2^2(\beta_2^2 - b_1^2)}/\beta_2\}$,
$s_{15} = \{a_2 = 0,\ b_2 = 0,\ \alpha_1 = \sqrt{\alpha_2^2(\beta_2^2 - b_1^2)}/\beta_2\}$,
$s_{16} = \{b_1 = b_2,\ \alpha_1 = 0,\ \alpha_2 = 0\}$ and
$s_{17} = \{\alpha_1 = 0,\ \alpha_2 = 0,\ \gamma_1 = -1/2\}$.

5.8. *SECTION*

All real solutions
$r_1 = \{\beta_1 = 0\}$,
$r_2 = \{a_2 = 0,\ \alpha_2 = 0\}$,

$r_3 = \{b_2 = 0,\ \beta_2 = 0\}$,
$r_4 = \{\alpha_2 = 0,\ \beta_2 = 0\}$ and
$r_5 = \{a_1 = \frac{4a_2\alpha_2 b_2\beta_2^3 + 2\beta_2^2\left(2a_2^2\left(b_2^2+\beta_1^2\right) - \alpha_2^2 b_2^2\right) - 4a_2\alpha_2 b_2\beta_2\left(b_2^2+2\beta_1^2\right) + \alpha_2^2\beta_2^4 + \alpha_2^2 b_2^2\left(b_2^2+4\beta_1^2\right)}{4\beta_2\left(b_2^2+\beta_2^2\right)(a_2\beta_2 - \alpha_2 b_2)}\}$.

5.9. *SECTION*

All real solutions
$s_1 = \{\gamma_2 = \frac{2a_2b_2(2\beta_2\gamma_1 - \beta_1 + \beta_2) + \alpha_2\beta_2(2\beta_2\gamma_1 - 2\beta_1 + \beta_2) + 4a_2\beta_1\beta_2(c_1 - c_2) + b_2^2(-(2\alpha_2\gamma_1 + \alpha_2)) + 4\alpha_2\beta_1 b_2(c_2 - c_1)}{4\beta_1(a_2b_2 + \alpha_2\beta_2)}\}$,
$s_2 = \{a_2 = -\alpha_2\beta_2/b_2,\ c_2 = \frac{2b_2\gamma_1 + b_2 + 4\beta_1 c_1}{4\beta_1}\}$,
$s_3 = \{a_2 = 0,\ \alpha_2 = 0\}$,
$s_4 = \{a_2 = \frac{\alpha_2(b_2 - \beta_2)(b_2 + \beta_2)}{2b_2\beta_2},\ \beta_1 = 0\}$,
$s_5 = \{\beta_1 = 0,\ \gamma_1 = -1/2\}$,
$s_6 = \{b_2 = 0,\ \beta_2 = 0\}$,
$s_7 = \{a_2 = 0,\ b_2 = 0,\ \alpha_2 = 0\}$,
$s_8 = \{b_2 = 0,\ c_2 = c_1,\ \alpha_2 = 0\}$,
$s_9 = \{b_2 = 0,\ \alpha_2 = 0,\ \beta_1 = 0\}$,
$s_{10} = \{b_2 = 0,\ \beta_1 = 0,\ \gamma_1 = -1/2\}$ and
$s_{11} = \{\alpha_2 = 0,\ \beta_1 = 0,\ \beta_2 = 0\}$.

5.10. *SECTION*

$s_1 = \{a_1 = \frac{\alpha_1\left(c_1^2 - \gamma_1(\gamma_1+1)\right)}{2\gamma_1 c_1 + c_1},\ b_1 = 0\}$,
$s_2 = \{a_2 = \frac{\alpha_2\left(c_2^2 - \gamma_2(\gamma_2+1)\right)}{2\gamma_2 c_2 + c_2},\ b_2 = 0\}$,
$s_3 = \{b_1 = 0,\ b_2 = 0\}$,
$s_4 = \{b_2 = 0,\ \alpha_1 = 0\}$,
$s_5 = \{\alpha_1 = 0,\ \alpha_2 = 0\}$,
$s_6 = \{\alpha_1 = 0,\ \gamma_1 = -1/2\}$,
$s_7 = \{b_1 = 0,\ \alpha_2 = 0\}$,
$s_8 = \{\alpha_1 = 0,\ \alpha_2 = 0\}$,
$s_9 = \{\alpha_2 = 0,\ \gamma_2 = -1/2\}$,
$s_{10} = \{b_1 = 0,\ c_1 = 0,\ \gamma_1 = -1\}$,
$s_{11} = \{b_1 = 0,\ c_1 = 0,\ \gamma_1 = 0\}$,
$s_{12} = \{b_2 = 0,\ c_2 = 0,\ \gamma_2 = -1\}$,
$s_{13} = \{b_2 = 0,\ c_2 = 0,\ \gamma_2 = 0\}$,
$s_{14} = \{b_1 = 0,\ c_1 = 0,\ \alpha_1 = 0\}$,
$s_{15} = \{b_1 = 0,\ \alpha_1 = 0,\ \gamma_1 = -1/2\}$,
$s_{16} = \{b_1 = b_2,\ \alpha_1 = 0,\ \gamma_1 = -1/2\}$,
$s_{17} = \{b_1 = 0,\ b_2 = 0,\ \alpha_1 = 0\}$,
$s_{18} = \{b_1 = 0,\ \alpha_1 = 0,\ \alpha_2 = 0\}$,
$s_{19} = \{b_1 = b_2,\ \alpha_2 = 0,\ \gamma_2 = -1/2\}$,
$s_{20} = \{b_1 = 0,\ b_2 = 0,\ \alpha_2 = 0\}$,
$s_{21} = \{b_2 = 0,\ c_2 = 0,\ \alpha_2 = 0\}$,
$s_{22} = \{b_2 = 0,\ \alpha_2 = 0,\ \gamma_2 = -1/2\}$,
$s_{23} = \{b_1 = b_2,\ \alpha_1 = 0,\ \alpha_2 = 0\}$,
$s_{24} = \{b_1 = b_2,\ c_1 = c_2,\ \gamma_1 = -1/2,\ \gamma_2 = -1/2\}$,
$s_{25} = \{a_2 = \frac{\alpha_2(2a_1\gamma_1 + a_1 + 2\alpha_1 c_2)}{2\alpha_1\gamma_2 + \alpha_1} - \frac{2\alpha_1 c_2}{2\gamma_1 + 1},\ b_1 = 0,\ b_2 = 0,\ c_1 = \frac{\alpha_1 c_2(2\gamma_2 + 1)}{2\alpha_2\gamma_1 + \alpha_2}\}$,
$s_{26} = \{a_1 = 0,\ b_1 = 0,\ c_1 = 0,\ \alpha_1 = 0\}$,
$s_{27} = \{b_1 = 0,\ b_2 = 0,\ \alpha_1 = 0,\ \gamma_1 = -1/2\}$,
$s_{28} = \{b_1 = 0,\ b_2 = 0,\ \alpha_2 = 0,\ \gamma_2 = -1/2\}$,
$s_{29} = \{a_2 = 0,\ b_2 = 0,\ c_2 = 0,\ \alpha_2 = 0\}$,

$s_{30} = \{a_2 = \frac{a_1\alpha_2(2\gamma_1+1)}{2\alpha_1\gamma_2+\alpha_1},\ b_1 = 0, b_2 = 0,\ c_1 = 0, c_2 = 0\},$
$s_{31} = \{a_2 = -\frac{2\alpha_2 c_1}{2\gamma_2+1},\ b_1 = 0,\ b_2 = 0,\ c_2 = 0,\ \gamma_1 = -1/2\},$
$s_{32} = \{a_1 = -\frac{2\alpha_1 c_2}{2\gamma_1+1},\ b_1 = 0,\ b_2 = 0,\ c_1 = 0,\ \gamma_2 = -1/2\},$
$s_{33} = \{b_1 = 0,\ b_2 = 0,\ c_1 = c_2,\ \gamma_1 = -1/2,\ \gamma_2 = -1/2\},$
$s_{34} = \{a_1 = 0,\ b_1 = 0,\ b_2 = 0,\ c_1 = 0,\ \alpha_1 = 0\},$
$s_{35} = \{a_2 = a_1,\ b_1 = b_2,\ c_1 = c_2,\ \alpha_1 = -\alpha_2,\ \gamma_1 = -\gamma_2 - 1\},$
$s_{36} = \{b_1 = b_2,\ c_1 = c_2,\ \alpha_1 = -\alpha_2,\ \gamma_1 = -1/2,\ \gamma_2 = -1/2\},$
$s_{37} = \{a_2 = \frac{2\alpha_2 c_2}{2\gamma_1+1} - \frac{2a_1\gamma_1+a_1-2\alpha_2 c_2}{2\gamma_2+1},\ b_1 = 0,\ b_2 = 0,\ c_1 = -\frac{2\gamma_2 c_2+c_2}{2\gamma_1+1},\ \alpha_1 = -\alpha_2\},$
$s_{38} = \{a_2 = a_1,\ b_1 = b_2,\ c_1 = c_2,\ \alpha_1 = \alpha_2,\ \gamma_1 = \gamma_2\},$
$s_{39} = \{b_1 = b_2,\ c_1 = c_2,\ \alpha_1 = \alpha_2,\ \gamma_1 = -1/2,\ \gamma_2 = -1/2\},$
$s_{40} = \{a_2 = \frac{2a_1\gamma_1+a_1+2\alpha_2 c_2}{2\gamma_2+1} - \frac{2\alpha_2 c_2}{2\gamma_1+1},\ b_1 = 0,\ b_2 = 0,\ c_1 = \frac{2\gamma_2 c_2+c_2}{2\gamma_1+1},\ \alpha_1 = \alpha_2\},$
$s_{41} = \{a_2 = 0,\ b_1 = 0,\ b_2 = 0,\ c_2 = 0,\ \alpha_2 = 0\},$
$s_{42} = \{a_2 = -\frac{2a_1\gamma_1+a_1}{2\gamma_2+1},\ b_1 = 0,\ b_2 = 0,\ c_1 = 0,\ c_2 = 0,\ \alpha_1 = -\alpha_2\},$
$s_{43} = \{a_2 = -\frac{2\alpha_2 c_1}{2\gamma_2+1}),\ b_1 = 0,\ b_2 = 0,\ c_2 = 0,\ \alpha_1 = -\alpha_2,\ \gamma_1 = -1/2\},$
$s_{44} = \{a_1 = \frac{2\alpha_2 c_2}{2\gamma_1+1},\ b_1 = 0,\ b_2 = 0,\ c_1 = 0,\ \alpha_1 = -\alpha_2,\ \gamma_2 = -1/2\},$
$s_{45} = \{b_1 = 0,\ b_2 = 0,\ c_1 = c_2,\ \alpha_1 = -\alpha_2,\ \gamma_1 = -1/2,\ \gamma_2 = -1/2\},$
$s_{46} = \{a_2 = \frac{2a_1\gamma_1+a_1}{2\gamma_2+1},\ b_1 = 0,\ b_2 = 0,\ c_1 = 0,\ c_2 = 0,\ \alpha_1 = \alpha_2\},$
$s_{47} = \{a_2 = -\frac{2\alpha_2 c_1}{2\gamma_2+1},\ b_1 = 0,\ b_2 = 0,\ c_2 = 0,\ \alpha_1 = \alpha_2,\ \gamma_1 = -1/2\},$
$s_{48} = \{a_1 = -\frac{2\alpha_2 c_2}{2\gamma_1+1},\ b_1 = 0,\ b_2 = 0,\ c_1 = 0,\ \alpha_1 = \alpha_2,\ \gamma_2 = -1/2\},$
$s_{49} = \{b_1 = 0,\ b_2 = 0,\ c_1 = c_2,\ \alpha_1 = \alpha_2,\ \gamma_1 = -1/2,\ \gamma_2 = -1/2\}.$

5.11. *SECTION*

All real solutions are
$r_1 = \{\alpha_1 = \frac{a_2(3b_1+8b_2)\beta_1\beta_2-\alpha_2 b_2(3b_1+2b_2)\beta_1}{6b_2^2\beta_2}\},$
$r_2 = \{b_1 = 0,\ b_2 = 0\},$
$r_3 = \{b_2 = 0,\ \beta_1 = 0\},$
$r_4 = \{b_2 = 0,\ \beta_2 = 0\},$
$r_5 = \{\beta_1 = 0,\ \beta_2 = 0\},$
$r_6 = \{b_1 = -2b_2/3,\ \beta_2 = 0\},$
$r_7 = \{\alpha_2 = 0,\ \beta_2 = 0\}$ and
$r_8 = \{a_2 = 0,\ b_2 = 0\}.$

5.12. *SECTION*

All real solutions are
$s_1 = \{\gamma_2 = \frac{3b_2^2(8\beta_2\gamma_1+\beta_1+4\beta_2)-8\beta_1\beta_2 b_2 c_2+3\beta_1\beta_2^2}{16b_2^2\beta_1}\},$
$s_2 = \{b_2 = 0,\ \beta_1 = 0\},$
$s_3 = \{b_2 = 0,\ \beta_2 = 0\},$
$s_4 = \{\beta_1 = 0,\ \gamma_1 = -1/2\}$ and
$s_5 = \{\beta_1 = 0,\ \beta_2 = 0\}.$

5.13. *SECTION*

All sets of real solutions
$v_1 = \{\beta_2 = \frac{b_2(8b_2\gamma_1+b_1(6\gamma_1+3)+4b_2-6\beta_1 c_1-8\beta_1 c_2)}{3\beta_1}\},$
$v_2 = \{b_2 = 0,\ \beta_1 = 0\},$
$v_3 = \{b_1 = -4b_2/3,\ \beta_1 = 0\}$ and
$v_4 = \{\beta_1 = 0,\ \gamma_1 = -1/2\}.$

5.14. *SECTION*

All sets of real solutions

$w_1 = \{c_2 = \frac{(15b_1+56b_2)\beta_1(2\gamma_1+1) - \frac{\sqrt{3}\mathcal{S}}{b_1^2+\beta_1^2} - 30\beta_1^2 c_1}{112\beta_1^2}\}$,
$w_2 = \{c_2 = \frac{\sqrt{3}\mathcal{S}+\beta_1(b_1^2+\beta_1^2)(15b_1(2\gamma_1+1)+56b_2(2\gamma_1+1)-30\beta_1 c_1)}{112\beta_1^2(b_1^2+\beta_1^2)}\}$,
$w_3 = \{\alpha_2 = \frac{a_2(8b_2\gamma_1+b_1(6\gamma_1+3)+4b_2-6\beta_1 c_1-8\beta_1 c_2)}{3\beta_1}\}$,
$w_4 = \{a_2 = 0,\ \beta_1 = 0\}$,
$w_5 = \{b_1 = 0,\ \beta_1 = 0\}$,
$w_6 = \{b_2 = -3b_1/4,\ \beta_1 = 0\}$,
$w_7 = \{b_2 = -3b_1/16,\ \beta_1 = 0\}$ and
$w_8 = \{\beta_1 = 0,\ \gamma_1 = -1/2\}$.
whith
$\mathcal{S} = \beta_1^2(b_1^2+\beta_1^2)(36\beta_1 b_1^3 c_1(2\gamma_1+1) + b_1^2(448(2b_2\gamma_1+b_2)^2 - 3\beta_1^2(12c_1^2+12\gamma_1(\gamma_1+1)+31)) + 4\beta_1 b_1 c_1(2\gamma_1 + 1)(9\beta_1^2 - 448b_2^2) + 448b_2^2\beta_1^2(4c_1^2+1) - 9b_1^4(2\gamma_1+1)^2 - 12\beta_1^4(3c_1^2+7))$.

5.15. *SECTION*

The polynomial of degree two of the Subcase 1.1 in the proof of statement (c) of Theorem 2 is

$$648c_1^3\beta_1^8 + 19656c_1\beta_1^8 + 1296c_1^3\gamma_1\beta_1^8 + 39312c_1\gamma_1\beta_1^8 - 15120b_2c_1^4\beta_1^7 - 972b_1c_1^2\beta_1^7 - 504b_2c_1^2\beta_1^7 - 3888b_1c_1^2\gamma_1^2\beta_1^7 - 15120b_2c_1^2\gamma_1^2\beta_1^7 -$$
$$39312b_1\gamma_1^2\beta_1^7 + 7056b_2\gamma_1^2\beta_1^7 - 9828b_1\beta_1^7 + 3528b_2\beta_1^7 - 3888b_1c_1^2\gamma_1\beta_1^7 - 15120b_2c_1^2\gamma_1\beta_1^7 - 39312b_1\gamma_1\beta_1^7 + 7056b_2\gamma_1\beta_1^7 + 1296b_1^2c_1^3\beta_1^6 -$$
$$334656b_2^2c_1^3\beta_1^6 + 30240b_1b_2c_1^3\beta_1^6 + 3888b_1^2c_1\gamma_1^3\beta_1^6 + 30240b_1b_2c_1\gamma_1^3\beta_1^6 + 5832b_1^2c_1\gamma_1^2\beta_1^6 + 45360b_1b_2c_1\gamma_1^2\beta_1^6 + 39798b_1^2c_1\beta_1^6 -$$
$$83664b_2^2c_1\beta_1^6 +$$
$$504b_1b_2c_1\beta_1^6 + 2592b_1^2c_1^3\gamma_1\beta_1^6 - 669312b_2^2c_1^3\gamma_1\beta_1^6 + 60480b_1b_2c_1^3\gamma_1\beta_1^6 + 81540b_1^2c_1\gamma_1\beta_1^6 - 167328b_2^2c_1\gamma_1\beta_1^6 + 16128b_1b_2c_1\gamma_1\beta_1^6 -$$
$$150528b_2^3c_1^4\beta_1^5 - 15120b_1^2b_2c_1^4\beta_1^5 - 1296b_1^3\gamma_1^4\beta_1^5 - 15120b_1^2b_2\gamma_1^4\beta_1^5 - 19737b_1^3\beta_1^5 - 18816b_2^3\beta_1^5 - 2592b_1^3\gamma_1^3\beta_1^5 - 30240b_1^2b_2\gamma_1^3\beta_1^5 +$$
$$41832b_1b_2^2\beta_1^5 - 1944b_1^3c_1^2\beta_1^5 - 112896b_2^3c_1^2\beta_1^5 + 501984b_1b_2^2c_1^2\beta_1^5 - 26964b_1^2b_2c_1^2\beta_1^5 - 80568b_1^3\gamma_1^2\beta_1^5 - 37632b_2^3\gamma_1^2\beta_1^5 + 167328b_1b_2^2\gamma_1^2\beta_1^5 -$$
$$7776b_1^3c_1^2\gamma_1^2\beta_1^5 - 150528b_2^3c_1^2\gamma_1^2\beta_1^5 + 2007936b_1b_2^2c_1^2\gamma_1^2\beta_1^5 - 120960b_1^2b_2c_1^2\gamma_1^2\beta_1^5 - 5292b_1^2b_2\gamma_1^2\beta_1^5 + 5166b_1^2b_2\beta_1^5 - 79272b_1^3\gamma_1\beta_1^5 -$$
$$37632b_2^3\gamma_1\beta_1^5 + 167328b_1b_2^2\gamma_1\beta_1^5 - 7776b_1^3c_1^2\gamma_1\beta_1^5 - 150528b_2^3c_1^2\gamma_1\beta_1^5 + 2007936b_1b_2^2c_1^2\gamma_1\beta_1^5 - 120960b_1^2b_2c_1^2\gamma_1\beta_1^5 + 9828b_1^2b_2\gamma_1\beta_1^5 +$$
$$648b_1^4c_1^3\beta_1^4 + 301056b_1b_2^3c_1^3\beta_1^4 - 334656b_1^2b_2^2c_1^3\beta_1^4 + 30240b_1^3b_2c_1^3\beta_1^4 + 7776b_1^4c_1\gamma_1^3\beta_1^4 + 301056b_1b_2^3c_1\gamma_1^3\beta_1^4 - 2007936b_1^2b_2^2c_1\gamma_1^3\beta_1^4 +$$
$$120960b_1^3b_2c_1\gamma_1^3\beta_1^4 + 1332\sqrt{3}\sqrt{R_1}c_1^2\beta_1^4 + 11664b_1^4c_1\gamma_1^2\beta_1^4 + 451584b_1b_2^3c_1\gamma_1^2\beta_1^4 - 3011904b_1^2b_2^2c_1\gamma_1^2\beta_1^4 + 181440b_1^3b_2c_1\gamma_1^2\beta_1^4 -$$
$$252\sqrt{3}\sqrt{R_1}\beta_1^4 - 504\sqrt{3}\gamma_1\sqrt{R_1}\beta_1^4 + 20628b_1^4c_1\beta_1^4 + 112896b_1b_2^3c_1\beta_1^4 - 334656b_1^2b_2^2c_1\beta_1^4 + 11844b_1^3b_2c_1\beta_1^4 + 1296b_1^4c_1^3\gamma_1\beta_1^4 +$$
$$602112b_1b_2^3c_1^3\gamma_1\beta_1^4 - 669312b_1^2b_2^2c_1^3\gamma_1\beta_1^4 + 60480b_1^3b_2c_1^3\gamma_1\beta_1^4 + 2664\sqrt{3}\sqrt{R_1}c_1^2\gamma_1\beta_1^4 + 45144b_1^4c_1\gamma_1\beta_1^4 + 376320b_1b_2^3c_1\gamma_1\beta_1^4 -$$
$$1673280b_1^2b_2^2c_1\gamma_1\beta_1^4 + 84168b_1^3b_2c_1\gamma_1\beta_1^4 - 9990b_1^5\beta_1^3 - 2592b_1^5\gamma_1^4\beta_1^3 - 150528b_1^2b_2^3\gamma_1^4\beta_1^3 + 669312b_1^3b_2^2\gamma_1^4\beta_1^3 - 45360b_1^4b_2\gamma_1^4\beta_1^3 -$$
$$37632b_1^2b_2^3\beta_1^3 + 1848\sqrt{3}\sqrt{R_1}b_2c_1^3\beta_1^3 - 5184b_1^5\gamma_1^3\beta_1^3 - 301056b_1^2b_2^3\gamma_1^3\beta_1^3 + 1338624b_1^3b_2^2\gamma_1^3\beta_1^3 - 90720b_1^4b_2\gamma_1^3\beta_1^3 + 83664b_1^3b_2^2\beta_1^3 -$$
$$972b_1^5c_1^2\beta_1^3 - 263424b_1^2b_2^3c_1^2\beta_1^3 + 501984b_1^3b_2^2c_1^2\beta_1^3 - 26460b_1^4b_2c_1^2\beta_1^3 - 43200b_1^5\gamma_1^2\beta_1^3 - 338688b_1^2b_2^3\gamma_1^2\beta_1^3 + 1171296b_1^3b_2^2\gamma_1^2\beta_1^3 -$$
$$3888b_1^5c_1^2\gamma_1^2\beta_1^3 - 1053696b_1^2b_2^3c_1^2\gamma_1^2\beta_1^3 + 2007936b_1^3b_2^2c_1^2\gamma_1^2\beta_1^3 - 105840b_1^4b_2c_1^2\gamma_1^2\beta_1^3 - 57708b_1^4b_2\gamma_1^2\beta_1^3 + 1848\sqrt{3}\sqrt{R_1}b_2c_1\gamma_1^2\beta_1^3 -$$
$$5328\sqrt{3}b_1c_1\gamma_1^2\sqrt{R_1}\beta_1^3 - 1332\sqrt{3}b_1c_1\sqrt{R_1}\beta_1^3 - 5328\sqrt{3}b_1c_1\gamma_1\sqrt{R_1}\beta_1^3 - 252b_1^4b_2\beta_1^3 + 924\sqrt{3}\sqrt{R_1}b_2c_1\beta_1^3 - 40608b_1^5\gamma_1\beta_1^3 -$$
$$188160b_1^2b_2^3\gamma_1\beta_1^3 + 501984b_1^3b_2^2\gamma_1\beta_1^3 - 3888b_1^5c_1^2\gamma_1\beta_1^3 - 1053696b_1^2b_2^3c_1^2\gamma_1\beta_1^3 + 2007936b_1^3b_2^2c_1^2\gamma_1\beta_1^3 - 105840b_1^4b_2c_1^2\gamma_1\beta_1^3 -$$
$$12348b_1^4b_2\gamma_1\beta_1^3 + 1848\sqrt{3}\sqrt{R_1}b_2c_1\gamma_1\beta_1^3 + 2664\sqrt{3}\sqrt{R_1}b_1^2\gamma_1^3\beta_1^2 + 3888b_1^6c_1\gamma_1^3\beta_1^2 + 903168b_1^3b_2^3c_1\gamma_1^3\beta_1^2 - 2007936b_1^4b_2^2c_1\gamma_1^3\beta_1^2 +$$
$$90720b_1^5b_2c_1\gamma_1^3\beta_1^2 + 81\sqrt{3}\sqrt{R_1}b_1^2\beta_1^2 + 952\sqrt{3}\sqrt{R_1}b_2^2\beta_1^2 + 1332\sqrt{3}\sqrt{R_1}b_1^2c_1^2\beta_1^2 + 3808\sqrt{3}\sqrt{R_1}b_2^2c_1^2\beta_1^2 + 3996\sqrt{3}\sqrt{R_1}b_1^2\gamma_1^2\beta_1^2 +$$
$$5832b_1^6c_1\gamma_1^2\beta_1^2 + 1354752b_1^3b_2^3c_1\gamma_1^2\beta_1^2 - 3011904b_1^4b_2^2c_1\gamma_1^2\beta_1^2 + 136080b_1^5b_2c_1\gamma_1^2\beta_1^2 - 1848\sqrt{3}b_1b_2\gamma_1^3\sqrt{R_1}\beta_1^2 - 2772\sqrt{3}b_1b_2c_1^2\sqrt{R_1}\beta_1^2 -$$
$$2772\sqrt{3}b_1b_2\gamma_1^2\sqrt{R_1}\beta_1^2 - 462\sqrt{3}b_1b_2\sqrt{R_1}\beta_1^2 - 5544\sqrt{3}b_1b_2c_1^2\gamma_1\sqrt{R_1}\beta_1^2 - 1848\sqrt{3}b_1b_2\gamma_1\sqrt{R_1}\beta_1^2 + 486b_1^6c_1\beta_1^2 + 112896b_1^3b_2^3c_1\beta_1^2 -$$
$$250992b_1^4b_2^2c_1\beta_1^2 + 11340b_1^5b_2c_1\beta_1^2 + 1494\sqrt{3}\sqrt{R_1}b_1^2\gamma_1\beta_1^2 + 1904\sqrt{3}\sqrt{R_1}b_2^2\gamma_1\beta_1^2 +$$
$$2664\sqrt{3}\sqrt{R_1}b_1^2c_1^2\gamma_1\beta_1^2 + 7616\sqrt{3}\sqrt{R_1}b_2^2c_1^2\gamma_1\beta_1^2 + 2916b_1^6c_1\gamma_1\beta_1^2 + 677376b_1^3b_2^3c_1\gamma_1\beta_1^2 - 1505952b_1^4b_2^2c_1\gamma_1\beta_1^2 + 68040b_1^5b_2c_1\gamma_1\beta_1^2 -$$
$$81b_1^7\beta_1 - 1296b_1^7\gamma_1^4\beta_1 - 301056b_1^4b_2^3\gamma_1^4\beta_1 + 669312b_1^5b_2^2\gamma_1^4\beta_1 - 30240b_1^6b_2\gamma_1^4\beta_1 - 18816b_1^4b_2^3\beta_1 - 2592b_1^7\gamma_1^3\beta_1 - 602112b_1^4b_2^3\gamma_1^3\beta_1 +$$
$$1338624b_1^5b_2^2\gamma_1^3\beta_1 - 60480b_1^6b_2\gamma_1^3\beta_1 + 41832b_1^5b_2^2\beta_1 - 1944b_1^7\gamma_1^2\beta_1 - 451584b_1^4b_2^3\gamma_1^2\beta_1 + 1003968b_1^5b_2^2\gamma_1^2\beta_1 - 45360b_1^6b_2\gamma_1^2\beta_1 +$$
$$7392\sqrt{3}\sqrt{R_1}b_1^2b_2c_1\gamma_1^2\beta_1 - 5328\sqrt{3}b_1^3c_1\gamma_1^2\sqrt{R_1}\beta_1 - 15232\sqrt{3}b_1b_2^2c_1\gamma_1^2\sqrt{R_1}\beta_1 - 1332\sqrt{3}b_1^3c_1\sqrt{R_1}\beta_1 - 3808\sqrt{3}b_1b_2^2c_1\sqrt{R_1}\beta_1 -$$
$$5328\sqrt{3}b_1^3c_1\gamma_1\sqrt{R_1}\beta_1 - 15232\sqrt{3}b_1b_2^2c_1\gamma_1\sqrt{R_1}\beta_1 - 1890b_1^6b_2\beta_1 + 1848\sqrt{3}\sqrt{R_1}b_1^2b_2c_1\beta_1 - 648b_1^7\gamma_1\beta_1 - 150528b_1^4b_2^3\gamma_1\beta_1 +$$
$$334656b_1^5b_2^2\gamma_1\beta_1 - 15120b_1^6b_2\gamma_1\beta_1 + 7392\sqrt{3}\sqrt{R_1}b_1^2b_2c_1\gamma_1\beta_1 + 333\sqrt{3}\sqrt{R_1}b_1^4 + 2664\sqrt{3}\sqrt{R_1}b_1^4\gamma_1^3 + 7616\sqrt{3}\sqrt{R_1}b_1^2b_2^2\gamma_1^3 +$$
$$952\sqrt{3}\sqrt{R_1}b_1^2b_2^2 + (-15120b_2c_1^2\beta_1^9 + 7056b_2\beta_1^9 + 15120b_1b_2c_1\beta_1^8 + 30240b_1b_2c_1\gamma_1\beta_1^8 - 37632b_2^3\beta_1^7 - 150528b_2^3c_1^2\beta_1^7 -$$
$$30240b_1^2b_2c_1^2\beta_1^7 - 15120b_1^2b_2\gamma_1^2\beta_1^7 + 10332b_1^2b_2\beta_1^7 - 15120b_1^2b_2\gamma_1\beta_1^7 + 150528b_1b_2^3c_1\beta_1^6 + 30240b_1^3b_2c_1\beta_1^6 + 301056b_1b_2^3c_1\gamma_1\beta_1^6 +$$
$$60480b_1^3b_2c_1\gamma_1\beta_1^6 - 75264b_1^2b_2^3\beta_1^5 - 150528b_1^2b_2^3c_1^2\beta_1^5 - 15120b_1^4b_2c_1^2\beta_1^5 - 150528b_1^2b_2^3\gamma_1^2\beta_1^5 - 30240b_1^4b_2\gamma_1^2\beta_1^5 - 504b_1^4b_2\beta_1^5 +$$
$$1848\sqrt{3}\sqrt{R_1}b_2c_1\beta_1^5 - 150528b_1^2b_2^3\gamma_1\beta_1^5 - 30240b_1^4b_2\gamma_1\beta_1^5 - 924\sqrt{3}b_1b_2\sqrt{R_1}\beta_1^4 - 1848\sqrt{3}b_1b_2\gamma_1\sqrt{R_1}\beta_1^4 + 150528b_1^3b_2^3c_1\beta_1^4 +$$
$$15120b_1^5b_2c_1\beta_1^4 + 301056b_1^3b_2^3c_1\gamma_1\beta_1^4 + 30240b_1^5b_2c_1\gamma_1\beta_1^4 - 37632b_1^4b_2^3\beta_1^3 - 150528b_1^4b_2^3\gamma_1^2\beta_1^3 - 15120b_1^6b_2\gamma_1^2\beta_1^3 - 3780b_1^6b_2\beta_1^3 +$$
$$1848\sqrt{3}\sqrt{R_1}b_1^2b_2c_1\beta_1^3 - 150528b_1^4b_2^3\gamma_1\beta_1^3 - 15120b_1^6b_2\gamma_1\beta_1^3 - 924\sqrt{3}b_1^3b_2\sqrt{R_1}\beta_1^2 - 1848\sqrt{3}b_1^3b_2\gamma_1\sqrt{R_1}\beta_1^2)y_2^2 + 3996\sqrt{3}\sqrt{R_1}b_1^4\gamma_1^2 +$$
$$11424\sqrt{3}\sqrt{R_1}b_1^2b_2^2\gamma_1^2 - 3696\sqrt{3}b_1^3b_2\gamma_1^3\sqrt{R_1} - 5544\sqrt{3}b_1^3b_2\gamma_1^2\sqrt{R_1} - 462\sqrt{3}b_1^3b_2\sqrt{R_1} - 2772\sqrt{3}b_1^3b_2\gamma_1\sqrt{R_1} + (1296c_1^3\beta_1^9 +$$
$$39312c_1\beta_1^9 - 1944b_1c_1^2\beta_1^8 - 15120b_2c_1^2\beta_1^8 - 19656b_1\beta_1^8 + 7056b_2\beta_1^8 - 3888b_1c_1^2\gamma_1\beta_1^8 - 30240b_2c_1^2\gamma_1\beta_1^8 - 39312b_1\gamma_1\beta_1^8 +$$

$14112b_2\gamma_1\beta_1^8 + 2592b_1^2c_1^3\beta_1^7 - 669312b_2^2c_1^3\beta_1^7 + 3888b_1^2c_1\gamma_1^2\beta_1^7 + 60480b_1b_2c_1\gamma_1^2\beta_1^7 + 79596b_1^2c_1\beta_1^7 - 167328b_2^2c_1\beta_1^7 +$
$15120b_1b_2c_1\beta_1^7 + 3888b_1^2c_1\gamma_1\beta_1^7 + 60480b_1b_2c_1\gamma_1\beta_1^7 - 39474b_1^3\beta_1^6 - 37632b_2^3\beta_1^6 - 1296b_1^3\gamma_1^3\beta_1^6 - 30240b_1^2b_2\gamma_1^3\beta_1^6 + 83664b_1b_2^2\beta_1^6 -$
$3888b_1^3c_1^2\beta_1^6 - 150528b_2^3c_1^2\beta_1^6 + 1003968b_1b_2^2c_1^2\beta_1^6 - 30240b_1^2b_2c_1^2\beta_1^6 - 1944b_1^3\gamma_1^2\beta_1^6 - 45360b_1^2b_2\gamma_1^2\beta_1^6 + 10332b_1^2b_2\beta_1^6 -$
$79596b_1^3\gamma_1\beta_1^6 - 75264b_2^3\gamma_1\beta_1^6 + 167328b_1b_2^2\gamma_1\beta_1^6 - 7776b_1^3c_1^2\gamma_1\beta_1^6 - 301056b_2^3c_1^2\gamma_1\beta_1^6 + 2007936b_1b_2^2c_1^2\gamma_1\beta_1^6 - 60480b_1^2b_2c_1^2\gamma_1\beta_1^6 +$
$5544b_1^2b_2\gamma_1\beta_1^6 + 1296b_1^4c_1^3\beta_1^5 - 669312b_1^2b_2^2c_1^3\beta_1^5 + 2664\sqrt{3}\sqrt{R_1}c_1^2\beta_1^5 + 7776b_1^4c_1\gamma_1^2\beta_1^5 + 602112b_1b_2^3c_1\gamma_1^2\beta_1^5 - 2007936b_1^2b_2^2c_1\gamma_1^2\beta_1^5 +$
$120960b_1^3b_2c_1\gamma_1^2\beta_1^5 - 504\sqrt{3}\sqrt{R_1}\beta_1^5 + 41256b_1^4c_1\beta_1^5 + 150528b_1b_2^3c_1\beta_1^5 - 669312b_1^2b_2^2c_1\beta_1^5 + 30240b_1^3b_2c_1\beta_1^5 + 7776b_1^4c_1\gamma_1\beta_1^5 +$
$602112b_1b_2^3c_1\gamma_1\beta_1^5 - 2007936b_1^2b_2^2c_1\gamma_1\beta_1^5 + 120960b_1^3b_2c_1\gamma_1\beta_1^5 - 19980b_1^5\beta_1^4 - 75264b_1^2b_2^3\beta_1^4 - 2592b_1^5\gamma_1^3\beta_1^4 - 301056b_1^2b_2^3\gamma_1^3\beta_1^4 +$
$669312b_1^3b_2^2\gamma_1^3\beta_1^4 - 60480b_1^4b_2\gamma_1^3\beta_1^4 + 167328b_1^3b_2^2\beta_1^4 - 1944b_1^5c_1^2\beta_1^4 - 150528b_1^2b_2^3c_1^2\beta_1^4 + 1003968b_1^3b_2^2c_1^2\beta_1^4 - 15120b_1^4b_2c_1^2\beta_1^4 -$
$3888b_1^5\gamma_1^2\beta_1^4 - 451584b_1^2b_2^3\gamma_1^2\beta_1^4 + 1003968b_1^3b_2^2\gamma_1^2\beta_1^4 - 90720b_1^4b_2\gamma_1^2\beta_1^4 - 2664\sqrt{3}b_1c_1\sqrt{R_1}\beta_1^4 - 5328\sqrt{3}b_1c_1\gamma_1\sqrt{R_1}\beta_1^4 -$
$504b_1^4b_2\beta_1^4 + 1848\sqrt{3}\sqrt{R_1}b_2c_1\beta_1^4 - 41256b_1^5\gamma_1\beta_1^4 - 301056b_1^2b_2^3\gamma_1\beta_1^4 + 669312b_1^3b_2^2\gamma_1\beta_1^4 - 3888b_1^5c_1^2\gamma_1\beta_1^4 - 301056b_1^2b_2^3c_1^2\gamma_1\beta_1^4 +$
$2007936b_1^3b_2^2c_1^2\gamma_1\beta_1^4 - 30240b_1^4b_2c_1^2\gamma_1\beta_1^4 - 31248b_1^4b_2\gamma_1\beta_1^4 + 3696\sqrt{3}\sqrt{R_1}b_2c_1\gamma_1\beta_1^4 + 162\sqrt{3}\sqrt{R_1}b_1^2\beta_1^3 + 1904\sqrt{3}\sqrt{R_1}b_2^2\beta_1^3 +$
$2664\sqrt{3}\sqrt{R_1}b_1^2c_1^2\beta_1^3 + 7616\sqrt{3}\sqrt{R_1}b_2^2c_1^2\beta_1^3 + 2664\sqrt{3}\sqrt{R_1}b_1^2\gamma_1^2\beta_1^3 + 3888b_1^6c_1\gamma_1^2\beta_1^3 + 602112b_1^3b_2^3c_1\gamma_1^2\beta_1^3 - 2007936b_1^4b_2^2c_1\gamma_1^2\beta_1^3 +$
$60480b_1^5b_2c_1\gamma_1^2\beta_1^3 - 3696\sqrt{3}b_1b_2\gamma_1^2\sqrt{R_1}\beta_1^3 - 924\sqrt{3}b_1b_2\sqrt{R_1}\beta_1^3 - 3696\sqrt{3}b_1b_2\gamma_1\sqrt{R_1}\beta_1^3 + 972b_1^6c_1\beta_1^3 + 150528b_1^3b_2^3c_1\beta_1^3 -$
$501984b_1^4b_2^2c_1\beta_1^3 + 15120b_1^5b_2c_1\beta_1^3 + 2664\sqrt{3}\sqrt{R_1}b_1^2\gamma_1\beta_1^3 + 3888b_1^6c_1\gamma_1\beta_1^3 + 602112b_1^3b_2^3c_1\gamma_1\beta_1^3 - 2007936b_1^4b_2^2c_1\gamma_1\beta_1^3 +$
$60480b_1^5b_2c_1\gamma_1\beta_1^3 - 162b_1^7\beta_1^2 - 37632b_1^4b_2^3\beta_1^2 - 1296b_1^7\gamma_1^3\beta_1^2 - 301056b_1^4b_2^3\gamma_1^3\beta_1^2 + 669312b_1^5b_2^2\gamma_1^3\beta_1^2 - 30240b_1^6b_2\gamma_1^3\beta_1^2 +$
$83664b_1^5b_2^2\beta_1^2 - 1944b_1^7\gamma_1^2\beta_1^2 - 451584b_1^4b_2^3\gamma_1^2\beta_1^2 + 1003968b_1^5b_2^2\gamma_1^2\beta_1^2 - 45360b_1^6b_2\gamma_1^2\beta_1^2 - 2664\sqrt{3}b_1^3c_1\sqrt{R_1}\beta_1^2 - 7616\sqrt{3}b_1b_2^2c_1\sqrt{R_1}\beta_1^2 -$
$5328\sqrt{3}b_1^3c_1\gamma_1\sqrt{R_1}\beta_1^2 - 15232\sqrt{3}b_1b_2^2c_1\gamma_1\sqrt{R_1}\beta_1^2 - 3780b_1^6b_2\beta_1^2 + 1848\sqrt{3}\sqrt{R_1}b_1^2b_2c_1\beta_1^2 - 972b_1^7\gamma_1\beta_1^2 - 225792b_1^4b_2^3\gamma_1\beta_1^2 +$
$501984b_1^5b_2^2\gamma_1\beta_1^2 - 22680b_1^6b_2\gamma_1\beta_1^2 + 3696\sqrt{3}\sqrt{R_1}b_1^2b_2c_1\gamma_1\beta_1^2 + 666\sqrt{3}\sqrt{R_1}b_1^4\beta_1 + 1904\sqrt{3}\sqrt{R_1}b_1^2b_2^2\beta_1 + 2664\sqrt{3}\sqrt{R_1}b_1^4\gamma_1^2\beta_1 +$
$7616\sqrt{3}\sqrt{R_1}b_1^2b_2^2\gamma_1^2\beta_1 - 3696\sqrt{3}b_1^3b_2\gamma_1^2\sqrt{R_1}\beta_1 - 924\sqrt{3}b_1^3b_2\sqrt{R_1}\beta_1 - 3696\sqrt{3}b_1^3b_2\gamma_1\sqrt{R_1}\beta_1 + 2664\sqrt{3}\sqrt{R_1}b_1^4\gamma_1\beta_1 +$
$7616\sqrt{3}\sqrt{R_1}b_1^2b_2^2\gamma_1\beta_1)y_2 + 1998\sqrt{3}\sqrt{R_1}b_1^4\gamma_1 + 5712\sqrt{3}\sqrt{R_1}b_1^2b_2^2\gamma_1 = 0,$
where
$R_1 = \beta_1^2(b_1^2+\beta_1^2)(-9(2\gamma_1+1)^2b_1^4 + 36c_1\beta_1(2\gamma_1+1)b_1^3 + (448(2\gamma_1b_2+b_2)^2 - 3\beta_1^2(12c_1^2+12\gamma_1(\gamma_1+1)+31))b_1^2 + 4c_1\beta_1(9\beta_1^2 -$
$448b_2^2)(2\gamma_1+1)b_1 - 12(3c_1^2+7)\beta_1^4 + 448b_2^2(4c_1^2+1)\beta_1^2).$

5.16. *SECTION*

The polynomial of degree two of the Subcase 1.2 in the proof of statement (c) of Theorem 2 is
$-648c_1^3\beta_1^8 - 19656c_1\beta_1^8 - 1296c_1^3\gamma_1\beta_1^8 - 39312c_1\gamma_1\beta_1^8 + 15120b_2c_1^4\beta_1^7 + 972b_1c_1^2\beta_1^7 + 504b_2c_1^2\beta_1^7 + 3888b_1c_1^2\gamma_1^2\beta_1^7 + 15120b_2c_1^2\gamma_1^2\beta_1^7 +$
$39312b_1\gamma_1^2\beta_1^7 - 7056b_2\gamma_1^2\beta_1^7 + 9828b_1\beta_1^7 - 3528b_2\beta_1^7 + 3888b_1c_1^2\gamma_1\beta_1^7 + 15120b_2c_1^2\gamma_1\beta_1^7 + 39312b_1\gamma_1\beta_1^7 - 7056b_2\gamma_1\beta_1^7 - 1296b_1^2c_1^3\beta_1^6 +$
$334656b_2^2c_1^3\beta_1^6 - 30240b_1b_2c_1^3\beta_1^6 - 3888b_1^2c_1\gamma_1^3\beta_1^6 - 30240b_1b_2c_1\gamma_1^3\beta_1^6 - 5832b_1^2c_1\gamma_1^2\beta_1^6 - 45360b_1b_2c_1\gamma_1^2\beta_1^6 - 39798b_1^2c_1\beta_1^6 +$
$83664b_2^2c_1\beta_1^6 -$
$504b_1b_2c_1\beta_1^6 - 2592b_1^2c_1^3\gamma_1\beta_1^6 + 669312b_2^2c_1^3\gamma_1\beta_1^6 - 60480b_1b_2c_1^3\gamma_1\beta_1^6 - 81540b_1^2c_1\gamma_1\beta_1^6 + 167328b_2^2c_1\gamma_1\beta_1^6 - 16128b_1b_2c_1\gamma_1\beta_1^6 +$
$150528b_2^3c_1^4\beta_1^5 + 15120b_1^2b_2c_1^4\beta_1^5 + 1296b_1^3\gamma_1^4\beta_1^5 + 15120b_1^2b_2\gamma_1^4\beta_1^5 + 19737b_1^3\beta_1^5 + 18816b_2^3\beta_1^5 + 2592b_1^3\gamma_1^3\beta_1^5 + 30240b_1^2b_2\gamma_1^3\beta_1^5 -$
$41832b_1b_2^2\beta_1^5 + 1944b_1^3c_1^2\beta_1^5 + 112896b_2^3c_1^2\beta_1^5 - 501984b_1b_2^2c_1^2\beta_1^5 + 26964b_1^2b_2c_1^2\beta_1^5 + 80568b_1^3\gamma_1^2\beta_1^5 + 37632b_2^3\gamma_1^2\beta_1^5 - 167328b_1b_2^2\gamma_1^2\beta_1^5 +$
$7776b_1^3c_1^2\gamma_1^2\beta_1^5 + 150528b_2^3c_1^2\gamma_1^2\beta_1^5 - 2007936b_1b_2^2c_1^2\gamma_1^2\beta_1^5 + 120960b_1^2b_2c_1^2\gamma_1^2\beta_1^5 + 5292b_1^2b_2\gamma_1^2\beta_1^5 - 5166b_1^2b_2\beta_1^5 + 79272b_1^3\gamma_1\beta_1^5 +$
$37632b_2^3\gamma_1\beta_1^5 - 167328b_1b_2^2\gamma_1\beta_1^5 + 7776b_1^3c_1^2\gamma_1\beta_1^5 + 150528b_2^3c_1^2\gamma_1\beta_1^5 - 2007936b_1b_2^2c_1^2\gamma_1\beta_1^5 + 120960b_1^2b_2c_1^2\gamma_1\beta_1^5 - 9828b_1^2b_2\gamma_1\beta_1^5 -$
$648b_1^4c_1^3\beta_1^4 - 301056b_1b_2^3c_1^3\beta_1^4 + 334656b_1^2b_2^2c_1^3\beta_1^4 - 30240b_1^3b_2c_1^3\beta_1^4 - 7776b_1^4c_1\gamma_1^3\beta_1^4 - 301056b_1b_2^3c_1\gamma_1^3\beta_1^4 + 2007936b_1^2b_2^2c_1\gamma_1^3\beta_1^4 -$
$120960b_1^3b_2c_1\gamma_1^3\beta_1^4 + 1332\sqrt{3}\sqrt{R_2}c_1^2\beta_1^4 - 11664b_1^4c_1\gamma_1^2\beta_1^4 - 451584b_1b_2^3c_1\gamma_1^2\beta_1^4 + 3011904b_1^2b_2^2c_1\gamma_1^2\beta_1^4 - 181440b_1^3b_2c_1\gamma_1^2\beta_1^4 -$
$252\sqrt{3}\sqrt{R_2}\beta_1^4 - 504\sqrt{3}\gamma_1\sqrt{R_2}\beta_1^4 - 20628b_1^4c_1\beta_1^4 - 112896b_1b_2^3c_1\beta_1^4 + 334656b_1^2b_2^2c_1\beta_1^4 - 11844b_1^3b_2c_1\beta_1^4 - 1296b_1^4c_1^3\gamma_1\beta_1^4 -$
$602112b_1b_2^3c_1^3\gamma_1\beta_1^4 + 669312b_1^2b_2^2c_1^3\gamma_1\beta_1^4 - 60480b_1^3b_2c_1^3\gamma_1\beta_1^4 + 2664\sqrt{3}\sqrt{R_2}c_1^2\gamma_1\beta_1^4 - 45144b_1^4c_1\gamma_1\beta_1^4 - 376320b_1b_2^3c_1\gamma_1\beta_1^4 +$
$1673280b_1^2b_2^2c_1\gamma_1\beta_1^4 - 84168b_1^3b_2c_1\gamma_1\beta_1^4 + 9990b_1^5\beta_1^3 + 2592b_1^5\gamma_1^4\beta_1^3 + 150528b_1^2b_2^3\gamma_1^4\beta_1^3 - 669312b_1^3b_2^2\gamma_1^4\beta_1^3 + 45360b_1^4b_2\gamma_1^4\beta_1^3 +$
$37632b_1^2b_2^3\beta_1^3 + 1848\sqrt{3}\sqrt{R_2}b_2c_1^3\beta_1^3 + 5184b_1^5\gamma_1^3\beta_1^3 + 301056b_1^2b_2^3\gamma_1^3\beta_1^3 - 1338624b_1^3b_2^2\gamma_1^3\beta_1^3 + 90720b_1^4b_2\gamma_1^3\beta_1^3 - 83664b_1^3b_2^2\beta_1^3 +$
$972b_1^5c_1^2\beta_1^3 + 263424b_1^2b_2^3c_1^2\beta_1^3 - 501984b_1^3b_2^2c_1^2\beta_1^3 + 26460b_1^4b_2c_1^2\beta_1^3 + 43200b_1^5\gamma_1^2\beta_1^3 + 338688b_1^2b_2^3\gamma_1^2\beta_1^3 - 1171296b_1^3b_2^2\gamma_1^2\beta_1^3 +$
$3888b_1^5c_1^2\gamma_1^2\beta_1^3 + 1053696b_1^2b_2^3c_1^2\gamma_1^2\beta_1^3 - 2007936b_1^3b_2^2c_1^2\gamma_1^2\beta_1^3 + 105840b_1^4b_2c_1^2\gamma_1^2\beta_1^3 + 57708b_1^4b_2\gamma_1^2\beta_1^3 + 1848\sqrt{3}\sqrt{R_2}b_2c_1\gamma_1^2\beta_1^3 -$
$5328\sqrt{3}b_1c_1\gamma_1^2\sqrt{R_2}\beta_1^3 - 1332\sqrt{3}b_1c_1\sqrt{R_2}\beta_1^3 - 5328\sqrt{3}b_1c_1\gamma_1\sqrt{R_2}\beta_1^3 + 252b_1^4b_2\beta_1^3 + 924\sqrt{3}\sqrt{R_2}b_2c_1\beta_1^3 + 40608b_1^5\gamma_1\beta_1^3 +$
$188160b_1^2b_2^3\gamma_1\beta_1^3 - 501984b_1^3b_2^2\gamma_1\beta_1^3 + 3888b_1^5c_1^2\gamma_1\beta_1^3 + 1053696b_1^2b_2^3c_1^2\gamma_1\beta_1^3 - 2007936b_1^3b_2^2c_1^2\gamma_1\beta_1^3 + 105840b_1^4b_2c_1^2\gamma_1\beta_1^3 +$
$12348b_1^4b_2\gamma_1\beta_1^3 + 1848\sqrt{3}\sqrt{R_2}b_2c_1\gamma_1\beta_1^3 + 2664\sqrt{3}\sqrt{R_2}b_1^2\gamma_1^3\beta_1^2 - 3888b_1^6c_1\gamma_1^3\beta_1^2 - 903168b_1^3b_2^3c_1\gamma_1^3\beta_1^2 + 2007936b_1^4b_2^2c_1\gamma_1^3\beta_1^2 -$
$90720b_1^5b_2c_1\gamma_1^3\beta_1^2 + 81\sqrt{3}\sqrt{R_2}b_1^2\beta_1^2 + 952\sqrt{3}\sqrt{R_2}b_2^2\beta_1^2 + 1332\sqrt{3}\sqrt{R_2}b_1^2c_1^2\beta_1^2 + 3808\sqrt{3}\sqrt{R_2}b_2^2c_1^2\beta_1^2 + 3996\sqrt{3}\sqrt{R_2}b_1^2\gamma_1^2\beta_1^2 -$
$5832b_1^6c_1\gamma_1^2\beta_1^2 - 1354752b_1^3b_2^3c_1\gamma_1^2\beta_1^2 + 3011904b_1^4b_2^2c_1\gamma_1^2\beta_1^2 - 136080b_1^5b_2c_1\gamma_1^2\beta_1^2 - 1848\sqrt{3}b_1b_2\gamma_1^3\sqrt{R_2}\beta_1^2 - 2772\sqrt{3}b_1b_2c_1^2\sqrt{R_2}\beta_1^2 -$
$2772\sqrt{3}b_1b_2\gamma_1^2\sqrt{R_2}\beta_1^2 - 462\sqrt{3}b_1b_2\sqrt{R_2}\beta_1^2 - 5544\sqrt{3}b_1b_2c_1^2\gamma_1\sqrt{R_2}\beta_1^2 - 1848\sqrt{3}b_1b_2\gamma_1\sqrt{R_2}\beta_1^2 - 486b_1^6c_1\beta_1^2 - 112896b_1^3b_2^3c_1\beta_1^2 +$
$250992b_1^4b_2^2c_1\beta_1^2 - 11340b_1^5b_2c_1\beta_1^2 + 1494\sqrt{3}\sqrt{R_2}b_1^2\gamma_1\beta_1^2 + 1904\sqrt{3}\sqrt{R_2}b_2^2\gamma_1\beta_1^2 + 2664\sqrt{3}\sqrt{R_2}b_1^2c_1^2\gamma_1\beta_1^2 + 7616\sqrt{3}\sqrt{R_2}b_2^2c_1^2\gamma_1\beta_1^2 -$
$2916b_1^6c_1\gamma_1\beta_1^2 - 677376b_1^3b_2^3c_1\gamma_1\beta_1^2 + 1505952b_1^4b_2^2c_1\gamma_1\beta_1^2 - 68040b_1^5b_2c_1\gamma_1\beta_1^2 + 81b_1^7\beta_1 + 1296b_1^7\gamma_1^4\beta_1 +$
$301056b_1^4b_2^3\gamma_1^4\beta_1 - 669312b_1^5b_2^2\gamma_1^4\beta_1 + 30240b_1^6b_2\gamma_1^4\beta_1 + 18816b_1^4b_2^3\beta_1 + 2592b_1^7\gamma_1^3\beta_1 + 602112b_1^4b_2^3\gamma_1^3\beta_1 - 1338624b_1^5b_2^2\gamma_1^3\beta_1 +$
$60480b_1^6b_2\gamma_1^3\beta_1 - 41832b_1^5b_2^2\beta_1 + 1944b_1^7\gamma_1^2\beta_1 + 451584b_1^4b_2^3\gamma_1^2\beta_1 - 1003968b_1^5b_2^2\gamma_1^2\beta_1 + 45360b_1^6b_2\gamma_1^2\beta_1 + 7392\sqrt{3}\sqrt{R_2}b_1^2b_2c_1\gamma_1^2\beta_1 -$
$5328\sqrt{3}b_1^3c_1\gamma_1^2\sqrt{R_2}\beta_1 - 15232\sqrt{3}b_1b_2^2c_1\gamma_1^2\sqrt{R_2}\beta_1 - 1332\sqrt{3}b_1^3c_1\sqrt{R_2}\beta_1 - 3808\sqrt{3}b_1b_2^2c_1\sqrt{R_2}\beta_1 - 5328\sqrt{3}b_1^3c_1\gamma_1\sqrt{R_2}\beta_1 -$
$15232\sqrt{3}b_1b_2^2c_1\gamma_1\sqrt{R_2}\beta_1 + 1890b_1^6b_2\beta_1 + 1848\sqrt{3}\sqrt{R_2}b_1^2b_2c_1\beta_1 + 648b_1^7\gamma_1\beta_1 + 150528b_1^4b_2^3\gamma_1\beta_1 - 334656b_1^5b_2^2\gamma_1\beta_1 +$
$15120b_1^6b_2\gamma_1\beta_1 + 7392\sqrt{3}\sqrt{R_2}b_1^2b_2c_1\gamma_1\beta_1 + 333\sqrt{3}\sqrt{R_2}b_1^4 + 2664\sqrt{3}\sqrt{R_2}b_1^4\gamma_1^3 + 7616\sqrt{3}\sqrt{R_2}b_1^2b_2^2\gamma_1^3 + 952\sqrt{3}\sqrt{R_2}b_1^2b_2^2 +$
$(15120b_2c_1^2\beta_1^9 - 7056b_2\beta_1^9 - 15120b_1b_2c_1\beta_1^8 - 30240b_1b_2c_1\gamma_1\beta_1^8 + 37632b_2^3\beta_1^7 + 150528b_2^3c_1^2\beta_1^7 + 30240b_1^2b_2c_1^2\beta_1^7 + 15120b_1^2b_2\gamma_1^2\beta_1^7 -$
$10332b_1^2b_2\beta_1^7 + 15120b_1^2b_2\gamma_1\beta_1^7 - 150528b_1b_2^3c_1\beta_1^6 - 30240b_1^3b_2c_1\beta_1^6 - 301056b_1b_2^3c_1\gamma_1\beta_1^6 - 60480b_1^3b_2c_1\gamma_1\beta_1^6 + 75264b_1^2b_2^3\beta_1^5 +$
$150528b_1^2b_2^3c_1^2\beta_1^5 + 15120b_1^4b_2c_1^2\beta_1^5 + 150528b_1^2b_2^3\gamma_1^2\beta_1^5 + 30240b_1^4b_2\gamma_1^2\beta_1^5 + 504b_1^4b_2\beta_1^5 + 1848\sqrt{3}\sqrt{R_2}b_2c_1\beta_1^5 + 150528b_1^2b_2^3\gamma_1\beta_1^5 +$
$30240b_1^4b_2\gamma_1\beta_1^5 - 924\sqrt{3}b_1b_2\sqrt{R_2}\beta_1^4 - 1848\sqrt{3}b_1b_2\gamma_1\sqrt{R_2}\beta_1^4 - 150528b_1^3b_2^3c_1\beta_1^4 - 15120b_1^5b_2c_1\beta_1^4 - 301056b_1^3b_2^3c_1\gamma_1\beta_1^4 -$

$30240b_1^5b_2c_1\gamma_1\beta_1^4+37632b_1^4b_2^3\beta_1^3+150528b_1^4b_2^3\gamma_1^2\beta_1^3+15120b_1^6b_2\gamma_1^2\beta_1^3+3780b_1^6b_2\beta_1^3+1848\sqrt{3}\sqrt{R_2}b_1^2b_2c_1\beta_1^3+150528b_1^4b_2^3\gamma_1\beta_1^3+15120b_1^6b_2\gamma_1\beta_1^3-924\sqrt{3}b_1^3b_2\sqrt{R_2}\beta_1^2-1848\sqrt{3}b_1^3b_2\gamma_1\sqrt{R_2}\beta_1^2)y_2^2+3996\sqrt{3}\sqrt{R_2}b_1^4\gamma_1^2+11424\sqrt{3}\sqrt{R_2}b_1^2b_2^2\gamma_1^2-3696\sqrt{3}b_1^3b_2\gamma_1^3\sqrt{R_2}-5544\sqrt{3}b_1^3b_2\gamma_1^2\sqrt{R_2}-462\sqrt{3}b_1^3b_2\sqrt{R_2}-2772\sqrt{3}b_1^3b_2\gamma_1\sqrt{R_2}+(-1296c_1^3\beta_1^9-39312c_1\beta_1^9+1944b_1c_1^2\beta_1^8+15120b_2c_1^2\beta_1^8+19656b_1\beta_1^8-7056b_2\beta_1^8+3888b_1c_1^2\gamma_1\beta_1^8+30240b_2c_1^2\gamma_1\beta_1^8+39312b_1\gamma_1\beta_1^8-14112b_2\gamma_1\beta_1^8-2592b_1^2c_1^3\beta_1^7+669312b_2^2c_1^3\beta_1^7-3888b_1^2c_1\gamma_1^2\beta_1^7-60480b_1b_2c_1\gamma_1^2\beta_1^7-79596b_1^2c_1\beta_1^7+167328b_2^2c_1\beta_1^7-15120b_1b_2c_1\beta_1^7-3888b_1^2c_1\gamma_1\beta_1^7-60480b_1b_2c_1\gamma_1\beta_1^7+39474b_1^3\beta_1^6+37632b_2^3\beta_1^6+1296b_1^3\gamma_1^3\beta_1^6+30240b_1^2b_2\gamma_1^3\beta_1^6-83664b_1b_2^2\beta_1^6+3888b_1^3c_1^2\beta_1^6+150528b_2^3c_1^2\beta_1^6-1003968b_1b_2^2c_1^2\beta_1^6+30240b_1^2b_2c_1^2\beta_1^6+1944b_1^3\gamma_1^2\beta_1^6+45360b_1^2b_2\gamma_1^2\beta_1^6-10332b_1^2b_2\beta_1^6+79596b_1^3\gamma_1\beta_1^6+75264b_2^3\gamma_1\beta_1^6-167328b_1b_2^2\gamma_1\beta_1^6+7776b_1^3c_1^2\gamma_1\beta_1^6+301056b_2^3c_1^2\gamma_1\beta_1^6-2007936b_1b_2^2c_1^2\gamma_1\beta_1^6+60480b_1^2b_2c_1^2\gamma_1\beta_1^6-5544b_1^2b_2\gamma_1\beta_1^6-1296b_1^4c_1^3\beta_1^5+669312b_1^2b_2^2c_1^3\beta_1^5+2664\sqrt{3}\sqrt{R_2}c_1^2\beta_1^5-7776b_1^4c_1\gamma_1^2\beta_1^5-602112b_1b_2^3c_1\gamma_1^2\beta_1^5+2007936b_1^2b_2^2c_1\gamma_1^2\beta_1^5-120960b_1^3b_2c_1\gamma_1^2\beta_1^5-504\sqrt{3}\sqrt{R_2}\beta_1^5-41256b_1^4c_1\beta_1^5-150528b_1b_2^3c_1\beta_1^5+669312b_1^2b_2^2c_1\beta_1^5-30240b_1^3b_2c_1\beta_1^5-7776b_1^4c_1\gamma_1\beta_1^5-602112b_1b_2^3c_1\gamma_1\beta_1^5+2007936b_1^2b_2^2c_1\gamma_1\beta_1^5-120960b_1^3b_2c_1\gamma_1\beta_1^5+19980b_1^5\beta_1^4+75264b_1^2b_2^3\beta_1^4+2592b_1^5\gamma_1^3\beta_1^4+301056b_1^2b_2^3\gamma_1^3\beta_1^4-669312b_1^3b_2^2\gamma_1^3\beta_1^4+60480b_1^4b_2\gamma_1^3\beta_1^4-167328b_1^3b_2^2\beta_1^4+1944b_1^5c_1^2\beta_1^4+150528b_1^2b_2^3c_1^2\beta_1^4-1003968b_1^3b_2^2c_1^2\beta_1^4+15120b_1^4b_2c_1^2\beta_1^4+3888b_1^5\gamma_1^2\beta_1^4+451584b_1^2b_2^3\gamma_1^2\beta_1^4-1003968b_1^3b_2^2\gamma_1^2\beta_1^4+90720b_1^4b_2\gamma_1^2\beta_1^4-2664\sqrt{3}b_1c_1\sqrt{R_2}\beta_1^4-5328\sqrt{3}b_1c_1\gamma_1\sqrt{R_2}\beta_1^4+504b_1^4b_2\beta_1^4+1848\sqrt{3}\sqrt{R_2}b_2c_1\beta_1^4+41256b_1^5\gamma_1\beta_1^4+301056b_1^2b_2^3\gamma_1\beta_1^4-669312b_1^3b_2^2\gamma_1\beta_1^4+3888b_1^5c_1^2\gamma_1\beta_1^4+301056b_1^2b_2^3c_1^2\gamma_1\beta_1^4-2007936b_1^3b_2^2c_1^2\gamma_1\beta_1^4+30240b_1^4b_2c_1^2\gamma_1\beta_1^4+31248b_1^4b_2\gamma_1\beta_1^4+3696\sqrt{3}\sqrt{R_2}b_2c_1\gamma_1\beta_1^4+162\sqrt{3}\sqrt{R_2}b_1^2\beta_1^3+1904\sqrt{3}\sqrt{R_2}b_2^2\beta_1^3+2664\sqrt{3}\sqrt{R_2}b_1^2c_1^2\beta_1^3+7616\sqrt{3}\sqrt{R_2}b_2^2c_1^2\beta_1^3+2664\sqrt{3}\sqrt{R_2}b_1^2\gamma_1^2\beta_1^3-3888b_1^6c_1\gamma_1^2\beta_1^3-602112b_1^3b_2^3c_1\gamma_1^2\beta_1^3+2007936b_1^4b_2^2c_1\gamma_1^2\beta_1^3-60480b_1^5b_2c_1\gamma_1^2\beta_1^3-3696\sqrt{3}b_1b_2\gamma_1^2\sqrt{R_2}\beta_1^3-924\sqrt{3}b_1b_2\sqrt{R_2}\beta_1^3-3696\sqrt{3}b_1b_2\gamma_1\sqrt{R_2}\beta_1^3-972b_1^6c_1\beta_1^3-150528b_1^3b_2^3c_1\beta_1^3+501984b_1^4b_2^2c_1\beta_1^3-15120b_1^5b_2c_1\beta_1^3+2664\sqrt{3}\sqrt{R_2}b_1^2\gamma_1\beta_1^3-3888b_1^6c_1\gamma_1\beta_1^3-602112b_1^3b_2^3c_1\gamma_1\beta_1^3+2007936b_1^4b_2^2c_1\gamma_1\beta_1^3-60480b_1^5b_2c_1\gamma_1\beta_1^3+162b_1^7\beta_1^2+37632b_1^4b_2^3\beta_1^2+1296b_1^7\gamma_1^3\beta_1^2+301056b_1^4b_2^3\gamma_1^3\beta_1^2-669312b_1^5b_2^2\gamma_1^3\beta_1^2+30240b_1^6b_2\gamma_1^3\beta_1^2-83664b_1^5b_2^2\beta_1^2+1944b_1^7\gamma_1^2\beta_1^2+451584b_1^4b_2^3\gamma_1^2\beta_1^2-1003968b_1^5b_2^2\gamma_1^2\beta_1^2+45360b_1^6b_2\gamma_1^2\beta_1^2-2664\sqrt{3}b_1^3c_1\sqrt{R_2}\beta_1^2-7616\sqrt{3}b_1b_2^2c_1\sqrt{R_2}\beta_1^2-5328\sqrt{3}b_1^3c_1\gamma_1\sqrt{R_2}\beta_1^2-15232\sqrt{3}b_1b_2^2c_1\gamma_1\sqrt{R_2}\beta_1^2+3780b_1^6b_2\beta_1^2+1848\sqrt{3}\sqrt{R_2}b_1^2b_2c_1\beta_1^2+972b_1^7\gamma_1\beta_1^2+225792b_1^4b_2^3\gamma_1\beta_1^2-501984b_1^5b_2^2\gamma_1\beta_1^2+22680b_1^6b_2\gamma_1\beta_1^2+3696\sqrt{3}\sqrt{R_2}b_1^2b_2c_1\gamma_1\beta_1^2+666\sqrt{3}\sqrt{R_2}b_1^4\beta_1+1904\sqrt{3}\sqrt{R_2}b_1^2b_2^2\beta_1+2664\sqrt{3}\sqrt{R_2}b_1^4\gamma_1^2\beta_1+7616\sqrt{3}\sqrt{R_2}b_1^2b_2^2\gamma_1^2\beta_1-3696\sqrt{3}b_1^3b_2\gamma_1^2\sqrt{R_2}\beta_1-924\sqrt{3}b_1^3b_2\sqrt{R_2}\beta_1-3696\sqrt{3}b_1^3b_2\gamma_1\sqrt{R_2}\beta_1+2664\sqrt{3}\sqrt{R_2}b_1^4\gamma_1\beta_1+7616\sqrt{3}\sqrt{R_2}b_1^2b_2^2\gamma_1\beta_1)y_2+1998\sqrt{3}\sqrt{R_2}b_1^4\gamma_1+5712\sqrt{3}\sqrt{R_2}b_1^2b_2^2\gamma_1=0.$

where

$R_2=\beta_1^2(b_1^2+\beta_1^2)(-9(2\gamma_1+1)^2b_1^4+36c_1\beta_1(2\gamma_1+1)b_1^3+(448(2\gamma_1b_2+b_2)^2-3\beta_1^2(12c_1^2+12\gamma_1(\gamma_1+1)+31))b_1^2+4c_1\beta_1(9\beta_1^2-448b_2^2)(2\gamma_1+1)b_1-12(3c_1^2+7)\beta_1^4+448b_2^2(4c_1^2+1)\beta_1^2).$

5.17. *SECTION*

All sets of real solutions

$s_1=\{b_1=\beta_1\}$,

$s_2=\{\alpha_1=\beta_1,\ \alpha_2=\beta_1\}$,

$s_3=\{\alpha_1=\beta_1,\ \gamma_1=-1/2\}$,

$s_4=\{\alpha_2=\beta_1,\ \gamma_2=3/16\}$,

$s_5=\{c_1=-4c_2/3,\ \gamma_1=-1/2,\ \gamma_2=3/16\}$ and

$s_6=\{a_2=-3a_1/4,\ c_1=-4c_2/3,\ \gamma_1=-\frac{2\alpha_2}{3\alpha_1}-1/2,\ \gamma_2=3/16-\frac{\alpha_2^2}{3\alpha_1^2}\}$.

5.18. *SECTION*

All solutions are

$s_1=\{a_1=0\}$,

$s_2=\{\gamma_1=\frac{2b_2\beta_1\left(-2\beta_2b_2(9c_1+8c_2)+b_2^2(16\gamma_2-3)-12\beta_2^2\right)+3b_1\left(b_2^2(\beta_1(16\gamma_2-3)-6\beta_2)+8\beta_1\beta_2b_2c_2-3\beta_1\beta_2^2\right)}{36b_1b_2^2\beta_2}\}$,

$s_3=\{b_1=0, c_1=\frac{1}{18}(\frac{b_2(16\gamma_2-3)}{\beta_2}-\frac{12\beta_2}{b_2}-16c_2)\}$,

$s_5=\{b_2=0,\beta_1=0\}$,

$s_6=\{b_2=0,\beta_2=0\}$,

$s_7=\{b_1=0,\beta_1=0\}$,

$s_8=\{\beta_1=0,\beta_2=0\}$,

$s_9=\{b_1=-2b_2/3,\beta_2=0\}$ and

$s_{10}=\{\beta_2=0,\gamma_2=3/16\}$.

5.19. *SECTION*

All solutions are

$u_1=\{\beta_1=0\}$,

$u_2=\{\beta_1=-\frac{\sqrt{2a_1b_2(3b_1+2b_2)-3a_2b_1^2}}{\sqrt{3}\sqrt{a_2}}\}$,

$u_3=\{\beta_1=\frac{\sqrt{2a_1b_2(3b_1+2b_2)-3a_2b_1^2}}{\sqrt{3}\sqrt{a_2}}\}$,

$u_4=\{a_1=0,a_2=0\}$,

$u_5 = \{a_2 = 0, b_1 = -2b_2/3\}$ and
$u_6 = \{a_2 = 0, b_2 = 0\}$.

5.20. *SECTION*

All solutions are
$v_1 = \{a_1 = 0\}$,
$v_2 = \{a_2 = \frac{2a_1 b_2(3b_1+2b_2)}{3b_1^2}\}$,
$v_3 = \{b_2 = -3b_1/2\}$,
$v_4 = \{c_1 = \frac{3b_1(8\beta_2 b_2 c_2+b_2^2(16\gamma_2-3)-3\beta_2^2)+4b_2(-16\beta_2 b_2 c_2+b_2^2(16\gamma_2-3)-12\beta_2^2)}{72b_2^2\beta_2}\}$,
$v_5 = \{b_1 = -4b_2/3,\ \beta_2 = 0\}$ and
$v_6 = \{\beta_2 = 0,\ \gamma_2 = 3/16\}$.

5.21. *SECTION*

All solutions are
$w_1 = \{a_2 = \frac{2(2a_1 b_2^2+3a_1 b_1 b_2)}{3b_1^2}\}$,
$w_2 = \{a_1 = 0, b_1 = 0\}$,
$w_3 = \{b_1 = 0, b_2 = 0\}$,
$w_4 = \{b_1 = \frac{4b_2 a_2(32\beta_2 b_2^3 c_2(3-16\gamma_2)+\beta_2^2 b_2^2(256c_2^2-384\gamma_2+153)+384\beta_2^3 b_2 c_2+b_2^4(3-16\gamma_2)^2+144\beta_2^4)}{9a_1(32\beta_2 b_2^3 c_2(3-16\gamma_2)+2\beta_2^2 b_2^2(128c_2^2+24\gamma_2+9)-48\beta_2^3 b_2 c_2+b_2^4(3-16\gamma_2)^2+9\beta_2^4)} - \frac{2}{3}b_2\}$,
$w_5 = \{a_1 = 0, a_2 = 0\}$,
$w_6 = \{a_1 = 0, b_2 = 0\}$,
$w_7 = \{b_2 = 0, \beta_2 = 0\}$ and
$w_8 = \{\beta_2 = 0, \gamma_2 = 3/16\}$.

5.22. *SECTION*

All solutions are
$z_1 = \{a_2 = -\frac{3a_1\left(32\beta_2 b_2^3 c_2(3-16\gamma_2)+8\beta_2^2 b_2^2\left(32c_2^2+6\gamma_2+9\right)-48\beta_2^3 b_2 c_2+b_2^4(3-16\gamma_2)^2+63\beta_2^4\right)}{4\left(32\beta_2 b_2^3 c_2(3-16\gamma_2)+\beta_2^2 b_2^2\left(256c_2^2-384\gamma_2+153\right)+384\beta_2^3 b_2 c_2+b_2^4(3-16\gamma_2)^2+144\beta_2^4\right)}\}$,
$z_2 = \{b_2 = 0\}$,
$z_3 = \{b_2 = 0, \beta_2 = 0\}$ and
$z_4 = \{\beta_2 = 0, \gamma_2 = 3/16\}$.

5.23. *SECTION*

Solving the second equation of (24) we get one of the following sets of real solutions
$v_1 = \{a_1 = 0\}$,
$v_2 = \{a_2 = \frac{2a_1 b_2(3b_1+2b_2)}{3b_1^2}\}$,
$v_3 = \{b_2 = -3b_1/2\}$,
$v_4 = \{c_1 = \frac{3b_1(8\beta_2 b_2 c_2+b_2^2(16\gamma_2-3)-3\beta_2^2)+4b_2(-16\beta_2 b_2 c_2+b_2^2(16\gamma_2-3)-12\beta_2^2)}{72b_2^2\beta_2}\}$,
$v_5 = \{b_1 = -4b_2/3,\ \beta_2 = 0\}$ and
$v_6 = \{\beta_2 = 0,\ \gamma_2 = 3/16\}$.
The only allowed solution is v_3 then we have

$$c_1 = \frac{3b_1(8\beta_2 b_2 c_2 + b_2^2(16\gamma_2 - 3) - 3\beta_2^2) + 4b_2(-16\beta_2 b_2 c_2 + b_2^2(16\gamma_2 - 3) - 12\beta_2^2)}{72b_2^2\beta_2}.$$

The first equation of (24) gives one of the following sets of real solutions
$w_1 = \{a_2 = \frac{2(2a_1 b_2^2+3a_1 b_1 b_2)}{3b_1^2}\}$,
$w_2 = \{a_1 = 0, b_1 = 0\}$,
$w_3 = \{b_1 = 0, b_2 = 0\}$,
$w_4 = \{b_1 = \frac{8b_2 a_2(32\beta_2 b_2^3 c_2(3-16\gamma_2)+\beta_2^2 b_2^2(256c_2^2-384\gamma_2+153)+384\beta_2^3 b_2 c_2+b_2^4(3-16\gamma_2)^2+144\beta_2^4)}{9a_1(32\beta_2 b_2^3 c_2(3-16\gamma_2)+2\beta_2^2 b_2^2(128c_2^2+24\gamma_2+9)-48\beta_2^3 b_2 c_2+b_2^4(3-16\gamma_2)^2+9\beta_2^4)} - \frac{2}{3}b_2\}$,
$w_5 = \{a_1 = 0, a_2 = 0\}$,
$w_6 = \{a_1 = 0, b_2 = 0\}$,
$w_7 = \{b_2 = 0, \beta_2 = 0\}$ and

$w_8 = \{\beta_2 = 0, \gamma_2 = 3/16\}$,
The only allowed solution is w_4 for which the value of b_1 is given as follows

$$\frac{8b_2a_2(32\beta_2b_2^3c_2(3-16\gamma_2)+\beta_2^2b_2^2(256c_2^2-384\gamma_2+153)+384\beta_2^3b_2c_2+b_2^4(3-16\gamma_2)^2+144\beta_2^4)}{9a_1(32\beta_2b_2^3c_2(3-16\gamma_2)+2\beta_2^2b_2^2(128c_2^2+24\gamma_2+9)-48\beta_2^3b_2c_2+b_2^4(3-16\gamma_2)^2+9\beta_2^4)}-\frac{2}{3}b_2.$$

Solving the fourth equation of (24) gives one of the following sets of real solutions
$z_1 = \{a_2 = -\frac{3a_1\left(32\beta_2b_2^3c_2(3-16\gamma_2)+8\beta_2^2b_2^2\left(32c_2^2+6\gamma_2+9\right)-48\beta_2^3b_2c_2+b_2^4(3-16\gamma_2)^2+63\beta_2^4\right)}{4\left(32\beta_2b_2^3c_2(3-16\gamma_2)+\beta_2^2b_2^2\left(256c_2^2-384\gamma_2+153\right)+384\beta_2^3b_2c_2+b_2^4(3-16\gamma_2)^2+144\beta_2^4\right)}\}$,
$z_2 = \{b_2 = 0\}$,
$z_3 = \{b_2 = 0, \beta_2 = 0\}$ and
$z_4 = \{\beta_2 = 0, \gamma_2 = 3/16\}$.
The only allowed solution is z_1, so the value of a_2 is fixed.

Now we have a continuous piecewise differential systems (8)-(10). We solve the algebraic system (15) with respect to y_1 and y_2, and we get one of the same sets of solutions s_0, s_1, s_2 and s_3 as in **Subcase 2.2.1.1**. Then the continuous piecewise differential systems (8)-(10) can have at most one limit cycle.

5.24. *SECTION*

All sets of solutions are
$s_1 = \{a_1 = 0\}$,
$s_2 = \{b_2 = 0\}$,
$s_3 = \{\gamma_2 = \frac{3b_2^2(8\beta_2\gamma_1+\beta_1+4\beta_2)-8\beta_1\beta_2b_2c_2+3\beta_1\beta_2^2}{16b_2^2\beta_1}\}$,
$s_4 = \{b_2 = 0, \beta_1 = 0\}$,
$s_5 = \{b_2 = 0, \beta_2 = 0\}$,
$s_6 = \{\beta_1 = 0, \gamma_1 = -1/2\}$ and
$s_7 = \{\beta_1 = 0, \beta_2 = 0\}$.

5.25. *SECTION*

All sets of real solutions
$s_1 = \{a_1 = a_2,\ \beta_1 = a_2\}$,
$s_2 = \{b_2 = a_2,\ \beta_1 = a_2\}$,
$s_3 = \{a_1 = a_2,\ \alpha_2 = a_2\}$,
$s_4 = \{b_2 = a_2,\ \beta_1 = a_2\}$,
$s_5 = \{c_1 = a_2,\ \beta_1 = a_2\}$,
$s_6 = \{\beta_1 = a_2,\ \gamma_1 = -1/2\}$,
$s_7 = \{\beta_2 = \frac{\alpha_2\left(4b_2^2+9\beta_1^2\right)}{36a_1b_2},\ \gamma_2 = -\frac{6b_2c_1\gamma_1+3b_2c_1+4\beta_1c_2^2}{3\beta_1}\}$,
$s_8 = \{b_2 = a_2,\ \alpha_2 = a_2,\ \gamma_2 = -4c_2^2/3\}$,
$s_9 = \{a_1 = a_2,\ \alpha_2 = a_2,\ \gamma_2 = -\frac{6b_2c_1\gamma_1+3b_2c_1+4\beta_1c_2^2}{3\beta_1}\}$,
$s_{10} = \{c_1 = a_2,\ \beta_1 = a_2,\ \beta_2 = \frac{\alpha_2b_2}{9a_1}\}$,
$s_{11} = \{\beta_1 = a_2,\ \beta_2 = \frac{\alpha_2b_2}{9a_1},\ \gamma_1 = -1/2\}$,
$s_{12} = \{a_1 = a_2,\ \alpha_2 = a_2,\ \beta_1 = a_2,\ \gamma_1 = -1/2\}$,
$s_{13} = \{a_1 = a_2,\ c_1 = a_2,\ \alpha_2 = a_2,\ \beta_1 = a_2\}$.

5.26. *SECTION*

All solutions are
$s_1 = \{a_1 = 0,\ \beta_1 = 0\}$,
$s_2 = \{a_1 = \frac{1}{36}a_2(-\frac{9\beta_1^2}{b_2^2}-64),\ \gamma_2 = \frac{3}{16}(1-\frac{c_1(2\beta_2\gamma_1+\beta_2)}{\beta_1c_2})\}$,
$s_3 = \{a_1 = 0,\ a_2 = 0\}$,
$s_4 = \{c_1 = 0,\ \beta_1 = 0\}$,
$s_5 = \{\beta_1 = 0,\ \beta_2 = 0\}$,
$s_6 = \{\beta_1 = 0,\ \gamma_1 = -1/2\}$,
$s_7 = \{a_2 = 0,\ b_2 = 0,\ \gamma_2 = \frac{3}{16}(1-\frac{c_1(2\beta_2\gamma_1+\beta_2)}{\beta_1c_2})\}$,
$s_8 = \{b_2 = 0,\ c_1 = 0,\ \beta_1 = 0\}$,
$s_9 = \{b_2 = 0,\ \beta_1 = 0,\ \beta_2 = 0\}$,
$s_{10} = \{b_2 = 0,\ \beta_1 = 0,\ \gamma_1 = -1/2\}$,
$s_{11} = \{a_1 = \frac{1}{36}a_2(-\frac{9\beta_1^2}{b_2^2}-64),\ c_2 = 0,\ \gamma_1 = -1/2\}$,

$s_{12} = \{a_1 = \frac{1}{36}a_2(-\frac{9\beta_1^2}{b_2^2} - 64),\ c_1 = 0,\ c_2 = 0\}$,
$s_{13} = \{a_1 = \frac{1}{36}a_2(-\frac{9\beta_1^2}{b_2^2} - 64),\ c_2 = 0,\ \beta_2 = 0\}$,
$s_{14} = \{a_1 = -16a_2/9,\ \beta_1 = 0,\ \gamma_1 = -1/2\}$,
$s_{15} = \{a_1 = -16a_2/9,\ c_1 = 0,\ \beta_1 = 0\}$,
$s_{16} = \{a_1 = -16a_2/9,\ \beta_1 = 0,\ \beta_2 = 0\}$,
$s_{17} = \{a_2 = 0,\ b_2 = 0,\ c_1 = 0,\ c_2 = 0\}$,
$s_{18} = \{a_2 = 0,\ b_2 = 0,\ c_2 = 0,\ \beta_2 = 0\}$ and
$s_{19} = \{a_2 = 0,\ b_2 = 0,\ c_2 = 0,\ \gamma_1 = -1/2\}$.

5.27. *SECTION*

All solutions are
$u_1 = \{a_2 = 0\}$,
$u_2 = \{\beta_2 = 0\}$,
$u_3 = \{\beta_1 = 0,\ \gamma_1 = -1/2\}$,
$u_4 = \{c_1 = -8c_2/3,\ \frac{\sqrt{b_2^2\beta_1^2(128b_2^2-9\beta_1^2)(16b_2^2+9\beta_1^2)}}{128b_2^2\beta_1-9\beta_1^3},\ \gamma_1 = -\frac{3\beta_1^2(16b_2^2+9\beta_1^2)}{8\sqrt{b_2^2\beta_1^2(128b_2^2-9\beta_1^2)(16b_2^2+9\beta_1^2)}} + \frac{\beta_1 c_2}{b_2} - \frac{1}{2}\}$,
$u_5 = \{c_1 = -8c_2/3,\ \beta_2 = \frac{\sqrt{b_2^2\beta_1^2(128b_2^2-9\beta_1^2)(16b_2^2+9\beta_1^2)}}{9\beta_1^3-128b_2^2\beta_1},\ \gamma_1 = \frac{3\beta_1^2(16b_2^2+9\beta_1^2)}{8\sqrt{b_2^2\beta_1^2(128b_2^2-9\beta_1^2)(16b_2^2+9\beta_1^2)}} + \frac{\beta_1 c_2}{b_2} - \frac{1}{2}\}$ and
$u_6 = \{c_1 = -8c_2/3,\ \beta_1 = 0,\ \gamma_1 = -1/2\}$.

5.28. *SECTION*

Solving the first equation of (15) with respect to y_2 after substituting in it only c_1, we get

$$y_2 = -\frac{128b_2(2\gamma_1 c_2 + c_2) + 64b_2^2(2\gamma_1 y_1 + y_1) + \beta_1\left(-128c_2^2 + 18\gamma_1(\gamma_1+1) + 9y_1(2\beta_1\gamma_1 + \beta_1)\right)}{(64b_2^2 + 9\beta_1^2)(2\gamma_1 + 2\beta_1 y_1 + 1)}.$$

Replacing this value of y_2 into the second equation of (15) and solving it with respect to y_2 after invoking the remaining parameters β_2 and γ_1, we get a polynomial of degree four which has the four roots

$$y_1 = -\frac{1}{2}3\sqrt{3}\sqrt{\frac{1}{128b_2^2-9\beta_1^2}} + \frac{3\beta_1(16b_2^2+9\beta_1^2)}{8\sqrt{b_2^2\beta_1^2(128b_2^2-9\beta_1^2)(16b_2^2+9\beta_1^2)}} - \frac{c_2}{b_2},$$
$$y_1 = -\frac{1}{2}3\sqrt{3}\sqrt{\frac{1}{128b_2^2-9\beta_1^2}} + \frac{3\beta_1(16b_2^2+9\beta_1^2)}{8\sqrt{b_2^2\beta_1^2(128b_2^2-9\beta_1^2)(16b_2^2+9\beta_1^2)}} - \frac{c_2}{b_2},$$
$$y_1 = \frac{3}{2}\sqrt{3}\sqrt{\frac{1}{128b_2^2-9\beta_1^2}} + \frac{3\beta_1(16b_2^2+9\beta_1^2)}{8\sqrt{b_2^2\beta_1^2(128b_2^2-9\beta_1^2)(16b_2^2+9\beta_1^2)}} - \frac{c_2}{b_2} \text{ and}$$
$$y_1 = \frac{3}{2}\sqrt{3}\sqrt{\frac{1}{128b_2^2-9\beta_1^2}} + \frac{3\beta_1(16b_2^2+9\beta_1^2)}{8\sqrt{b_2^2\beta_1^2(128b_2^2-9\beta_1^2)(16b_2^2+9\beta_1^2)}} - \frac{c_2}{b_2}.$$

Replacing these values of y_1 into y_2, all cases give $y_2 = y_1$ which does not give limit cycles for the piecewise differential systems (8)-(11) in this subcase.

5.29. *SECTION*

Solving the first equation of (15) with respect to y_2 after substituting in it c_1, we get

$$y_2 = -\frac{128b_2(2\gamma_1 c_2 + c_2) + 64b_2^2(2\gamma_1 y_1 + y_1) + \beta_1\left(-128c_2^2 + 18\gamma_1(\gamma_1+1) + 9y_1(2\beta_1\gamma_1 + \beta_1)\right)}{(64b_2^2 + 9\beta_1^2)(2\gamma_1 + 2\beta_1 y_1 + 1)}.$$

Replacing this value of y_2 into the second equation of (15) and solving it with respect to y_2 after invoking the remaining parameters β_2 and γ_1, we get a polynomial of degree four which has the four roots

$$y_1 = -\frac{1}{2}3\sqrt{3}\sqrt{\frac{1}{128b_2^2-9\beta_1^2}} + \frac{3\beta_1(16b_2^2+9\beta_1^2)}{8\sqrt{b_2^2\beta_1^2(128b_2^2-9\beta_1^2)(16b_2^2+9\beta_1^2)}} - \frac{c_2}{b_2},$$
$$y_1 = -\frac{1}{2}3\sqrt{3}\sqrt{\frac{1}{128b_2^2-9\beta_1^2}} + \frac{3\beta_1(16b_2^2+9\beta_1^2)}{8\sqrt{b_2^2\beta_1^2(128b_2^2-9\beta_1^2)(16b_2^2+9\beta_1^2)}} - \frac{c_2}{b_2},$$
$$y_1 = \frac{3}{2}\sqrt{3}\sqrt{\frac{1}{128b_2^2-9\beta_1^2}} + \frac{3\beta_1(16b_2^2+9\beta_1^2)}{8\sqrt{b_2^2\beta_1^2(128b_2^2-9\beta_1^2)(16b_2^2+9\beta_1^2)}} - \frac{c_2}{b_2} \text{ and}$$
$$y_1 = \frac{3}{2}\sqrt{3}\sqrt{\frac{1}{128b_2^2-9\beta_1^2}} + \frac{3\beta_1(16b_2^2+9\beta_1^2)}{8\sqrt{b_2^2\beta_1^2(128b_2^2-9\beta_1^2)(16b_2^2+9\beta_1^2)}} - \frac{c_2}{b_2}.$$

Replacing these values of y_1 into y_2, all cases cases give $y_2 = y_1$ which does not give limit cycles for the piecewise differential systems (8)-(11) in this subcase.

5.30. *SECTION*

All solutions are
$s_1 = \{a_2 = 0\}$,
$s_2 = \{\beta_1 = 0\}$,
$s_3 = \{\beta_2 = 0, \gamma_2 = 3/16\}$,
$s_4 = \{\beta_1 = -\frac{4}{3}\sqrt{b_2^2\left(12c_1^2 - 1\right)}, \beta_2 = 0, \gamma_2 = 3/16\}$ and
$s_5 = \{\beta_1 = \frac{4}{3}\sqrt{b_2^2\left(12c_1^2 - 1\right)}, \beta_2 = 0, \gamma_2 = 3/16\}$.

5.31. *SECTION*

All solutions are
$u_1 = \{a_2 = 0\}$,
$u_2 = \{\beta_1 = 0,\ \gamma_1 = -1/2\}$,
$u_3 = \{\beta_1 = 0,\ \beta_2 = 0\}$,
$u_4 = \{\beta_2 = 0,\ \gamma_2 = 3/16\}$,
$u_5 = \{\beta_1 = 0,\ \beta_2 = 0,\ \gamma_2 = -9\gamma_1(\gamma_1 + 1)/16\}$,
$u_6 = \{\beta_1 = 0,\ \gamma_1 = -1/2,\ \gamma_2 = 9/64\}$,
$u_7 = \{\beta_2 = \frac{16\beta_1 b_2^4 + 9\beta_1^3 b_2^2}{\sqrt{b_2^2\beta_1^2\left(128b_2^2 - 9\beta_1^2\right)\left(16b_2^2 + 9\beta_1^2\right)}},\ \gamma_1 = -\frac{-36\beta_1^2 b_2^2 + 3\sqrt{b_2^2\beta_1^2\left(1008\beta_1^2 b_2^2 + 2048b_2^4 - 81\beta_1^4\right)} + 512b_2^4}{1024b_2^4 - 72b_2^2\beta_1^2}$,
$\gamma_2 = \frac{54b_2^2}{9\beta_1^2 - 128b_2^2} + \frac{9}{16}\}$,
$u_8 = \{\beta_2 = \frac{\sqrt{b_2^2\beta_1^2\left(128b_2^2 - 9\beta_1^2\right)\left(16b_2^2 + 9\beta_1^2\right)}}{9\beta_1^3 - 128b_2^2\beta_1},\ \gamma_1 = \frac{36\beta_1^2 b_2^2 + 3\sqrt{b_2^2\beta_1^2\left(128b_2^2 - 9\beta_1^2\right)\left(16b_2^2 + 9\beta_1^2\right)} - 512b_2^4}{8b_2^2\left(128b_2^2 - 9\beta_1^2\right)}$,
$\gamma_2 = \frac{54b_2^2}{9\beta_1^2 - 128b_2^2} + \frac{9}{16}\}$,
$u_9 = \{\beta_1 = -8\sqrt{2}b_2/3,\ \beta_2 = 0,\ \gamma_1 = 1,\ \gamma_2 = 3/16\}$ and
$u_{10} = \{\beta_1 = 8\sqrt{2}b_2/3,\ \beta_2 = 0,\ \gamma_1 = 1,\ \gamma_2 = 3/16\}$.

5.32. *SECTION*

All solutions are
$s_1 = \{a_1 = 0,\ a_2 = 0,\ \beta_1 = 0\}$,
$s_2 = \{a_1 = 0,\ c_1 = -\sqrt{\gamma_1}\sqrt{\gamma_1 + 1},\ \alpha_1 = 0, \beta_1 = 0\}$,
$s_3 = \{a_1 = 0,\ c_1 = \sqrt{\gamma_1}\sqrt{\gamma_1 + 1},\ \alpha_1 = 0,\ \beta_1 = 0\}$ and
$s_4 = \{\alpha_2 = \dfrac{a_2\left(6a_1c_1 + 16a_1c_2 + 6\alpha_1\gamma_1 + 3\alpha_1\right)}{3a_1},\ \beta_1 = 0,\ \beta_2 = 0, \gamma_2 = \dfrac{a_2(-c_1^2 + \gamma_1^2 + \gamma_1)}{a_1} - \dfrac{4c_2^2}{3}\}$.

5.33. *SECTION*

All solutions are $s_1 = \{a_2 = \alpha_2 b_2/\beta_2,\ c_2 = \frac{(8\gamma_2 + 3)(b_2^2 + \beta_2^2)}{24b_2\beta_2}\}$,
$s_2 = \{b_1 = 0,\ \beta_1 = 0\}$,
$s_3 = \{c_1 = [-3b_1^2(2\gamma_1 + 1)(4\alpha_2 b_2 - a_2\beta_2) + b_1(9a_1\beta_2 b_2(2\gamma_1 + 1) + \alpha_1\beta_2^2(8\gamma_2 + 3)$
$+ 4\alpha_1 b_2^2(8\gamma_2 + 3)) - \beta_1(3(2\gamma_1 + 1)(-a_2\beta_2\beta_1 + 4\alpha_2 b_2\beta_1 - 3\alpha_1 b_2\beta_2)$
$+ a_1(8\gamma_2 + 3)(4b_2^2 + \beta_2^2))]/[18b_2\beta_2(\alpha_1 b_1 - a_1\beta_1)]$,
$c_2 = \frac{(-3b_1^2(2\gamma_1+1)(\alpha_2 b_2 - a_2\beta_2) - \beta_1(3\beta_1(2\gamma_1+1)(\alpha_2 b_2 - a_2\beta_2) + a_1(8\gamma_2+3)(b_2^2+\beta_2^2)) + \alpha_1 b_1(8\gamma_2+3)(b_2^2+\beta_2^2)}{(24b_2\beta_2(\alpha_1 b_1 - a_1\beta_1)}\}$,
$s_4 = \{a_2 = \alpha_2 b_2/\beta_2,\ \alpha_1 = a_1\beta_1/b_1\}$,
$s_5 = \{\alpha_1 = a_1\beta_1/b_1,\ \gamma_1 = -1/2\}$,
$s_6 = \{b_2 = 0,\ \beta_2 = 0\}$,
$s_7 = \{a_1 = 0,\ a_2 = \alpha_2 b_2/\beta_2,\ \alpha_1 = 0\}$,
$s_8 = \{a_1 = 0, \alpha_1 = 0, \gamma_1 = -1/2\}$,
$s_9 = \{a_1 = 0,\ a_2 = \alpha_2 b_2/\beta_2,\ b_1 = 0\}$,
$s_{10} = \{a_1 = 0,\ b_1 = 0,\ \gamma_1 = -1/2\}$,
$s_{11} = \{a_2 = 0,\ b_2 = 0,\ \gamma_2 = -3/8\}$,
$s_{12} = \{b_1 = 0,\ b_2 = 0,\ \beta_1 = 0\}$,
$s_{13} = \{a_2 = 0,\ b_2 = 0,\ \alpha_1 = a_1\beta_1/b_1\}$,
$s_{14} = \{b_2 = 0,\ \alpha_1 = a_1\beta_1/b_1,\ \gamma_1 = -1/2\}$,
$s_{15} = \{b_2 = 0,\ c_1 = \frac{a_1\beta_2 b_1(6\gamma_1+3) - 3\alpha_2\beta_1(2\beta_1\gamma_1+\beta_1) + \beta_1\beta_2(-8a_1c_2 + 6\alpha_1\gamma_1 + 3\alpha_1) - 3b_1^2(2\alpha_2\gamma_1 + \alpha_2) + 8\alpha_1\beta_2 b_1 c_2}{6\beta_2(\alpha_1 b_1 - a_1\beta_1)}$,
$\gamma_2 = \frac{-3a_2(2\gamma_1+1)(b_1^2+\beta_1^2)}{8\beta_2(\alpha_1 b_1 - a_1\beta_1)} - 3/8\}$,

$s_{16} = \{a_2 = 0,\ c_2 = \frac{(8\gamma_2+3)(b_2^2+\beta_2^2)}{24b_2\beta_2},\ \alpha_2 = 0\},$
$s_{17} = \{b_1 = 0,\ \beta_1 = 0,\ \beta_2 = 0\},$
$s_{18} = \{\alpha_1 = a_1\beta_1/b_1,\ \alpha_2 = 0,\ \beta_2 = 0\},$
$s_{19} = \{\alpha_1 = a_1\beta_1/b_1,\ \beta_2 = 0,\ \gamma_1 = -1/2\},$
$s_{20} = \{\alpha_2 = 0,\ \beta_2 = 0,\ \gamma_2 = -3/8\},$
$s_{21} = \{c_1 = \frac{-3a_2(2\gamma_1+1)(b_1^2+\beta_1^2)+a_1b_2(b_1(6\gamma_1+3)-32\beta_1c_2)+\alpha_1b_2(32b_1c_2+6\beta_1\gamma_1+3\beta_1)}{6b_2(\alpha_1b_1-a_1\beta_1)},\ \beta_2 = 0,$
$\gamma_2 = \frac{3(\beta_1(\alpha_2\beta_1(2\gamma_1+1)+a_1b_2)+b_1^2(2\alpha_2\gamma_1+\alpha_2)-\alpha_1b_1b_2)}{8b_2(\alpha_1b_1-a_1\beta_1)}\},$
$s_{22} = \{a_1 = 0,\ a_2 = \alpha_2b_2/\beta_2,\ b_1 = 0,\ \alpha_1 = 0\},$
$s_{23} = \{a_1 = 0,\ b_1 = 0,\ \alpha_1 = 0,\ \gamma_1 = -1/2\},$
$s_{25} = \{a_1 = 0,\ b_2 = 0,\ \alpha_1 = 0,\ \gamma_1 = -1/2\},$
$s_{26} = \{a_1 = 0,\ a_2 = 0,\ b_1 = 0,\ b_2 = 0\},$
$s_{27} = \{a_1 = 0,\ b_1 = 0,\ b_2 = 0,\ \gamma_1 = -1/2\},$
$s_{28} = \{a_1 = 0,\ \alpha_1 = 0,\ \alpha_2 = 0,\ \beta_2 = 0\},$
$s_{29} = \{a_1 = 0,\ \alpha_1 = 0,\ \beta_2 = 0,\ \gamma_1 = -1/2\},$
$s_{30} = \{a_1 = 0,\ b_1 = 0,\ \alpha_2 = 0,\ \beta_2 = 0\},$
$s_{31} = \{a_1 = 0,\ b_1 = 0,\ \beta_2 = 0,\ \gamma_1 = -1/2\},$
$s_{32} = \{a_1 = 0,\ a_2 = 0,\ b_1 = 0,\ b_2 = 0,\ \alpha_1 = 0\},$
$s_{33} = \{a_1 = 0,\ b_1 = 0,\ b_2 = 0,\ \alpha_1 = 0,\ \gamma_1 = -1/2\},$
$s_{34} = \{a_1 = 0,\ b_1 = 0,\ \alpha_1 = 0,\ \alpha_2 = 0,\ \beta_2 = 0\}$ and
$s_{35} = \{a_1 = 0,\ b_1 = 0,\ \alpha_1 = 0,\ \beta_2 = 0,\ \gamma_1 = -1/2\}.$

5.34. *SECTION*

All solutions are $u_1 = \{\alpha_2 = \dfrac{\beta_2\left(4\beta_2^2(\alpha_1b_1 - a_1\beta_1) + 3a_2b_1^2\beta_1 + 8a_1b_2^2\beta_1 + 3a_2\beta_1^3 - 8\alpha_1b_1b_2^2\right)}{3b_2\beta_1(b_1^2+\beta_1^2)}\},$
$u_2 = \{b_2 = 0,\ \alpha_1 = \dfrac{\beta_1}{4b_1}\left(4a_1 - \frac{3a_2(b_1^2+\beta_1^2)}{\beta_2^2}\right)\},$
$u_3 = \{b_1 = 0,\ \beta_1 = 0\},$
$u_4 = \{b_2 = -\dfrac{\beta_2}{\sqrt{2}},\ \beta_1 = 0\},$
$u_5 = \{b_2 = \dfrac{\beta_2}{\sqrt{2}},\ \beta_1 = 0\},$
$u_6 = \{\alpha_1 = 0,\ \beta_1 = 0\},$
$u_7 = \{\beta_1 = 0,\ \beta_2 = 0\},$
$u_8 = \{b_2 = 0,\ \beta_2 = 0\}$ and
$u_9 = \{a_2 = \dfrac{4a_1\beta_2^2}{3\beta_1^2},\ b_1 = 0,\ b_2 = 0\}.$

5.35. *SECTION*

All solutions are
$v_1 = \{a_2 = \dfrac{4\beta_2^2(a_1\beta_1 - \alpha_1b_1) + b_2\left(6a_1\beta_1b_1 - 16a_1b_2\beta_1 + 3\alpha_1\beta_1^2 - 3\alpha_1b_1^2 + 16\alpha_1b_2b_1\right)}{3\beta_1(b_1^2+\beta_1^2)}\},$
$v_2 = \{\beta_1 = \dfrac{\alpha_1b_1}{a_1}\},$
$v_3 = \{\beta_2 = 0\},$
$v_4 = \{\beta_2 = -\sqrt{2}b_2\},$
$v_5 = \{\beta_2 = \sqrt{2}b_2\},$
$v_6 = \{a_1 = 0,\ b_1 = 0\},$
$v_7 = \{a_1 = 0,\ \alpha_1 = 0\},$
$v_8 = \{b_1 = 0,\ \beta_1 = 0\},$
$v_9 = \{\alpha_1 = 0,\ \beta_1 = 0\},$
$v_{10} = \{\beta_1 = 0,\ \beta_2 = -\sqrt{b_2(16b_2 - 3b_1)}/2\}$ and
$v_{11} = \{\beta_1 = 0,\ \beta_2 = \sqrt{b_2(16b_2 - 3b_1)}/2\}.$

5.36. *SECTION*

All solutions are $z_1 = \{\beta_1 = \frac{\alpha_1b_1}{a_1}\},$
$z_2 = \{\gamma_1 = \frac{(\beta_1(8\gamma_2+3)-4\beta_2)}{8\beta_2}$

$- \frac{9(-b_2^2\beta_1^2\beta_2^4(9b_1^2+32b_2^2+9\beta_1^2-16\beta_2^2)(\beta_2^2-2b_2^2)(4\beta_2^2(9b_1^2-32b_2^2+9\beta_1^2)+b_2^2(9b_1^2-256b_2^2+9\beta_1^2)-16\beta_2^4))^{1/2}}{8\beta_2^2(\beta_2^2-2b_2^2)(-4\beta_2^2(9b_1^2-32b_2^2+9\beta_1^2)+b_2^2(-9b_1^2+256b_2^2-9\beta_1^2)+16\beta_2^4)}\},$
$z_3 = \{\gamma_1 = \frac{(\beta_1(8\gamma_2+3)-4\beta_2)}{8\beta_2}$
$+ \frac{9(-b_2^2\beta_1^2\beta_2^4(9b_1^2+32b_2^2+9\beta_1^2-16\beta_2^2)(\beta_2^2-2b_2^2)(4\beta_2^2(9b_1^2-32b_2^2+9\beta_1^2)+b_2^2(9b_1^2-256b_2^2+9\beta_1^2)-16\beta_2^4))^{1/2}}{8\beta_2^2(\beta_2^2-2b_2^2)(-4\beta_2^2(9b_1^2-32b_2^2+9\beta_1^2)+b_2^2(-9b_1^2+256b_2^2-9\beta_1^2)+16\beta_2^4)}\},$
$z_4 = \{a_1 = 0,\ b_1 = 0\},$
$z_5 = \{a_1 = 0,\ \alpha_1 = 0\},$
$z_6 = \{b_1 = -\frac{1}{3}\sqrt{4\beta_2^2 - 9\beta_1^2},\ b_2 = 0\},$
$z_7 = \{b_1 = \frac{1}{3}\sqrt{4\beta_2^2 - 9\beta_1^2},\ b_2 = 0\},$
$z_8 = \{b_2 = -\frac{\beta_2}{\sqrt{2}},\ \beta_1 = 0\},$
$z_9 = \{b_2 = \frac{\beta_2}{\sqrt{2}},\ \beta_1 = 0\},$
$z_{10} = \{b_1 = -\frac{1}{3}\sqrt{\frac{32\beta_2^2}{3} - 9\beta_1^2},\ b_2 = -\frac{\beta_2}{\sqrt{6}}\},$
$z_{11} = \{b_1 = \frac{1}{3}\sqrt{\frac{32\beta_2^2}{3} - 9\beta_1^2},\ b_2 = -\frac{\beta_2}{\sqrt{6}}\},$
$z_{12} = \{b_1 = -\frac{1}{3}\sqrt{\frac{32\beta_2^2}{3} - 9\beta_1^2},\ b_2 = \frac{\beta_2}{\sqrt{6}}\},$
$z_{13} = \{b_1 = \frac{1}{3}\sqrt{\frac{32\beta_2^2}{3} - 9\beta_1^2},\ b_2 = \frac{\beta_2}{\sqrt{6}}\},$
$z_{14} = \{b_1 = -\frac{4(4b_2^2+\beta_2^2)}{3\sqrt{b_2^2+4\beta_2^2}},\ \beta_1 = 0\},$
$z_{15} = \{b_1 = \frac{4(4b_2^2+\beta_2^2)}{3\sqrt{b_2^2+4\beta_2^2}},\ \beta_1 = 0\},$
$z_{16} = \{b_1 = -\frac{1}{3}\sqrt{256b_2^2 - 9\beta_1^2},\ \beta_2 = 0\},$
$z_{17} = \{b_1 = \frac{1}{3}\sqrt{256b_2^2 - 9\beta_1^2},\ \beta_2 = 0\},$
$z_{18} = \{b_2 = 0,\ \beta_2 = 0\},$
$z_{19} = \{\beta_1 = 0,\ \beta_2 = 0\},$
$z_{20} = \{\beta_2 = 0,\ \gamma_2 = -\frac{3}{8}\},$
$z_{21} = \{b_1 = -\frac{41}{3}\sqrt{\frac{2}{3}}\beta_2,\ b_2 = -\frac{\beta_2}{\sqrt{6}},\ \beta_1 = 0\},$
$z_{22} = \{b_1 = \frac{4}{3}\sqrt{\frac{2}{3}}\beta_2,\ b_2 = -\frac{\beta_2}{\sqrt{6}},\ \beta_1 = 0\},$
$z_{23} = \{b_1 = -\frac{4}{3}\sqrt{\frac{2}{3}}\beta_2,\ b_2 = \frac{\beta_2}{\sqrt{6}},\ \beta_1 = 0\}$ and
$z_{24} = \{b_1 = \frac{4}{3}\sqrt{\frac{2}{3}}\beta_2,\ b_2 = \frac{\beta_2}{\sqrt{6}},\ \beta_1 = 0\}.$

5.37. *SECTION*

All solutions are $v_1 = \{b_1 = 0\}$,
$v_2 = \{\alpha_1 = 0\}$,
$v_3 = \{\alpha_2 = -\sqrt{2}a_2\}$,
$v_4 = \{\beta_2 = 0\}$ and
$v_5 = \{\beta_2 = -\frac{3b_1}{4\sqrt{2}}\}$.

5.38. *SECTION*

All solutions are $u_1 = \{\beta_2 = 0\}$,
$u_2 = \{a_1 = 3a_2(b_1^2 + \beta_1^2)/(4\beta_2^2) + \alpha_1 b_1/\beta_1,\ \alpha_2 = 3a_2 b_1/(2\beta_2) + \alpha_1\beta_2/\beta_1\}$,
$u_3 = \{a_1 = \alpha_1 b_1/\beta_1,\ a_2 = 0\}$,
$u_4 = \{b_1 = 0, \beta_1 = 0\}$,
$u_5 = \{\alpha_1 = 0, \beta_1 = 0\}$,
$u_6 = \{a_2 = 0, \beta_2 = 0\}$,
$u_7 = \{a_2 = 0, \alpha_1 = 0, \beta_1 = 0\}$ and
$u_8 = \{a_2 = 0, \beta_1 = 0, \beta_2 = 0\}$.

5.39. *SECTION*

All solutions are $u_1 = \{a_2 = 0\}$,
$u_2 = \{\beta_1 = 0\}$,

$u_3 = \{b_1 = 0,\ \beta_1 = -4\beta_2\sqrt{64c_2^2+9}/(3\sqrt{256c_2^2+9})\}$,
$u_4 = \{b_1 = 0,\ \beta_1 = 4\beta_2\sqrt{64c_2^2+9}/(3\sqrt{256c_2^2+9})\}$,
$u_5 = \{b_1 = -4\beta_2\sqrt{64c_2^2+9}/(3\sqrt{256c_2^2+9}),\ \beta_1 = 0\}$,
$u_6 = \{b_1 = 4\beta_2\sqrt{64c_2^2+9}/(3\sqrt{256c_2^2+9}),\ \beta_1 = 0\}$,
$u_7 = \{a_2 = 0,\ \beta_1 = -\sqrt{16\beta_2^2(64c_2^2+9) - 9b_1^2(256c_2^2+9)}/(3\sqrt{256c_2^2+9})\}$ and
$u_8 = \{a_2 = 0,\ \beta_1 = \sqrt{16\beta_2^2(64c_2^2+9) - 9b_1^2(256c_2^2+9)}/(3\sqrt{256c_2^2+9})\}$.

5.40. *SECTION*

All solutions are $u_1 = \{b_2 = 0\}$,
$u_2 = \{\alpha_2 = 0\}$,
$u_3 = \{b_1 = 0,\ \beta_1 = 0\}$,
$u_4 = \{b_2 = 3b_1/16,\ \beta_1 = 0\}$ and
$u_5 = \{\alpha_1 = 0,\ \beta_1 = 0\}$.

5.41. *SECTION*

All solutions are $u_1 = \{a_2 = 3a_1/16\}$,
$u_2 = \{b_1 = 0\}$,
$u_3 = \{\alpha_2 = 0\}$ and
$u_4 = \{\gamma_1 = -1/2\}$.